# Algebraic categorification and its applications, II

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Winter School "Geometry and physicf" January 17.24, 2015, Srni, Czech Republic

 $S_n$  — the symmetric group on  $\{1, 2, \ldots, n\}$ 

$$\mathbf{P}_n := \{\lambda = (\lambda_1, \dots, \lambda_k) : \lambda_1 \ge \dots \ge \lambda_k, \, \lambda_1 + \dots + \lambda_k = n\}$$

 $\lambda \in \mathbf{P}_n$  is called a partition of n, denoted  $\lambda \vdash n$ 

 $\mathcal{S}^{\lambda}$  — the Specht module associated to  $\lambda$ 

**Theorem**.  $\{S^{\lambda} : \lambda \vdash n\}$  is a cross-section of isomorphism classes of simple  $S_n$ -modules.

#### Examples:

- $S^{(n)}$  is the trivial module
- $S^{(1,1,\ldots,1)}$  is the sign module
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 $\mathcal{O}-\operatorname{BGG}\operatorname{category}\,\mathcal{O}$ 

 $\mathcal{O}_0$  — principal block of  $\mathcal{O}$ 

 $S_n$  — Weyl group of  $\mathfrak{g}$ 

 $M(\mu)$  — Verma module with highest weight  $\mu$ 

 $L(\mu)$  — unique simple quotient of  $M(\mu)$ 

Theorem.  $\{L(w) := L(w \cdot 0) : w \in S_n\}$  is a cross-section of isomorphism classes of simple objects in  $\mathcal{O}_0$ 

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Corollary. Gr( $\mathcal{O}_0$ )  $\cong \mathbb{Z}[S_n]$ .

Note.  $\{[L(w)] : w \in S_n\}$  is the natural basis in  $Gr(\mathcal{O}_0)$ .

 $\Delta(w) := M(w \cdot 0)$ 

Fact.  $\{[\Delta(w)] : w \in S_n\}$  is the standard basis in  $Gr(\mathcal{O}_0)$ .

**Reason**:  $[\Delta(x) : L(y)] \neq 0$  implies  $x \leq y$  and  $[\Delta(x) : L(x)] = 1$ .

Fact.  $\mathcal{O}_0$  has finite global dimension.

P(w) — the indecomposable projective cover of L(w)

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- $\Delta(w) \subset T(w)$  and the cokernel has a Verma flag;
- ▶ T(w) is self-dual.
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- Fact.  $\{[T(w)] : w \in S_n\}$  is a basis in  $Gr(\mathcal{O}_0)$ .

Reason: Extensions between Vermas are directed.

Question. Which bases in  $\mathbb{Z}[S_n]$  correspond to:

- $\blacktriangleright \{[L(w)]: w \in S_n\}?$
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Observation. For s simple reflection and  $w \in S_n$  there are s.e.s.  $\Delta(ws) \hookrightarrow \theta_s \Delta(w) \twoheadrightarrow \Delta(w)$  if ws > w,  $\Delta(w) \hookrightarrow \theta_s \Delta(w) \twoheadrightarrow \Delta(ws)$  if ws < w.

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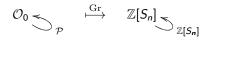
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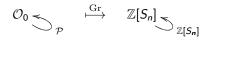
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#### Kazhdan-Lusztig basis

Note. The action of  $\mathcal{P}$  categorifies  $\mathbb{Z}[S_n]$  and not  $S_n$ .

Question. What is  $\{[\theta_w], w \in S_n\}$ ?

Answer. This is the Kazhdan-Lusztig basis.

Remark. This is equivalent to Kazhdan-Lusztig conjecture (=theorem).

Remark. Recent algebraic proof by Elias-Williamson.

Remark. To define Kazhdan-Lusztig basis one needs to deform  $\mathbb{Z}[S_n]$  to the Hecke algebra.

Categorically this means to introduce a grading on  $\mathcal{O}_0$ .

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grading:  $\deg(x_i) = 2$ 

 $S_n$  acts on  $\mathbb{C}[x_1, x_2, \ldots, x_n]$  by permuting indices

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 $\mathbb{C}[x_1, x_2, \dots, x_n]^{S_n}_+ = \bigoplus_{i>0} \mathbb{C}[x_1, x_2, \dots, x_n]^{s_n}_i$ 

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# Categorification of permutation modules

 $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  — composition of *n* 

 $\mathfrak{g}_{\lambda} \subset \mathfrak{g}$  — corresponding parabolic subalgebra.

 $W_{\lambda}$  — corresponding Young subgroup of  $S_n$ 

 $_{\lambda} \text{Long}$  — longest representatives in  $W_{\lambda} \setminus W$ 

 $\mathcal{X}_{\lambda}$  — Serre subcategory of  $\mathcal{O}_0$  generated by L(w),  $w \notin {}_{\lambda} \mathrm{Long}$ 

Fact:  $\mathcal{P}$  preserves  $\mathcal{X}_{\lambda}$ 

**Theorem**. [M.-Stroppel] The induced action of  $\mathcal{P}$  on  $\mathcal{O}_0/\mathcal{X}_{\lambda}$  categorifies the permutation module  $\operatorname{Ind}_{W_{\lambda}}^{W}$  triv.

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These two categorifications are indeed equivalent (using derived completion functors).

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Simple transitive categorification of a Specht module (using  $\mathcal{P}$ ) is unique up to equivalence.

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Note. Uses combinatorially defined subquotients of  $\mathcal{O}_0$ 

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Filtration of  $\mathbb{Z}[S_n]$  using Gelfand-Kirillov dimension of simples in  $\mathcal{O}$ 

Uniqueness of categorification allows to compare different categories of g-modules

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## Summary of bonuses provided by categorification

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# THANK YOU!!!

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