Symplectic Dirac operator for KK-theory

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Metaplectic group and SSW-representation

1. G metaplectic group associated to (V, ω_0, L, J), where (V, ω_0) is a symplectic vector space, L is a Lagrangian subspace of (V, ω) and J is a compatible almost complex structure, $\lambda : G \to Sp(V, \omega)$ covering map
2. $\sigma : G \to U(L^2(L)')$ the dual of the unitary Segal-Shale-Weil representation (as appear in Folland [4], Wallach [18]) See also Weil [19], Shale [16] for original definitions.
3. σ is not irreducible $L^2(L)' \simeq L^2_+(L)' \oplus L^2_-(L)'$, continuous duals of even and odd functions - irreducible
4. Set $H = L^2(L)$

Metaplectic structure

(M, ω) symplectic manifold

A principal G-bundle $\pi_P : \mathcal{P} \to M$ is called a metaplectic structure, if there exists a bundle homomorphism $\Lambda : \mathcal{P} \to \mathcal{Q}$ such that the diagram commutes

$$
\begin{tikzcd}
\mathcal{P} \times G \arrow[r] \arrow[dd, "\Lambda \times \lambda"] & \mathcal{P} \arrow[dr, "\pi_P"] \arrow[dd, "\Lambda"] & \\
& & M \\
\mathcal{Q} \times Sp(V, \omega) \arrow[r] & \mathcal{Q} \arrow[ur, "\pi_Q"'] &
\end{tikzcd}
$$

$\pi_Q : \mathcal{Q} \to M$ is the bundle of symplectic bases in each point of M

Forms with values in the SSW-representation

1. $H_k = \mathcal{C}^\infty(\mathcal{P}, \bigwedge^k V^* \otimes H')^G = \{f : \mathcal{P} \to \bigwedge^k V^* \otimes H' |$ $\sigma_k(g)(f(p)) = f(p \cdot g^{-1}), g \in G, p \in \mathcal{P}\}$
   where $\cdot$ denotes the action of G on the metaplectic structure and σ_k is the tensorial extension of σ and of the dual of covering homomorphism $\lambda : G \to Sp(V, \omega)$, i.e.,
   $$\sigma_k(g)(\alpha \otimes f) = \lambda(g)^* \alpha \otimes \sigma(g) f,$$
   $g \in G, \alpha \in \bigwedge^k V^*$, $f \in H'$. Note $\lambda(g)^* \in GL(V^*)$.
2. H_0 – Kostant's spinors, see Kostant [11].

An algebra of observables

Let us set
$\mathbb{A} = \langle \sigma(G) \rangle = \{ \sum_{i=1}^{r} \lambda_i \sigma(g_i) | \, \lambda_i \in C, \, g_i \in G, i = 1, \ldots, r, r \in N \}$
We have

Lemma

For any $n \in N$,

1. $\mathbb{A}$ *is an associative C-algebra*
2. $* : \sum_{i=1}^{r} \lambda_i \sigma_k(g_i) \mapsto \sum_{i=1}^{r} \overline{\lambda_i} \sigma_k(g_i)^*$ *defines an anti-involution on* $\mathbb{A}$
3. $\mathbb{A} \subseteq B(H)$

An algebra of observables

Definition

The completion of $\mathbb{A}$ in $(B(H), ||_{op})$ is called the algebra of observables.

We infer

Theorem

The algebra of observables is a C^*-subalgebra of the C^*-algebra $(B(L^2(L)), *, ||_{op})$.

An algebra of observables

Lemma

For $i = 1, 2$, let $\rho_i : \mathbb{A} \to B(V_i)$ be a representation of $\mathbb{A}$ on Banach space V_i which is continuous as a map of normed spaces $\mathbb{A} \times V_i \to V_i$, and $D : V_1 \to V_2$ be a continuous intertwiner, i.e., $D\rho_1(X) = \rho_2(X)D$ for any $X \in \mathbb{A}$. Suppose that the representations ρ_i have continuous extensions to A. Then D commutes with them as well.

Remark: If $V_i = L^2(L)'$ and $\rho_i(X)f = X(f)$, $X \in \mathbb{A}$ then the condition on the existence of the continuous extension is satisfied (automatically).

Certain $B(H)$-Hilbert modules

Let (M, ω) be a symplectic manifold with a fixed metaplectic structure and an adapted positive almost complex structure J

1. Let $X \in B(H)$, and $f \in H_k$. With $(f \cdot X)(p) = f(p) \circ X$ for any $p \in \mathcal{P}$, H_0 becomes a right $B(H)$-module.
2. Extending by $((\alpha \otimes f) \cdot X)(p) = \alpha(p) \otimes (f(p) \circ X)$, H_k becomes a right $B(H)$-module
3. For $\alpha \otimes f, \beta \otimes g \in H_k$ we define a $B(H)$-valued function $(,) : H_k \times H_k \to B(H)$, $(\alpha \otimes f, \beta \otimes g)(p) = h(\alpha, \beta)(p)(f(p)^\sharp \otimes g(p))$, $p \in \mathcal{P}$, where $f(p)^\sharp$ is the inner space dual vector of $f(p)$, $(f(p)^\sharp, v) = f(p)v$, $v \in H$.

Certain $B(H)$-Hilbert modules

4. $(f,g) = \int_{p \in M} (f,g)(p) \mathrm{d}p$, $f,g \in H_k$, is the Dunford-Pettis integral induced by the measure on M associated to the volume form and the operator norm on $B(H)$
5. $||f|| = \sqrt{|(f,f)|_{op}}$ norm on H_k
6. $H_k^{DF} = \{f \in H_k | \mathcal{P} \ni p \mapsto (f,f)(p)$ is Dunford-Pettis integrable$\}$
7. L_k denotes the norm completion of $(H_k^{DF}, ||\cdot||)$

Theorem

Spaces $(H_k^{DF}, (,))$ are pre-Hilbert C^*-modules and $(L_k, (,))$ are Hilbert $B(H)$-modules, $k = 0, \ldots, \dim M$.

Connection on symplectic manifolds

Definition

Any connection ∇ on a symplectic manifold (M, ω) which preserves the symplectic form is called a symplectic connection, i.e., $\nabla\omega = 0$. If moreover, ∇ is torsion-free, it is call a Fedosov connection.

Remark: Fedosov connections form an infinite dimensional affine space. See Tondeur [17], Gelfand, Retakh, Shubin [6].
Any symplectic manifold admits so-called complex metaplectic structure. Since one can pass to complexification, we'll not mention existence of metaplectic structures, and consider the complex version without explicit mention. (See Robinson, Rawnsley [15], Cahen, La Fuente Gravy, Gutt, Rawnsley [3].)

Covariant derivatives

If (M, ω) is a symplectic manifold, a symplectic connection ∇ induces a principal G-bundle connection ω on $\mathcal{P}$.

Lemma

Let M be a compact symplectic manifold equipped with a Fedosov connection. The covariant derivative d_ω is a homomorphism of Hilbert $B(H)$-modules H_0 and H_1. The extensions of d_ω to forms with values in H' is a Hilbert $B(H)$-homomorphism as well.

Symbols of PDO's are useful due to the Mishchenko–Fomenko elliptic operator theory (also in the infinite fiber case).

Lemma

On a symplectic manifold, the principal symbol of d_ω is given by $\sigma(d_\omega, \xi)f = \xi \wedge f$ for any $f \in H_k$ and $\xi \in T^*M$.

Based on theory of Mishchenko–Fomenko [5] for elliptic operators and Hodge theory for them (K [13, 14]), we deduce

Theorem

For a compact symplectic manifold with a flat symplectic connection, the complex $(L_k, d_{\omega|L_k})_{k\in Z}$ is an elliptic complex in the category of Hilbert $B(H)$-modules. Moreover, its cohomology groups are finitely generated projective $K(H)$-modules by restriction.

The product satisfies $(,) : L_k \times L_k \to K(H) \subset B(H)$.

When does Hodge theory holds? Categories and Hilbert modules

Definition

Let $(\mathcal{R}, +, \dagger)$ be and additive dagger category and $K(\mathcal{R})$ the category of complexes in $\mathcal{R}$. We say that $R = (R_k, d_k : R_k \to R_{k+1})_{k \in Z} \in K(\mathcal{R})$ is of Hodge type if and only if

$$R_k = \operatorname{Im} d_{k-1} \oplus \operatorname{Im} d_k^* \oplus \operatorname{Ker} \Delta_k,$$

where $\Delta_k = d_k^* d_k + d_{k-1} d_{k-1}^*$ is the associated Laplacian, and

$$\operatorname{Ker} \Delta_k \to H_k(R)$$

is an isomorphism.

For dagger and so-called correspondence categories, see Brinkmann, Puppe [2].

Examples of Hodge type complexes

1. Category of finite dimensional inner product vector spaces and linear maps. Each complex is of Hodge type.
2. C. of finite rank vector bundles over compact manifolds. Elliptic complexes are of Hodge type.
3. C. of Hilbert spaces and continuous maps. All complexes whose differentials have closed images. Especially, all complexes with Fredholm differentials.
4. C. of Hilbert modules over a compact algebra K and adjointable maps. All complexes with K-Fredholm Laplacians of H. t.
5. C. of Hilbert spaces and continuous maps. Each complexes whose Laplacians have self-adjoint parametrix of H. t.

Further example, Krýsl [14]:

6 Finitely generated projective A-Hilbert bundles over compact manifolds and bundle maps which are adjointable A-Hilbert module homomorphism. All elliptic complexes invariant over C^*-algebras of compact operators are of H. t.

Theorem

If (M, ω) is compact, $(L_k, d_{\omega|L_k})_{k \in Z}$ is an elliptic complex in the category of Hilbert $K(H)$-modules. In particular, the Hodge theory holds for it. The cohomology groups are finitely generated projective $K(H)$-modules.

Modified Habermann's construction

It is well known that $L^2(L)' \otimes V$ contains a complemented G-submodule isomorphic to $L^2(L)'$ with multiplicity one (Krýsl [12]). Denote the unique G-equivariant projection onto it by q.

Definition

For a symplectic manifold (M, ω) and a symplectic connection ∇, we set

$$(Df)(p) = q(d_\omega f)(p), p \in \mathcal{P}, f \in H_0$$

and call it the Habermann's principal Dirac operator.

See Habermann [7] and Habermann, Habermann [8]

Lemma

$D : L_0 \to L_0$ is an A-Hilbert module homomorphism.

Proof. Lemma + G-invariance of q. □

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