The Hecke Algebra and its Categorification

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The formalism of Hecke algebras

- G finite group, $(\mathbb{Z}G, *)$ group ring
- $B \subset G$ subgroup, ${}^B(\mathbb{Z}G)^B$ the B-biinvariant functions, stable under $*$ and under $*_B := */|B|$
- $\mathcal{H} = \mathcal{H}(G, B) := ({}^B(\mathbb{Z}G)^B, *_B)$ the **Hecke algebra**
- Unit element characteristic function $\underline{B} = 1_{\mathcal{H}}$ the of B
- $\mathbb{Z}$-basis of $\mathcal{H}$ are the characteristic functions $\underline{D}$ for D runnig over all double cosets
- For V a G-module V^B is an $\mathcal{H}$-module

The algebra of Hecke operators

- $G = \mathrm{GL}(2;\mathbb{R})^+$ acts on the upper half plane $\mathbb{H}^+$
- $G = \mathrm{GL}(2;\mathbb{R})^+$ acts on $\mathcal{O}^{\mathrm{an}}(\mathbb{H}^+)(\mathrm{d}z)^{\otimes k}$
- Take $B = \mathrm{SL}(2;\mathbb{Z})$
- $\mathcal{H}(G,B)$ acts on $\left(\mathcal{O}^{\mathrm{an}}(\mathbb{H}^+)(\mathrm{d}z)^{\otimes k}\right)^{\mathrm{SL}(2;\mathbb{Z})}$
- These are the classical Hecke operators acting on modular functions
- Sure these groups are infinite. Then should use the more general definition

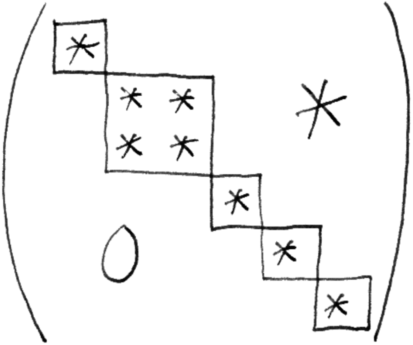
$$\mathcal{H}(G,B) := \mathrm{End}_{\mathbb{Z}}^{G}(\mathrm{prod}_{B}^{G}\,\mathbb{Z})^{\mathrm{opp}}$$

Towards the Iwahori-Matumoto Hecke algebras

- Take $G = \mathrm{GL}(n; \mathbb{F}_q)$ and B upper triangular matrices
- Bruhat decomposition $G = \bigsqcup_{x \in W} BxB$ for $W = \mathcal{S}_n$ the permutation matrices
- The $T_x := \underline{BxB}$ form a basis of the Hecke algebra

For $s \in S := \{(i, i+1) \text{ transposition} \mid 1 \leq i < n\}$ know $B \sqcup BsB := P_s \subset G$ is a subgroup and $|P_s/B| = q+1$

For example here is $P_{(2,3)}$:



Towards generators and relations of Hecke algebras

- For $s \in S$ know $B \sqcup BsB := P_s \subset G$ is a subgroup and $|P_s/B| = q + 1$
- Deduce $(T_s + 1)^2 = (q + 1)(T_s + 1)$ and thus $T_s^2 = (q - 1)T_s + q$ for $s \in S$
- Let $l(x)$ number of Fehlstände of x
- $BxB \times_B ByB \xrightarrow{\sim} BxyB$ if $l(x) + l(y) = l(xy)$
- Deduce $T_x T_y = T_{xy} \quad$ if $l(x) + l(y) = l(xy)$
- Deduce $T_x T_y = \sum_z c_{x,y}^z(q) T_z$ with $c_{x,y}^z$ polynomial in q

Generic Hecke algebra

- Generic Hecke algebra $\mathcal{H} := \bigoplus_{x \in W} \mathbb{Z}[q] T_x$ with $T_s^2 = (q-1)T_s + q$ for $s \in S$ and $T_x T_y = T_{xy}$ if $l(x) + l(y) = l(xy)$
- Called **Iwahori-Matsumoto** Hecke algebra
- Specializes to group ring $\mathbb{Z}\mathcal{S}_n$ for $q \mapsto 1$
- $\mathcal{H}$ might be thought of as quantization of $\mathbb{Z}\mathcal{S}_n$
- We want to discuss the categorification of $\mathcal{H}$
- One application is to refine knot polynomials to knot homology

Coxeter System

- A Coxeter System (W, S) is a group W with a finite subset $S \subset W$ such that W is generated by S subject to the only relations

  $$(st)^{m(s,t)} = e$$

  for some symmetric matrix $m : S \times S \to \mathbb{Z}_{\geq 1} \sqcup \{\infty\}$ which is 1 on the diagonal and > 1 off the diagonal
- For $x \in W$ put $l(x)$ minimal number of $s \in S$ needed to express x
- Any finite group W generated by reflections is always part of a Coxeter system (W, S)

- For any Coxeter System (W, S) there is a Hecke algebra $\mathcal{H} = \mathcal{H}(W, S) := \bigoplus_{x \in W} \mathbb{Z}[q] T_x$ with

$$T_s^2 = (q-1) T_s + q \text{ for } s \in S$$

$$T_x T_y = T_{xy} \text{ if } l(x) + l(y) = l(xy)$$

- Called **Iwahori-Matsumoto** Hecke algebra
- Specializes to group ring $\mathbb{Z}W$ for $q \mapsto 1$

Alternative description of the Hecke algebra $\mathcal{H}(W, S)$ as $\mathbb{Z}[q]$-ringalgebra with generators T_s for $s \in S$ subject to the relations $T_s^2 = (q-1)T_s + q$ and braid relations

$$T_s T_t \ldots = T_t T_s \ldots$$

with $m(s, t)$ factors on both sides

Categorification of Hecke algebra $\mathcal{H}(W, S)$ by bimodules

- Let (W, S) be a Coxeter system
- Choose a representation $W \looparrowright V$ wich is
  - finite dimensional
  - over an infinite field k with char $k \neq 2$
  - exactly the conjugates $t = wsw^{-1}$ of elements of $s \in S$ have fixed point spaces of codimension one
- Call such a representation **reflection faithful**
- Typical example: Symmetric group $\mathcal{S}_n$ permuting the coordinates of k^n

Categorification of Hecke algebra $\mathcal{H}(W, S)$ by bimodules

- Choose $W \looparrowright V$ reflection faithful representation
- Put $R := \mathcal{O}(V)$ a polynomial ring
- Let $R\text{-Mod}_{\mathbb{Z}}\text{-}R$ be the category of $\mathbb{Z}$-graded R-bimodules or more precisely $R \otimes_k R$-modules
- Let

  $$R\text{-Modbf}_{\mathbb{Z}}\text{-}R$$

  be the subcategory of graded bifinite bimodules
- Bifinite means finitely generated from the left and from the right

Categorification of Hecke algebra $\mathcal{H}(W, S)$ by bimodules

- Let $\langle R\text{-Modbf}_{\mathbb{Z}}\text{-}R\rangle$ be the split Grothendieck group
- It becomes a ring under $\otimes_R$
- **Categorification Theorem:** There is exactly one ring homomorphism

  $$\mathcal{E} : \mathcal{H} \to \langle R\text{-Modbf}_{\mathbb{Z}}\text{-}R\rangle$$

  such that we have $\mathcal{E}(T_s + 1) = \langle R \otimes_{R^s} R\rangle\ \forall s \in \mathcal{S}$ and $\mathcal{E}(q) = \langle R\langle -1\rangle\rangle$
- Notation $(M\langle n\rangle)_i = M_{i+n}$ for grading shift

Sketch of proof of bimodule-categorification

- Recall quadratic relation $T_s^2 = (q-1)T_s + q$
- Rewrite to $(T_s+1)^2 = (q+1)(T_s+1)$
- Need $\langle R \otimes_{R^s} R\rangle^2 = \langle R\langle -1\rangle \oplus R\rangle\langle R \otimes_{R^s} R\rangle$
- $(R \otimes_{R^s} R) \otimes_R (R \otimes_{R^s} R) \cong (R\langle -1\rangle \oplus R) \otimes_R (R \otimes_{R^s} R)$
- $R \otimes_{R^s} R \otimes_{R^s} R \cong (R \otimes_{R^s} R)\langle -1\rangle \oplus (R \otimes_{R^s} R)$
- Follows from recalling in the middle left
  $R = \alpha R^s \oplus R^s \cong R^s\langle -1\rangle \oplus R^s$
  with $\alpha \in V^*$ equation of V^s
- So only need to check braid relations for bimodules
- Need only to argue for dihedral groups. Omitted.

Categorification of Kazhdan-Lusztig basis

- Extend scalars in Hecke algebra $\mathcal{H}$ from $\mathbb{Z}[q]$ to $\mathbb{Z}[v, v^{-1}]$ by $q = v^{-2}$
- Kazhdan-Lusztig constructed a canonical basis $(C_x)_{x \in W}$ of $\mathcal{H}_v$ as a $\mathbb{Z}[v, v^{-1}]$-module
- Regrade R to sit only in even degrees to get categorification map $\mathcal{E} : \mathcal{H}_v \to \langle R\text{-Modbf}_{\mathbb{Z}}\text{-} R \rangle$
- **Indecomposable Bimodule Theorem:** There exist indecomposable bimodules $B_x \in R\text{-Modbf}_{\mathbb{Z}}\text{-} R$ such that $\mathcal{E}(C_x) = \langle B_x \rangle$
- In words: The elements of the Kazhdan-Lusztig canonical basis correspond under the categorification theorem to indecomposable bimodules

Definition of Kazhdan-Lusztig basis

- Put $H_x = v^{l(x)} T_x$
- $C_x \in H_x + \sum_y v\mathbb{Z}[v] H_y$ and $d(C_x) = C_x$ is selfdual, uniquely determines the canonical basis element C_x
- Duality $d : \mathcal{H}_v \to \mathcal{H}_v$ the unique ring automorphism, which fixes $H_s + v$ for $s \in S$ and maps $d : v \mapsto v^{-1}$
- In particular $C_s = H_s + v$ for $s \in S$ a simple reflection

Discussion of categorification of KL-basis

- Take simple reflections $s, t, \ldots, u \in S$
- Form the bimodules $R \otimes_{R^s} R \otimes_{R^t} R \ldots \otimes_{R^u} R$
- Krull-Schmid decompose those bimodules: Get very special indecomposable bimodules B_x categorifying the Kazhdan-Lusztig basis
- Call the graded bimodules $R \otimes_{R^s} R \otimes_{R^t} R \ldots \otimes_{R^u} R$ and all you get from them by taking finite direct sums, direct summands and grading shifts **special bimodules** and denote the monoidal category of those

$$R\text{-SMod}_{\mathbb{Z}}\text{-}R$$

Its indecomposables are precisely the $B_x\langle n\rangle$.

Positivity Corollaries of categorification

- $C_x C_y \in \sum_z \mathbb{N}[v, v^{-1}] C_z$ since $B_x \otimes_R B_y$ is an actual bimodule, decomposes as

$$B_x \otimes_R B_y = \bigoplus_{z,n} B_z \langle n \rangle^{m(z,n)}$$

- $C_x = \sum_y P_{x,y}(v) H_y$ with $P_{x,y}(v) \in \mathbb{Z}[v]$ the **Kazhdan-Lusztig polynomials**
- Coefficients of Kazhdan-Lusztig-Polynomials are non-negative, since they can be interpreted as $\operatorname{rk} \operatorname{Hom}_{R-R}(\mathcal{O}(\Gamma(x)), B_y)$
- Here $\Gamma(x) \subset V \times V$ is the graph of x and $\mathcal{O}(\Gamma(x))$ the regular functions on $\Gamma(x)$, a quotient of $\mathcal{O}(V \times V) = R \otimes R$. Put another way, $\mathcal{O}(\Gamma(x)) = R$ as left R-module with the right R-action twisted by x

- Example: $w_\circ \in W$ longest element of finite reflection group. $C_{w_\circ} = v^{l(w_\circ)} \sum_{x \in W} T_x = \sum_{x \in W} v^{l(w_\circ)-l(x)} H_x$

$$B_{w_\circ} = \mathcal{O}\left(\bigcup_{x \in W} \Gamma(x)\right)$$

  is the bimodule of all regular functions on the union of the graphs of all Weyl group elements $\Gamma(x) \subset V \times V$
- In general B_x is still supported on $\bigcup_{y \leq x} \Gamma(y)$
- If $C_x = \sum_{y \leq x} v^{l(x)-l(y)} H_y$, then $B_x = \mathcal{O}\left(\bigcup_{y \leq x} \Gamma(y)\right)$

Application to representation theory

- $\mathfrak{g}$ a semisimple complex Lie algebra, $Z \subset U(\mathfrak{g})$ the center of its enveloping algebra
- $\mathcal{M} \subset \mathfrak{g}$-Mod the category of all representations of $\mathfrak{g}$ locally finite under Z
- $\mathcal{P}$ the category of all functors $\mathcal{M} \to \mathcal{M}$ isomorphic to a direct summand of some functor $E \otimes_{\mathbb{C}}$ for E finite dimensional representation, so-called **projective functors**
- Equivalence of categories between $\{$indecomposable projective functors starting and ending with the trivial central character$\}$ and $\{\hat{B}_x \mid x \in W\} \subset \hat{R}$-Mod-$\hat{R}$

Application to representation theory, variant

- $\mathcal{O}_\circ \subset \mathfrak{g}$-Mod principal block of BGG-category $\mathcal{O}$
- Equivalence of categories between {indecomposable projectives of $\mathcal{O}_\circ$} and $\{B_x \otimes_R \mathbb{C} \mid x \in W\} \subset R$-Mod
- Gives new proof of KL-conjecture on Jordan-Hölder multiplicities of Verma modules
- $$\mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ) \cong \mathrm{Hot}^{\mathrm{b}}(\mathrm{proj}\,\mathcal{O}_\circ) \cong \mathrm{Hot}^{\mathrm{b}}(R\text{-SMod})$$
  for R-SMod $\subset R$-Mod the subcategory of all $B \otimes_R \mathbb{C}$ for $B \in R$-SMod-R
- Can define graded version $\mathcal{O}_\circ^{\mathbb{Z}}$ of $\mathcal{O}_\circ$ formally such that $\mathrm{proj}\,\mathcal{O}_\circ^{\mathbb{Z}} = R\text{-SMod}_{\mathbb{Z}}$

Categorification of $\mathbb{N}$

- k a field
- $\dim : \mathrm{Modf}_k \to \mathbb{N}$ "decategorification"
- Multiplication corresponds to tensor product

$$\dim(V \otimes W) = (\dim V)(\dim W)$$

Categorification of $\mathrm{Ens}(X, \mathbb{N}) = \mathrm{Maps}(X, \mathbb{N})$ for X a set

- k a field and $\mathrm{Mod}_k /X \supset \mathrm{Modf}_k /X$ sheaves on the discrete set X alias families $(\mathcal{F}_x)_{x \in X}$ of vector spaces respectively finitely generated vector spaces
- $\mathrm{Dim} : \mathrm{Modf}_k /X \to \mathrm{Ens}(X, \mathbb{N})$ "decategorification"
- Multiplication corresponds to tensor product

$$\mathrm{Dim}(\mathcal{F} \otimes \mathcal{G}) = (\mathrm{Dim}\,\mathcal{F})(\mathrm{Dim}\,\mathcal{G})$$

Categorification of maps

- $f : X \to Y$ map of finite sets leads to morphisms

$$\mathrm{Ens}(X, \mathbb{N}) \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{Ens}(Y, \mathbb{N})$$

  called pull-back and integration along the fibres
- $|X|1 = c_! c^* 1$ for $c : X \to \mathrm{pt}$ constant map

- $f : X \to Y$ map of finite sets leads to functors

$$\mathrm{Modf}_k\,/X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{Modf}_k\,/\,Y$$

  called pull-back and integration along the fibres
- $(f^*\mathcal{G})_x := \mathcal{G}_{f(x)}$ and $(f_!\mathcal{F})_y := \bigoplus_{x\in f^{-1}(y)} \mathcal{F}_x$
- Commutative diagrams

$$\begin{array}{ccc}
\mathrm{Modf}_k\,/X & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathrm{Modf}_k\,/\,Y \\
\mathrm{Dim}\downarrow & & \downarrow \mathrm{Dim} \\
\mathrm{Ens}(X,\mathbb{N}) & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathrm{Ens}(Y,\mathbb{N})
\end{array}$$

Grothendieck function-sheaf correspondence

- To $X_\circ$ variety over $\mathbb{F}_q$ and ℓ prime $\neq$ char $\mathbb{F}_q$ associate $\mathrm{Der}^c(X_\circ; \mathbb{Q}_l)$ triangulated $\mathbb{Q}_\ell$-category
- Called „cohomologically constructible complexes of étale sheaves on $X_\circ$“
- Define map

$$\mathrm{Tr} : \mathrm{Der}^c(X_\circ; \mathbb{Q}_l) \to \mathrm{Ens}(X_\circ(\mathbb{F}_q), \mathbb{Q}_l)$$

- $\mathrm{Tr}(\mathcal{F}_\circ) : x \mapsto \sum_i (-1)^i \,\mathrm{Tr}(\mathrm{F}_g^* \mid \mathcal{H}^i \mathcal{F}_x)$ with $\mathcal{F} := \mathcal{F}_\circ \times_{\mathbb{F}_q} \mathbb{F}$ sheaf on $X := X_\circ \times_{\mathbb{F}_q} \mathbb{F}$ and F_g Frobenius

Grothendieck function-sheaf correspondence

- To $f : X_\circ \to Y_\circ$ morphism of varieties over $\mathbb{F}_q$ associate triangulated functors $f_!, f^*$ fitting into a commutative diagram

$$\begin{array}{ccc}
\mathrm{Der}^c(X_\circ;\mathbb{Q}_l) & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathrm{Der}^c(Y_\circ;\mathbb{Q}_l) \\
\mathrm{Tr}\downarrow & & \downarrow\mathrm{Tr} \\
\mathrm{Ens}(X_\circ(\mathbb{F}_q),\mathbb{Q}_l) & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathrm{Ens}(Y_\circ(\mathbb{F}_q),\mathbb{Q}_l)
\end{array}$$

- For $c : X_\circ \to \mathrm{pt}_\circ$ this specializes to $|X_\circ(\mathbb{F}_q)|1 = c_! c^* 1 = c_! c^* \mathrm{Tr}(\mathbb{Q}_l) = \mathrm{Tr}(c_! c^* \mathbb{Q}_l) =$ $= \sum_i (-1)^i \operatorname{tr}(\mathrm{F}_g \,|\mathrm{H}^i_c(X;\mathbb{Q}_l))$ Grothendieck-Lefschetz

- Let G be a finite group. The multiplication in the group ring could for $f, g \in \mathrm{Ens}(G, \mathbb{Z})$ be written as

$$f * g = \mathrm{mult}_!((\mathrm{pr}_1^* f)(\mathrm{pr}_2^* g))$$

  with $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{mult} : G \times G \to G$ the projections and the multiplication.
- A natural candidate for the categorification of the group ring in case $G = G_\circ(\mathbb{F}_q)$ is thus $\mathrm{Der}^c(G_\circ; \mathbb{Q}_l)$ with the convolution functor

$$\mathcal{F} * \mathcal{G} := \mathrm{mult}_!((\mathrm{pr}_1^* \mathcal{F}) \otimes (\mathrm{pr}_2^* \mathcal{G}))$$

- Recall $G = \mathrm{GL}(n; \mathbb{F}_q)$ and B upper triangular matrices and
$$\mathcal{H}_q = ({}^B\mathrm{Ens}(G, \mathbb{Z})^B, */|B|)$$
functions on the group, B-invariant from both sides
- So a natural categorification ought to be some both sides equivariant derived category of étale sheaves
$$\mathrm{Der}^c_{B_\circ \times B_\circ}(G_\circ; \mathbb{Q}_l)$$
- Let's be a bit less perfect and try for the usual topology version of the equivariant derived category
$$\mathrm{Der}^c_{B\times B}(G; \mathbb{Q})$$
with $G = \mathrm{GL}(n; \mathbb{C})$ and metric topology

First discuss equivariant cohomology

- $G \looparrowright X$ topological group acting on topological space
- $\mathrm{H}^*(X/G)$ not a good concept
- For $f : X \to Y$ is a morphism of G-spaces, which is a fibration with contractible fibers, need not have $\mathrm{H}^*(Y/G) \xrightarrow{\sim} \mathrm{H}^*(X/G)$
- Example: $\mathbb{R} \twoheadrightarrow \mathrm{pt}$ with $\mathbb{Z}$-action
- Better concept $\mathrm{H}^*_G(X) := \mathrm{H}^*(\mathrm{E}G \times_G X)$ equivariant cohomology
- $\mathrm{E}G$ contractible with topologially free G-action, the universal bundle over the classifying space

Examples for equivariant cohomology

- $\mathrm{H}^*_G(X) := \mathrm{H}^*(\mathrm{E}G \times_G X)$
- $G \looparrowright X$ topological group acting freely on topological space, then $\mathrm{H}^*_G(X) = \mathrm{H}^*(X/G)$
- $\mathrm{H}^*_G(\mathrm{pt}) = \mathrm{H}^*(\mathrm{E}G \times_G \mathrm{pt}) = \mathrm{H}^*(\mathrm{E}G/G) = \mathrm{H}^*(\mathrm{B}G)$ the ring of characteristic classes
- $\mathrm{H}^*_{\mathbb{C}^\times}(\mathrm{pt}) = \mathrm{H}^*(\mathbb{P}^\infty\mathbb{C}) = \mathbb{Z}[t]$ with $\deg t = 2$
- $\mathrm{H}^*_B(\mathrm{pt}) = \mathbb{Z}[t_1, \ldots, t_n]$ with $\deg t_i = 2$ for $B \subset \mathrm{GL}(n; \mathbb{C})$ upper triangular matrices
- For $P \to X$ a principal G-bundle, pullback $\mathrm{H}^*_G(\mathrm{pt}) \to \mathrm{H}^*_G(P) = \mathrm{H}^*(P/G) = \mathrm{H}^*(X)$ gives its characteristic classes

Derived category for X a topological space

- $\mathrm{Der}(X) = \mathrm{Der}(\mathrm{Ab}\,/X)$ derived category of abelian sheaves on X
- $f : X \to Y$ continous map of locally compact Hausdorff spaces gives triangulated functors $f_! : \mathrm{Der}(X) \to \mathrm{Der}(Y)$ and $f^* : \mathrm{Der}(Y) \to \mathrm{Der}(X)$
- For $c : X \to \mathrm{pt}$ constant map, get $c_! c^* \mathbb{Z} = \mathrm{H}^*_c(X)$
- $c^*\mathbb{Z} =: \underline{X}$ the constant sheaf on X
- $\mathrm{Der}_X(\underline{X}, \underline{X}[*]) = \mathrm{H}^*(X)$ the cohomology ring of X
- $\mathrm{Der}_X(\underline{X}, \mathcal{F}[*]) = \mathrm{H}^*(X; \mathcal{F}) = \mathbb{H}\mathcal{F}$ (hyper)cohomology of the sheaf(complex) $\mathcal{F}$
- $\mathbb{H}\mathcal{F}$ is a $\mathrm{H}^*(X)$-module

Equivariant derived category

- $G \looparrowright X$ topological space with G-action
- $\mathrm{Der}_G(X) = \{\mathcal{F} \in \mathrm{Der}(\mathrm{E}G \times_G X) | \; \exists \mathcal{G} \in \mathrm{Der}(X) \text{ such that } p^*\mathcal{F} \cong q^*\mathcal{G}\}$

  with $\mathrm{E}G \times_G X \xleftarrow{p} \mathrm{E}G \times X \xrightarrow{q} X$
- For $\mathcal{F} \in \mathrm{Der}_G(X)$ get $\mathbb{H}_G\mathcal{F} \in \mathrm{H}_G^*(X)$ -Mod
- f^* and $f_!$ for equivariant maps of locally compact Hausdorff spaces
- $\mathrm{Der}_G(X) = \mathrm{Der}(X/G)$ in the case of a topologically free action
- $\mathrm{Der}_G(\mathrm{pt}) \subset \mathrm{dgDer}\text{-}(\mathrm{H}_G^*(\mathrm{pt}), d = 0)$ for G a complex connected algebraic group
- $\mathrm{Der}_B(\mathrm{pt}) \subset \mathrm{dgDer}\text{-}\,\mathbb{Z}[t_1, \ldots, t_n]$

The natural categorification of the Hecke algebra

- Again $G = \mathrm{GL}(n; \mathbb{C})$ with B the upper triangular matrices
- The natural categorification of the Hecke algebra $\mathcal{H} = ({}^{B}(\mathbb{Z}G)^{B}, *_B)$ is the constructible equivariant derived category with convolution

$$(\mathrm{Der}^{c}_{B\times B}(G), *_B)$$

- The convolution is

$$\mathcal{F} *_B \mathcal{G} := \mathrm{mult}_! \,\mathrm{desc}((\mathrm{pr}_1^* \mathcal{F}) \otimes (\mathrm{pr}_2^* \mathcal{G}))$$

$\mathrm{pr}_i : G \times G \to G$
$\mathrm{desc} : \mathrm{Der}^{c}_{B\times B\times B\times B}(G \times G) \to \mathrm{Der}^{c}_{B\times B}(G \times_B G)$
$\mathrm{mult} : G \times_B G \to G$

Now need intersection cohomology

- For $X \subset \mathbb{P}^n\mathbb{C}$ a smooth irreducible complex projective algebraic variety the cohomology $\mathrm{H}^*(X)$ has remarkable properties:
  - Poincaré duality
  - Hard Lefschetz
  - Hodge Diamond
  - Positivities
- For $X \subset \mathbb{P}^n\mathbb{C}$ an non-smooth irreducible complex projective algebraic variety **intersection cohomology** $\mathrm{IH}^*(X)$ continues to have these properties
- For X smooth, $\mathrm{IH}^*(X) = \mathrm{H}^*(X)$
- In general $\mathrm{IH}^*(X)$ is an $\mathrm{H}^*(X)$-module

Intersection cohomology complex

- For X irreducible complex algebraic variety can still define intersection cohomology $\mathrm{IH}^*(X)$
- Formally $\mathrm{IH}^*(X) = \mathbb{H}\mathcal{IC}_X$ for $\mathcal{IC}_X \in \mathrm{Der}(X)$ the **intersection cohomology complex**
- Aside: For $\mathcal{D}$-modules have the Riemann-Hilbert correspondence, a fully faithful triangulated functor

$$\mathrm{RH} : \mathrm{Der}^{\mathrm{b}}_{\mathrm{hol,reg}}(\mathcal{D}_X\text{-}\mathrm{Mod}^{\mathrm{qc}}) \hookrightarrow \mathrm{Der}(X)$$

- The unique simple $\mathcal{D}_X$-module restricting to $\mathcal{O}_U$ on any open smooth subset $U \Subset X$ gets mapped by RH to $\mathcal{IC}_X$

- Back to $G = \mathrm{GL}(n;\mathbb{C}) \supset B$ with $G = \bigsqcup_{x\in W} BxB$ for $W = \mathcal{S}_n$ the permutation matrices
- Consider $\mathcal{IC}_x =: i_{x!}\mathcal{IC}_{\overline{BxB}}$ for $i_x : \overline{BxB} \hookrightarrow G$ intersection cohomology complex of Schubert variety
- All finite direct sums of shifts of $\mathcal{IC}_x$ form an additive subcategory $\mathrm{Der}^{ss}_{B\times B}(G) \subset \mathrm{Der}_{B\times B}(G)$ of "perversely semisimple complexes"
- This subcategory is even stable under convolution, due to the so-called decomposition theorem
- **Theorem:** The functor of hypercohomology gives an equivalence of monoidal categories

$$\begin{array}{rccc} \mathbb{H}_{B\times B} : (\mathrm{Der}^{ss}_{B\times B}(G), *_B) & \stackrel{\approx}{\to} & (R\text{-}\mathrm{SMod}_{\mathbb{Z}}\text{-}\,R, \otimes_R) \\ \mathcal{IC}_x & \mapsto & B_x[\dim B] \end{array}$$

Here $\mathbb{H}_{B\times B} : (\mathrm{Der}^{ss}_{B\times B}(G), *_B) \xrightarrow{\approx} (R\text{-SMod}_{\mathbb{Z}}\text{-}R, \otimes_R)$ is defined using the identifications

$$\begin{array}{ccc}
\mathrm{H}^*_{B\times B}(\mathrm{pt}) & \twoheadrightarrow & \mathrm{H}^*_{B\times B}(G) \\
\wr\downarrow & & \downarrow\wr \\
\mathcal{O}(V\times V) & \twoheadrightarrow & \mathcal{O}\left(\bigcup_{x\in W}\Gamma(x)\right) \\
\wr\downarrow & & \downarrow\wr \\
R\otimes R & \twoheadrightarrow & R\otimes_{R^W} R
\end{array}$$

for $V = \mathrm{Lie}\, T$ and $T\subset B$ a maximal torus and degrees on $\mathcal{O}$ doubled to match cohomological degrees.

COMMERCIAL FOR TWO THEOREMS 6

- in [S, *Universelle...*, Math. Ann. **284** (1989)] tdo-case
- in [S, *The prime...*, Math. Z. **204** (1990)] general

G be a connected complex affine algebraic group,
B a closed subgroup, $X = G/B$ the homogeneous space,
$n = \dim X$ its dimension, $x \in G/B$ the natural base point,
V, W finite dimensional rational representations of B,
$\mathcal{V}, \mathcal{W}$ the sheaves of sections of the associated bundles.

Then the action leads to an G-equivariant isomorphism

$$\Gamma(X; \mathrm{Dif}(\mathcal{V}, \mathcal{W})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^n_x(X; \mathcal{V}), \mathrm{H}^n_x(X; \mathcal{W}))_B^{G\text{-alg}}$$

- Have $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \left(V \otimes_{\mathbb{C}} \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b})\right) \xrightarrow{\sim} \mathrm{H}^n_x(X; \mathcal{V})$
- Can replace G-alg by $\mathfrak{g}$-finite if G is simply connected
- B-compatibility is automatic for B connected

Summing up:

$$\begin{array}{ccccc} \langle \mathrm{Der}^{ss}_{B\times B}(G), *_B\rangle & \xrightarrow{\sim} & \langle R\text{-}\mathrm{SMod}_{\mathbb{Z}}\text{-} R, \otimes_R\rangle & \xleftarrow{\sim} & \mathcal{H}_v \\ \mathcal{IC}_x[-\dim B] & \mapsto & B_x & \leftarrow\!\shortmid & C_x \\ \text{intersection cohomology} & & \text{special bimodule} & & \text{canonical basis} \end{array}$$

Original motivation: Sheaf-function-correspondence

$$\begin{array}{ccc} (\mathrm{Der}_{B_\circ\times B_\circ}(G_\circ;\mathbb{Q}_\ell), *_{B_\circ}) & \to & \mathcal{H}_q \\ \mathcal{IC}_x & \mapsto & v^? C_x \end{array}$$

This was the starting point of Kazhdan-Lusztig

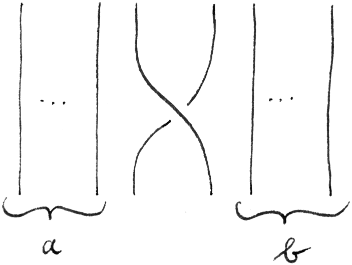
More categorification of the Hecke algebra

- Given X a complex algebraic variety can define variant $\mathrm{MDer}(X)$ of $\mathrm{Der}(X)$ with functors $f^*, f_!$ as before such that $\mathrm{MDer}(\mathrm{pt}) = \mathrm{Der}(\mathbb{C}\text{-Modf}_{\mathbb{Z}})$
- Joint with Matthias Wendt, Rahbar Virk, work in progress
- Based on new progress in motives by Ayoub, Cisinski-Deglise, Drew,...
- Variant of Hodge theory

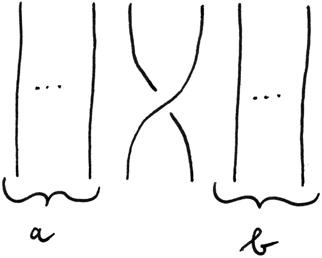
Our old equivalence can be upgraded further to

$$\begin{array}{ccc}
(\mathrm{Der}^{ss}_{B\times B}(G), *_B) & \overset{\approx}{\to} & (R\text{-}\mathrm{SMod}_{\mathbb{Z}}\text{-}R, \otimes_R) \\
\wr\downarrow & & \downarrow \\
(\mathrm{MDer}_{B\times B}(G)_{w=0}, *_B) & & \downarrow \\
\downarrow & & \downarrow \\
(\mathrm{MDer}_{B\times B}(G), *_B) & \overset{\approx}{\to} & (\mathrm{Hot}^{\mathrm{b}}(R\text{-}\mathrm{SMod}_{\mathbb{Z}}\text{-}R), \otimes_R)
\end{array}$$

Back to knot invariants (variation on Webster-Williamson)



$i_{s!}\underline{BsB} := T_s^!$



$i_{s*}\underline{BsB} := T_s^*$

- Take $s = (a+1, a+2) \in \mathcal{S}_n = W$ the transposition
- $T_s^!, T_s^* \in \mathrm{MDer}_{B\times B}(G)$
- Recall $\underline{BsB} \mapsto s$ under $\mathcal{H} \mapsto \mathbb{Z}W$ given by $q \mapsto 1$

- Given a braid Z, scan it from the top and convolve corresponding $T_s^!$, T_s^* with $* := *_B$ to get an object $M(Z) \in \mathrm{MDer}_{B\times B}(G)$
- For $M(Z)$ to be well-defined, use braid relations $T_s^! * T_t^! * T_s^! \cong T_t^! * T_s^! * T_t^!$ for $sts = tst$ and similarly for $st = ts$ and ! replaced by $*$
- These are geometrically clear, since $BsB \times_B BtB \times_B BsB \xrightarrow{\sim} BstsB$ by multiplication, so $T_s^! * T_t^! * T_s^! \cong i_{sts!}\underline{BstsB} = i_{tst!}\underline{BtstB} \cong T_t^! * T_s^! * T_t^!$ etc
- Also need $T_s^! * T_s^* \cong T_s^* * T_s^! \cong i_{e!}\underline{B} = i_{e*}\underline{B}$ unit object Calculation, but not so hard: only on $\mathbb{P}^1\mathbb{C}$

Calculation in bimodules

- $\mathrm{MDer}_{B\times B}(G) \xrightarrow{\approx} \mathrm{Hot}^{\mathrm{b}}(R\text{-SMod}_{\mathbb{Z}}\text{-}R)$
- $T_s^!$ maps to $\ldots \to 0 \to R \otimes_{R^s} R \twoheadrightarrow R \to 0 \to \ldots$ multiplication map
- T_s^* maps to $\ldots \to 0 \to R \hookrightarrow R \otimes_{R^s} R \to 0 \to \ldots$
  - Geometrically, need $\mathcal{O}(\Gamma(e)) \hookrightarrow \mathcal{O}(\Gamma(e) \cup \Gamma(s))$
  - Given by choosing linear function on $V \times V$, whose zero set intersects $\Gamma(e) \cup \Gamma(s)$ precisely in $\Gamma(s)$
  - Multiply a function on $\Gamma(e)$ with this linear function and extend by zero to $\Gamma(e) \cup \Gamma(s)$
- $M(Z)$ corresponds to $B(Z) \in \mathrm{Hot}^{\mathrm{b}}(R\text{-SMod}_{\mathbb{Z}}\text{-}R)$ the tensor product of these elementary complexes

To get an invariant of the knot $K(Z)$ obtained closing the braid Z procede as follows:

- Take at each stage of the bimodule complex $\ldots \to B(Z)^q \to B(Z)^{q+1} \to \ldots$ of bimodules the Hochschild homology
- Get for each j a complex of (graded) vector spaces $\ldots \to \mathrm{HH}_j(B(Z)^q) \to \mathrm{HH}_j\, B(Z)^{q+1}) \to \ldots$
- Take its cohomology groups $\mathcal{H}^q(\mathrm{HH}_j(B(Z)^*)$
- This is **Khovanov's triply graded knot homology:**
    - Choosen and fixed degree j of Hochschild homology
    - Degree q of cohomology of the resulting complex
    - Internal degree, the bimodules beeing graded
- It categorifies the HOMFLYPT polynomial, which can be gotten as some Euler characteristic

I am still lacking full geometric understanding of why this has to give a knot invariant. Webster-Williamson seem to understand it better. And the construction of MDer is very recent.

Recall relation to representation theory

- $\mathfrak{g}$ a semisimple complex Lie algebra, $Z \subset U(\mathfrak{g})$ the center of its enveloping algebra
- $\mathcal{M} \subset \mathfrak{g}$-Mod the category of all representations of $\mathfrak{g}$ locally finite under Z
- $\mathcal{P}$ the category of all functors $\mathcal{M} \to \mathcal{M}$ with split embedding in some functor $E \otimes_{\mathbb{C}}$ for $\dim_{\mathbb{C}} E < \infty$
- $\mathcal{M} = \prod_{\chi \in \operatorname{Max} Z} \mathcal{M}_\chi$ and $\mathcal{P} = \prod_{\chi, \psi \in \operatorname{Max} Z} \; {}_\psi\mathcal{P}_\chi$
- Equivalence of monoidal categories for $\chi = \operatorname{Ann}_Z \mathbb{C}$

$$\hat{\mathbb{V}} : {}_\chi\mathcal{P}_\chi \overset{\approx}{\to} \hat{S}\text{-SMod-}\hat{S}$$

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ as usual, $S := U(\mathfrak{h})$ polynomial ring

Construction of $\hat{\mathbb{V}} : {}_\chi\mathcal{P}_\chi \stackrel{\approx}{\to} \hat{S}$-SMod-$\hat{S}$

- Abbreviate $U := U(\mathfrak{g})$, recall $\chi = Z^+ = \mathrm{Ann}_Z\,\mathbb{C}$
- $U/U\chi^n$ form an inverse system in $\mathcal{M}_\chi$
- They also are of finite length as a U-bimodules
- For $P \in {}_\chi\mathcal{P}_\chi$ still $P(U/U\chi^n)$ naturally is a bimodule
- The $P(U/U\chi^n)$ are **Harish-Chandra bimodules**:
  By definition, these are the bimodules of finite length, which are in addition locally finite for the adjoint action of $\mathfrak{g}$.
- Call HCH the category of Harish-Chandra bimodules
- ${}_\chi\mathrm{HCH}_\chi$ has a unique simple object L of maximal Gelfand-Kirillov dimension
- There is an exact functor $\mathbb{V} : {}_\chi\mathrm{HCH}_\chi \to \mathbb{C}$-Modf with $L \mapsto \mathbb{C}$ and killing the other simples. It is essentially unique.

Construction of $\hat{\mathbb{V}} : {}_\chi\mathcal{P}_\chi \xrightarrow{\approx} \hat{S}$-SMod-$\hat{S}$, continued

- By functoriality, our exact functor $\mathbb{V}$ is even a functor $\mathbb{V} : {}_\chi\mathrm{HCH}_\chi \to Z$-Modf-$Z$
- Looking closer, our exact functor $\mathbb{V}$ is even a functor $\mathbb{V} : {}_\chi\mathrm{HCH}_\chi \to \hat{Z}$-Modf-$\hat{Z}$ for $\hat{Z} = Z_\chi^\wedge$
- Set
$$\hat{\mathbb{V}}P := \varprojlim_n \mathbb{V}(P(U/U\chi^n))$$
- Use natural isomorphism $\hat{Z} \xrightarrow{\sim} \hat{S}$ induced by unnormalized Harish-Chandra isomorphism $Z \xrightarrow{\sim} S^{(W\cdot)} \subset S$ with $S = \mathcal{O}(\mathfrak{h}^*)$ and W-action shifted to fix $-\rho$ determined by $\mathbb{C}_{-2\rho} \cong \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b})$ over $\mathfrak{h}$...

- Consider ${}_\chi\mathrm{HCH}^n_\chi := \{M \in {}_\chi\mathrm{HCH}_\chi \mid M\chi^n = 0\}$
- Has enough projectives: The $P(U/U\chi^n)$ for $P \in {}_\chi\mathcal{P}_\chi$
- Get by the above also equivalence $\mathbb{V} : \mathrm{proj}({}_\chi\mathrm{HCH}^n_\chi) \xrightarrow{\approx} S\text{-SMod-}S/(S^+)^n$
- In the case $n = 1$ have ${}_\chi\mathrm{HCH}^1_\chi \xrightarrow{\approx} \mathcal{O}_\circ$ equivalence with principal block of BGG-category by tensoring with dominant Verma $\otimes_U \Delta(0)$
- Proof of KL-conjectures using bimodules:

$$\begin{array}{ccccccc}
 & P_x & \mapsto & B_x \otimes_S \mathbb{C} & \leftarrow\!\shortmid & B_x & \\
Q \in & \mathrm{proj}(\mathcal{O}_\circ) & \xrightarrow{\approx} & S\text{-SMod} & \leftarrow & S\text{-SMod}_{\mathbb{Z}}\text{-}S & \ni B_x \\
\downarrow & \downarrow & & & & \downarrow & \downarrow \\
\sum_y (Q : \Delta_y) y \in & \mathbb{Z}[W] & \longleftarrow & \xleftarrow{v=1} & \longleftarrow & \mathcal{H}_v & \ni C_x
\end{array}$$

$$\Rightarrow \sum_y (P_x : \Delta_y) y = C_x(1) \Rightarrow [\Delta_y : L_x] = (P_x : \Delta_y) = P_{yx}(1)$$

Graded versions and Koszul duality

- Construct $\mathbb{Z}$-graded version $\mathcal{O}_\circ^{\mathbb{Z}}$ of $\mathcal{O}_\circ$ by declaring $\mathrm{proj}(\mathcal{O}_\circ^{\mathbb{Z}}) = S\text{-SMod}_{\mathbb{Z}}$
- Then $\sum_i [\Delta_y^{\mathbb{Z}} : L_x^{\mathbb{Z}}\langle i\rangle] v^i = \sum_i (P_x^{\mathbb{Z}} : \Delta_y^{\mathbb{Z}}\langle i\rangle) v^i = P_{yx}(v)$
- Characterization in joint recent work with Rottmaier: $\mathcal{O}_\circ^{\mathbb{Z}}$ is “the essentially unique $\mathbb{Z}$-graded version of the artinian category $\mathcal{O}_\circ$ compatible with the action of the center”
- Deduce $\mathrm{Hot}^{\mathrm{b}}(\mathrm{proj}(\mathcal{O}_\circ^{\mathbb{Z}})) = \mathrm{Hot}^{\mathrm{b}}(S\text{-SMod}_{\mathbb{Z}})$
- Thus get Koszul duality K triangulated functor

$$\begin{array}{ccccc}
\mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ^{\mathbb{Z}}) & \stackrel{\approx}{\to} & \mathrm{Hot}^{\mathrm{b}}(S\text{-SMod}_{\mathbb{Z}}) & \stackrel{\approx}{\to} & \mathrm{MDer}_{N\times B}(G) \\
K \downarrow & & & & \downarrow \wr \\
\mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ^{\mathbb{Z}}) & \stackrel{\sim}{\leftarrow} & \stackrel{\sim}{\leftarrow} & \stackrel{\sim}{\leftarrow} & \mathrm{MDer}_N(G/B) \\
\downarrow & & & & \downarrow \\
\mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ) & \stackrel{\sim}{\leftarrow} & \mathrm{Der}^{\mathrm{b}}_N(\mathcal{D}_{G/B}\text{-Mod}^{\mathrm{qc}}) & \stackrel{\sim}{\leftarrow} & \mathrm{Der}_N(G/B)
\end{array}$$

Kozsul duality K preceded by $\mathcal{O}$-duality d, properties:

- $Kd : \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ^{\mathbb{Z}}) \to \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_\circ^{\mathbb{Z}})$ triangulated contravariant
- $\Delta_x^{\mathbb{Z}} \mapsto \Delta_{w_\circ x}^{\mathbb{Z}}$
- $L_x^{\mathbb{Z}} \mapsto P_{w_\circ x}^{\mathbb{Z}}$
- $P_x^{\mathbb{Z}} \mapsto L_{w_\circ x}^{\mathbb{Z}}$
- $Kd(M[n]) \cong (KdM)[-n]$
- $Kd(M\langle n\rangle) \cong (KdM)[n]\langle n\rangle$
- Funny formulas $\sum_i \dim \mathrm{Ext}^i_{\mathcal{O}}(\Delta_x, L_y) = [\Delta_{w_\circ x} : L_{w_\circ y}]$
- Kd gives $\mathrm{Der}(\Delta_x^{\mathbb{Z}}, L_y^{\mathbb{Z}}[i]\langle j\rangle) = \mathrm{Der}(P_{w_\circ y}^{\mathbb{Z}}[-i+j]\langle j\rangle, \Delta_{w_\circ x}^{\mathbb{Z}})$
- This explains these funny formulas

Other things on Koszul duality

- Variant exchanging parabolic and singular category $\mathcal{O}$
- Variant from parabolic-singular to singular-parabolic
- BGG-resolution of simple Verma corresponds to Verma flag of antidominant projective
- More natural from Langlands philosophy point of view

Variant for Harish-Chandra modules

- Consider $\overline{\mathrm{HCH}}$ the category of U-bimodules M such that every vector is killed by some χ^n from right and left and $\{v \in M \mid \chi v = 0\}$ is of finite length
- Has enough injectives and finite homological dimension
- Using $\mathbb{V}$ and some duality get contravariant equivalence $\mathrm{inj}\,\overline{\mathrm{HCH}} \xrightarrow{\approx} S\text{-SMod-}S$
- Define $\mathbb{Z}$**-graded version** $\overline{\mathrm{HCH}}_{\mathbb{Z}}$ of $\overline{\mathrm{HCH}}$ by declaring $\mathrm{inj}\,\overline{\mathrm{HCH}}_{\mathbb{Z}} := (S\text{-SMod}_{\mathbb{Z}}\text{-}S)^{\mathrm{opp}}$
- Deduce $\mathrm{Hot}^{\mathrm{b}}(\mathrm{inj}\,\overline{\mathrm{HCH}}_{\mathbb{Z}}) \xrightarrow{\approx} \mathrm{Hot}^{\mathrm{b}}(S\text{-SMod}_{\mathbb{Z}}\text{-}S)^{\mathrm{opp}}$
- Get $\mathrm{Der}^{\mathrm{b}}(\overline{\mathrm{HCH}}_{\mathbb{Z}}) \xrightarrow{\approx} \mathrm{MDer}_{B^\vee \times B^\vee}(G^\vee)^{\mathrm{opp}}$ Koszul duality
- Need dual group $G^\vee$ since $S = \mathcal{O}(\mathfrak{h}^*)$ but $R = \mathcal{O}(\mathfrak{h})$