Compactification, boundary calculus, and applications: Hypersurface and boundary calculus

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Some motivation (partly naïve) via a general question: **Taming big spaces:** Suppose we have an infinite space (i.e. non-compact manifold, geodesically complete):

How do we deal with the “far region”? Can we make a notion of “infinity” that is mathematically useful? If so what geometry does it have? Are there many ways to do such things, or is any success essentially unique?

A **Compactification** of a non-compact (“large”) topological space \( M \) is an embedding of \( M \) as a dense subset of a compact (“small”) space \( \tilde{M} \): So \( M \hookrightarrow \tilde{M} \) injective cts and a homeo. onto its image.
Example: A compactification of Minkowski space

\[ \mathbb{R} \times S^{n-1} \]

**Figure:** The standard embedding of \( n \)-dimensional Minkowski space \( M^n \) into the Einstein cylinder. This is conformal: \( g_{\text{Mink}} = \Omega^2 g_{\text{Lorentzian cyln}} \)

**Questions:** Is this essentially the only way to conformally compactify \( M \)? Is it forced that \( i^\pm \) and \( i^0 \) are points? That \( \mathcal{I} \) is an open subset of \( \partial M \)?
Penrose’s “generalisation” and conformal infinity

Definition

A smooth (time- and space-orientable) spacetime \((M_+, g_+)\) is called **asymptotically simple** if there exists another smooth Lorentzian manifold \((M, g)\) such that

1. \(M_+\) is an open submanifold of \(M\) with smooth boundary \(\partial M_+ = \mathcal{I}\);
2. there exists a smooth scalar field \(\Omega\) on \(M\), such that \(g = \Omega^2 g_+\) on \(M_+\), and so that \(\Omega = 0\), \(d\Omega \neq 0\) on \(\mathcal{I}\);
3. every null geodesic in \(\overline{M}\) acquires a future and a past endpoint on \(\mathcal{I}\).

An asymptotically simple spacetime is called **asymptotically flat** if in addition \(\text{Ric}^{g_+} = 0\) in a neighbourhood of \(\mathcal{I}\).

Questions: How would we re-discover the Einstein cylinder compactification or this useful definition? Treat other geometries similarly?
Compactification: $M \hookrightarrow \overline{M}$ smooth injective, $M$ open dense. (In general $M$ may be a manifold with boundary, a manifold with corners, . . .

**Question:** What is a right way to do this when geometry is involved?

In many simple cases the result is a manifold with boundary $\overline{M}$ so that $M$ is the interior and $\partial M$ has codimension 1.

**Questions:** How do we find the geometry on $\partial M$?

**Boundary calculus:** Relate the geometries/fields on $\partial M$ and $M$?

**Applications:** 1. Discovery new links between different geometries; geometric tools PDE boundary problems; invariant theory and invariant operators; representation theory; scattering and non-local operators; AdS/CFT correspondence in physics
A flat model – compactifying Euclidean space

Stereographic projection \( p : S^{n+1} \setminus \{N\} \to \mathbb{E}^{n+1} \) is a diffeomorphism (even \( C^\infty \))

and \( p^{-1} : \mathbb{E}^{n+1} \to S^{n+1} \) is a conformal embedding. That is it preserves angles and circles are mapped to circles.

**BUT:** the conformal infinity of this compactification is the one point \( N \in S^{n+1} \) – so this cannot encode much information about \( \mathbb{E}^{n+1} \), or analysis on \( \mathbb{E}^{n+1} \). For example:

- The **Euclidean group** action on \( \mathbb{E}^{n+1} \) extends to \( S^{n+1} \), but it acts trivially on \( N \).
- It is clearly a bad compactification for **Euclidean scattering**.

Q: Other ways to compactify?
Answering these questions has several steps. The first step is linked to another problem:

**Problem:** Suppose a Lie group $H$ acts on a manifold $X$ with a finite number of orbits. Then: (i) understand and relate the different (Klein) geometries on the orbits; and (ii) construct and treat a well defined curved version of this theory.

E.g. **Example** $H := SO(n, 1)$ orbits $X = \text{the Klein Ball}$:

**Note:** The Klein ball is a compactification of $\mathbb{H}^n$ linked to projective geometry.

Rod Gover. background: G-., Waldon, arXiv:1506.02723, and Compactification and boundary calc
$H = SO(n,1)$ orbits on the sphere

$\mathbb{R}^{n+1} \times \mathbb{A} \xrightarrow{G} \mathbb{S}^{n+1}$

standard vol.

Symmetry reduction by $h$: $\Rightarrow$ North polar cap is projective compactification of $\mathbb{H}^n$; $\tau = 0$ projective $\infty$ with conformal str.

$S^n = \mathbb{P}_+(\mathbb{R}^{n+1} \setminus \{0\})$ is model of flat projective geometry.

NB: Embeddings relate the orbits – but these encoded in $H \hookrightarrow G$. 

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Conformal compactification of $\mathbb{H}^{n+1}$ – the Poincaré ball

Escher’s circle limit

$\mathbb{H}^2 = \mathbb{H}^2 + \partial \mathbb{H}^2$

The embedding gives the compactification

$\mathbb{H}^{n+1}$ embedded conformally in Euclidean $\mathbb{E}^{n+1}$

$S^n = \partial \mathbb{H}^{n+1}$

$g_+ = \frac{4}{(1-|x|^2)^2} \sum^{n+1} dx_i^2$

$\overline{\mathbb{H}}^{n+1} = \mathbb{H}^{n+1} + \partial \mathbb{H}^{n+1}$

Rod Gover. background: G-, Waldon, arXiv:1506.02723, and
Compactification and boundary calc
Poincaré compactification via via $\mathbb{P}_+(\text{nullcone})$

Conformal compactification of $\mathbb{H}^{n+1}$ by symmetry breaking:

\[
\tilde{\sigma} = 0 \quad \tilde{\sigma} = 1
\]

i.e. $\mathbb{R}^{n+3}$

with

\[
\begin{pmatrix}
-1 & & \\
& 1 & \\
& & \ddots \\
& & & 1
\end{pmatrix}
\]

\[
\tilde{\sigma} = I_A X^A
\]

$S^{n+1} = \partial \mathbb{H}^{n+1}$

$S^{n+1} = \mathbb{P}_+(\mathcal{N}_+ \subset \mathbb{R}^{n+3} \setminus \{0\})$ is model of flat conformal geometry.

$G := SO_o(n + 2, 1)$ acts transitively. $I \in \mathbb{R}^{n+3}$, spacelike $h(I, I) = 1$

Symmetry reduction by $I$: $\Rightarrow H = SO_o(n + 1, 1)$ orbits. Right hemi. is conformal compactification $\overline{M}_c$ of $\mathbb{H}^{n+1}$; $\sigma = 0$ conformal $\infty$ with conformal str.
Bigger groups: $H$ vs $G$

In each example above there is implicitly a larger group $G \supset H$:

- **Poincaré ball and compactifying boundary** arise as two $H = SO_+ (n+1,1)$ orbits on $S^{n+1} = G/P$ where $G = SO_+ (n+2, 1)$ and $P$ maximal parabolic in $G$.

The larger homogeneous space $G/P$ encodes how the orbits smoothly fit together – i.e. the conformal compactification. Similarly:

- **Stereographic (conformal) compactification** of $\mathbb{E}^{n+1}$ arises as two $H = \text{Euclidean group}$ orbits on $S^{n+1} = G/P$ – with same $G$ and $P$. I.e. $\text{Stereo. encoded by } H \hookrightarrow G$

- **Klein (projective) compactification** of $\mathbb{H}^{n+1}$ arises as two $H = SO_+ (n+1,1)$ orbits on $S^{n+1} = G/P$ where now $G = SL(n+2)$ and $P$ maximal parabolic in $SL(n+2)$.

- Projective compactification of $\mathbb{E}^{n+1}$ arises as two $\cdots$ (Exercise!)
Curving homogeneous spaces

For a Lie group $G$ and closed Lie group $P$, homogeneous spaces $G/P$ are **geometries** in the sense of **Klein**. There are often canonical curved generalisations:

### Theorem (Cartan, Tanaka, ···)

If $P$ is a parabolic subgroup of a semisimple Lie group $G$ then there is a **canonical** notion of geometry

\[
G \leftarrow P \\
\downarrow \\
M \\
\downarrow \\
G/P
\]

modelled on

\[
G \leftarrow P \\
\downarrow \\
G/P
\]

where $G$ is equipped with a Cartan connection $\omega$ – viz. a suitably equivariant Lie($\tilde{G}$)-valued 1-form, cf. Maurer-Cartan form on $\tilde{G}$.

E.g. Conformal geometry, projective DG, CR geometry, ···

For **conformal DG**: $G = SO_o(p + 1, q + 1)$, and $P$ subgroup stabilising a ray in $\mathbb{R}^{p+q+2}$.
If we have a representation $V$ of the group $G$ then we have an associated **vector bundle** $G \times_P V$ with a **linear connection** $\nabla$. This is the associated **tractor connection**. In fact for $(G, \omega)$ modelled on $(G, P)$, with $G$ semi-simple, $P$ parabolic:

**Theorem (Čap+G.)**

*Cartan bundle $\mathcal{G} + \text{connection } \omega \iff \text{Tractor bundle and tractor connection}.*

Then:

**parallel tractors lead to curved analogues of orbit decompositions.**

The point is that even on the model we can think of the orbit decomposition as arising from a parallel tractor of some type. Then the corresponding parallel tractor leads to a corresponding stratification of the curved manifold.
Theorem (Curved orbit decomposition - Čap, G., Hammerl)

Suppose $(G, \omega) \to M$ is a Cartan geometry (modelled on $G \to G/P$) endowed with a parallel tractor field $h$ giving a Cartan holonomy reduction with holonomy group $H$. Then:

1. $M$ is canonically stratified $M = \bigcup_{i \in H \backslash G/P} M_i$ in a way locally diffeomorphic to the the $H$-orbit decomposition of $G/P$; and
2. there $\exists$ a Cartan geometry on $M_i$ of the same type as the model.

Thus there is a general way to define a curved analogue of an orbit decomposition of a homogeneous space.
Curving the conformal compactification of $\mathbb{H}^{n+1}$

Recall the $H = SO_o(n + 1, 1)$ orbits on conformal sphere $G/P$, where $G = SO_o(n + 2, 1)$, $H$ fixes $I \in \mathbb{R}^{n+3}$ spacelike:

$$S^n = \partial \mathbb{H}^{n+1}$$

Curved: A conformal manifold has a canonical Cartan bundle $G$ modeled on $(G, P)$. If this supports a parallel spacelike tractor $I$ then the curved orbit theorem (plus some interpretation) states either $M$ Einstein or $M$ stratifies into disjoint union $M = M_- \cup M_0 \cup M_+$ and $M_0$ is a separating hypersurface. Moreover $M \setminus M_\mp$ is a compactification of the Einstein $M_\pm$. 

**Rod Gover.** background: G-., Waldon, arXiv:1506.02723, and Compactification and boundary calc
Compactification Programme

Given some non-compact geometry of interest (e.g. pseudo-Riemannian):

**Part 1 (homogeneous):** Identify a homogeneous model $X_i = H/K$ of the geometry as an open $H < G$ orbit $M$ in a compact homogeneous space $X = G/P$. (E.g. $G$ semi-simple and $P$ parabolic.) Then the topological closure $\overline{X}_i \subset X$ is a compactification of $X_i$.

**Part 2 (curved I):** Given a compact Cartan geometry $(\mathcal{G}, \omega) \to M$ modelled on $G \to G/P$, with a Cartan holonomy reduction with holonomy group $H$ and an open curved orbit $M_i$ (with same Cartan geometry type as $X_i$), then $\overline{M}_i$ is its compactification.

**Part 3 (curved II):** Typically the geometry on $M_i$ has restrictions on e.g. Einstein or symmetries, . . . (as a normal solution of a BGG equation holds on $M$). In some cases we can drop restrictions yet still exploit the Cartan/tractor machinery.
Henceforth in these talks, **conformal compactification** of pseudo-Riemannian manifold \((M^{n+1}, g_+)\) is a manifold \(\overline{M}\) with boundary \(\partial M\) s.t.:

- \(\exists \overline{g}\) on \(\overline{M}\), with
- \(g_+ = r^{-2}\overline{g}\), where \(r\) a defining function for \(\partial M\).

\(\Rightarrow\) canonical **conformal structure on boundary**: \((\partial M, [\overline{g}|_{\partial M}]\) (where \(dr\) not null).

- Called a **Poincaré-Einstein** metric if also \(g_+\) negative **Einstein**.

This is the case rediscovered above by a parallel spacelike tractor \(l\) on \(\overline{M} = M_+ \cup M_0\).
It is natural to seek generalising (even) less rigid notions than holonomy reductions. Above we saw the Poincare Ball compactification of $\mathbb{H}^{n+1}$ generalises via curved orbit decomposition methods to a notion of conformal compactification of complete Einstein spaces – this recovers the usual notion of Poincaré-Einstein spaces, but in a new and very useful way.

We want to drop the “Einstein” in this and similar cases.

But we want to do this in a way that retains the Cartan tractor picture – as this encodes deep information. Here in particular we will attempt to treat conformally compact manifolds in general (almost).
A conformal \(n+1\)-manifold \((n \geq 2)\) is the structure \((M, c)\) where

- \(M\) is an \(d = n + 1\)-manifold,
- \(c\) is a conformal equivalence class of signature \((p, q)\) metrics,

i.e. \(g, \hat{g} \in c \iff \hat{g} = \Omega^2 g\) and \(C^\infty(M) \ni \Omega > 0\).

Because there is no distinguished metric on \((M, c)\) an important role is played by the **density bundles**. Note \((\Lambda^{n+1} TM)^2\) is an oriented real line bundle \(\mathcal{K}\). We write \(\mathcal{E}[w]\) for the roots

\[
\mathcal{E}[w] = \mathcal{K}^{\frac{w}{2n+2}}, \quad \text{so} \quad \mathcal{K} = \mathcal{E}[2n+2],
\]

\(\mathcal{E}[0] : = \mathcal{E}\) (the trivial bundle with fibre \(\mathbb{R}\)), and \(\mathcal{E}_+[w]\) for the positive elements. With this notation there is tautologically a **conformal metric**

\[
g \in S^2 T^* M[2], \quad \text{so that} \quad g^\sigma := \sigma^{-2} g \in c, \quad \sigma \in \Gamma(\mathcal{E}_+[1]),
\]

and

\[
\otimes^{n+1} g : (\Lambda^{n+1} TM)^2 \xrightarrow{\sim} \mathcal{E}[2n+2].
\]
In view of the 1-1 relation between sections $\sigma$ of $\mathcal{E}_+[1]$ and metrics $g^{\sigma}$ in $\mathfrak{c}$ (via $g^{\sigma} := \sigma^{-2} g \in \mathfrak{c}$) be call

$$\sigma \in \Gamma(\mathcal{E}_+[1])$$

a **strict scale**. Such sections provide the “symmetry breaking” which reduces us from conformal geometry to pseudo-Riemannian. Since any Levi-Civita connection $\nabla^g$, for $g \in \mathfrak{c}$, acts on $\mathcal{E}[2n + 2]$ via the isomorphism $\otimes^{n+1} g : (\Lambda^{n+1} TM)^2 \xrightarrow{\sim} \mathcal{E}[2n + 2]$ it follows easily that

$$\nabla^{g^{\sigma}} \sigma = 0.$$ 

So $\sigma$ is parallel for the Levi-Civita connection it determines.
The tractor connection

On a conformal manifold \((\overline{M}, c)\) there is no distinguished connection on \(\mathcal{T}M\). But we have the conformally invariant \textbf{tractor bundle} \(\mathcal{T}\) and \textbf{connection} \(\nabla^\mathcal{T}\). Given \(\overline{g} \in c\) this is given by

\[
\mathcal{T} \overline{g} = \mathcal{E}[1] \oplus \mathcal{T}^* M[1] \oplus \mathcal{E}[-1], \quad \mathcal{E}[1] := (\Lambda^{n+1} \mathcal{T} M)^{\frac{2}{2(n+1)}}
\]

\[
\nabla^\mathcal{T}_a (\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a, \nabla \mu_b + P_{ab} \sigma + g_{ab} \rho, \nabla \rho - P_{ab} \mu^b),
\]

and \(\nabla^\mathcal{T}\) preserves a conformally invariant \textbf{tractor metric} \(h\)

\[
\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma \rho + \mu_b \mu^b = h(V, V).
\]

There is also a second order \textbf{Thomas operator}:

\[
\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \equiv \begin{pmatrix}
(n + 2w - 2)wf \\
(n + 2w - 2) \nabla_a f \\
-(\Delta f + wJf)
\end{pmatrix}
\]

where \(J\) is \(\text{trace} \overline{g}(P_{ab})\), so a number times \(\text{Sc}(\overline{g})\).
Parallel standard tractors

Note that from the formula

\[ \nabla_a^T (\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a, \nabla \mu_b + P_{ab} \sigma + g_{ab} \rho, \nabla \rho - P_{ab} \mu^b), \]

if \( I_A \equiv (\sigma, \mu, \rho) \) is a parallel tractor then \( \mu_a = \nabla_a \sigma \), and \( \rho = - (\Delta \sigma + \omega J \sigma) \). This gives the first statement of:

**Proposition**

*\( I \) parallel implies \( I_A = \frac{1}{d} D_A \sigma \). So \( I \neq 0 \) \( \Rightarrow \) \( \sigma \) is nonvanishing on an open dense set \( M_{\sigma \neq 0} \). On \( M_{\sigma \neq 0} \), \( g^o = \sigma^{-2} g \) is Einstein. Conversely if \( g^o = \sigma^{-2} g \) is Einstein then \( I := \frac{1}{d} D \sigma \) is parallel.*

**Proof.**

On \( M_{\sigma \neq 0} \) we have locally \( \pm \sigma \in \Gamma(E_+[1]) \) so \( \mu_a = \nabla_a \sigma = 0 \) for \( \nabla = \nabla^{g^o} \). Thus

\[ P_{ab} + \rho g_{ab} = 0. \]

The converse is easy.

So we say \((M, c)\) with parallel \( I \neq 0 \) is **almost Einstein**.
Concerning $M_0 = \mathcal{Z}(\sigma)$. (Here and throughout $I^2 = I^A I_A$.)

**Theorem**

The curved orbit decomposition of an almost Einstein manifold $(M, c, I)$ is according to the strict sign of $\sigma = I_A X^A$. The zero locus satisfies:

- If $I^2 \neq 0$ (i.e. $g^o$ Einstein and not Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or is a smoothly embedded separating hypersurface.
- If $I^2 = 0$ (i.e. $g^o$ Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or, after excluding isolated points from $\mathcal{Z}(\sigma)$, is a smooth embedded hypersurface.

**Proof.**

The local aspects follow from the general curved orbit theorem. Using the above formulae they are also easily recovered directly and one sees the separating statement.
Thus if $I^2 \neq 0$ we have the picture:

$M \setminus M_{\pm}$ is evidently conformally compact and hence **Poincaré-Einstein**. Conversely all Poincaré-Einstein manifolds arise this way.
Almost pseudo-Riemannian geometry

We want now to drop the Einstein condition and understand e.g. general conformally compact manifolds.

For convenience we say that a structure $(M^d, c, \sigma)$ where $\sigma \in \Gamma(\mathcal{E}[1])$ is almost pseudo-Riemannian if the scale tractor

$$I_A := \frac{1}{d} D_A \sigma$$

is nowhere zero. Note then that $\sigma$ is non-zero on an open dense set, since $D_A \sigma$ encodes part of the 2-jet of $\sigma$. So on an almost pseudo-Riemannian manifold there is the pseudo-Riemannian metric $g^o = \sigma^{-2} g$ on the same open dense set. In the following the notation $I$ will always refer to a scale tractor, so $I = \frac{1}{d} D \sigma$, for some $\sigma \in \Gamma(\mathcal{E}[1])$. Then we often mention $I$ instead of $\sigma$ and refer to $(M, c, I)$ as an almost pseudo-Riemannian manifold. Evidently:

**Lemma**

A conf. compact mfld is an almost Riemannian manifold $(\overline{M}, c, \sigma)$ with boundary $(\overline{M} = M_+ \cup \partial M_+)$ such that $\sigma$ defines $\partial M_+$.
Now recall from the formula for $I$ and the metric we have

$$I^A I_A =: I^2 = g^{ab} (\nabla_a \sigma)(\nabla_b \sigma) - \frac{2}{d} \sigma (J + \Delta) \sigma$$  \hspace{1cm} (1)

where $g$ is any metric from $c$ and $\nabla$ its Levi-Civita connection. This is well-defined everywhere on an almost pseudo-Riemannian manifold, while where $\sigma$ is non-zero, it computes

$$I^2 = - \frac{2}{d} \int g^o = - \frac{\text{Sc}^g}{d(d-1)} \hspace{1cm} \text{where} \hspace{1cm} g^o = \sigma^{-2} g.$$

Thus $I^2$ gives a **generalisation of the scalar curvature** (up to a constant factor $-1/d(d-1)$); it is canonical and smoothly extends the scalar curvature to include the zero set of $\sigma$. We shall use the term **ASC manifold** (where ASC means almost scalar constant) to mean an almost pseudo-Riemannian manifold with $I^2 = \text{constant}$. Since the tractor connection preserves $h$, then $I$ parallel implies $I^2 = \text{constant}$. So an almost Einstein manifold is ASC, just as Einstein manifolds have constant scalar curvature.
Non-zero generalised scalar curvature.

Much of the almost Einstein curved orbit picture remains in the almost pseudo-Riemannian setting when $I^2$ is non-vanishing:

**Theorem**

Let $(M, c, I)$ be an almost pseudo-Riemannian manifold with $I^2$ nowhere zero. Then $Z(\sigma)$, if not empty, is a smooth embedded separating hypersurface. This has a spacelike (resp. timelike) normal if $g^\sigma$ has negative scalar (resp. positive) scalar curvature. If $c$ has Riemannian signature and $I^2 < 0$ then $Z(\sigma)$ is empty.

**Key aspect of Proof.**

From $I^2 \overset{g}{=} g^{ab}(\nabla_a \sigma)(\nabla_b \sigma) - \frac{2}{d}\sigma(J + \Delta)\sigma$: Along $Z(\sigma)$ we have

$$I^2 = g^{ab}(\nabla_a \sigma)(\nabla_b \sigma).$$

in particular $\nabla \sigma$ is nowhere zero on $Z(\sigma)$, and so $\sigma$ is a defining density. Thus $Z(\sigma)$ is a smoothly embedded hypersurface by the implicit function theorem.

**Summary:** A conformal manifold equipped with a scale tractor $I = \frac{1}{d} D \sigma$, with $I^2$ nowhere zero has $I$ nowhere zero and so is almost pseudo-Riemannian. Where $\sigma = X^A I_A$ is nonzero (almost everywhere) there is the pseudo-Riemannian metric $g^o = \sigma^{-2} g$, and $\sigma$ is a defining density for the separating hypersurface $M_0 = Z(\sigma)$.

Thus we again have a stratification

$$M = M_- \cup M_0 \cup M_+.$$ 

Moreover $(M, c, I) \setminus M_{\mp}$ is conformally compact, as any scale $\tau \in \Gamma(E_+[1])$ gives $\overline{g} = \tau^{-2} g \in c$, and $r := \tau^{-1} \sigma$ is a defining function for $M_0$ in $\overline{M}_{\pm} = M_{\pm} \cup M_0$. It is clear all conformally compact manifolds with scalar curvature bounded away from zero arise this way.

We want to develop a boundary calculus for these conformally compact manifolds. First we digress to understand hypersurfaces.
Compactification and boundary calc
Part III: Hypersurfaces in conformal manifolds

Part IV: Geometry of conformal infinity
To treat boundary calculus we need to understand the mathematics of hypersurfaces.

**Defn:** hypersurface $\Sigma$ in a manifold $M$ means a smoothly embedded codimension 1 submanifold of $(M, c)$.

- we restrict to $\Sigma$ with the property that the any conormal field along $\Sigma$ is nowhere null (i.e. to nondegenerate hypersurfaces).

Then:

- restriction of any $g \in c$ gives metric $\bar{g}$ on $\Sigma \sim c$ induces $\bar{c}$ on $\Sigma$.
- It is natural to work with a weight 1 co-normal $n_a$ along $\Sigma$ satisfying $g^{ab}n_a n_b = \pm 1$. 

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Basic hypersurface invariants

For \( g \in \mathfrak{c} \), the **second fundamental form** \( L_{ab} \) is the restriction of \( \nabla_a n_b \) to \( T\Sigma \times T\Sigma \subset (TM \times TM)|_\Sigma \), where \( \nabla = \nabla^g \); i.e.

\[
L_{ab} := \nabla_a n_b \mp n_a n^c \nabla_c n_b \quad \text{along} \quad \Sigma.
\]

This is not conformally invariant. But under a conformal rescaling, \( g \mapsto \hat{g} = e^{2\omega} g \), \( L_{ab} \) transforms according to

\[
L_{ab}^\hat{g} = L_{ab}^g + \hat{g}_{ab} \Upsilon_c n^c,
\]

where \( \Upsilon = d\omega \)

Thus:

**Proposition**

*The trace-free part of the second fundamental form*

\[
\hat{L}_{ab} = L_{ab} - H \hat{g}_{ab}, \quad \text{where,} \quad H := \frac{1}{d-1} \hat{g}^{cd} L_{cd}
\]

is conformally invariant.

Here \( d = n + 1 \) is the dimension of the ambient manifold \( M \).
Evidently, under a conformal rescaling $g \mapsto \hat{g} = e^{2\omega} g$, the mean curvature $H^g$ transforms to $H^{\hat{g}} = H^g + n^a \gamma_a$. Thus we obtain a conformally invariant section $N$ of $\mathcal{T}|_\Sigma$

$$N_A \overset{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix},$$

and $h(N, N) = \pm 1$ along $\Sigma$. This is the normal tractor of Bailey-Eastwood-G. Differentiating $N$ tangentially along $\Sigma$ using $\nabla^\mathcal{T}$, we obtain the following result.

**Proposition (Conformal Shape operator)**

$$\mathcal{L}_{aB} := \nabla_a N_B \overset{g_{cb}}{=} \begin{pmatrix} 0 \\ \hat{\mathcal{L}}_{ab} \\ -\frac{1}{d-2} \nabla^b \hat{\mathcal{L}}_{ab} \end{pmatrix}$$

where $\nabla$ is the pullback to $\Sigma$ of the ambient tractor connection. Thus $\Sigma$ is totally umbilic iff $N$ is parallel along $\Sigma$. 

Rod Gover. background: G-, Waldon, arXiv:1506.02723, and Compactification and boundary calc
The classical **Gauss formula**

\[ \nabla_a v^b = \nabla_a v^b \mp n^b L_{ac} v^c \quad v \in T\Sigma \subset TM, \]

is the basis of Riemannian hypersurface calculus.

We want the conformal analogue. First we need this:

**Proposition (Branson-G., Grant)**

There is a natural conformally invariant (isometric) isomorphism

\[ T|_{\Sigma} \supset N^\perp \overset{\sim}{\rightarrow} \overline{T} = \text{std tractor bdle of } (\Sigma, \overline{c}). \]

**Proof.**

Calculating in a scale \( g \) on \( M \) the tractor bundle \( T \), and hence also \( N^\perp \), decomposes into a triple. Then the mapping of the isomorphism is

\[
[N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \mp H n_b \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\overline{T}]_{g}.
\]
The tractor Gauss equation

The above reveals two connections on $\mathcal{T} \cong N^\perp$ that we can compare. Namely the **intrinsic tractor connection** $\nabla^{\mathcal{T}}$ determined by $(\Sigma, \bar{c})$, and the **projected ambient tractor connection** $\tilde{\nabla}$. The latter is defined by

$$\tilde{\nabla}_a U^B := \Pi^B_C (\Pi^c_a \nabla^c U^C) \quad U \in \Gamma(N^\perp) \text{ extended arb. off } \Sigma$$

where $\Pi^B_C$ and $\Pi^c_a$ are the orthog. projections due to $N$ and $n$. Including the tractor derivative of $\Pi^B_C$ gives:

**Proposition (Tractor Gauss formula – Stafford, Vyatkin)**

$$\nabla_a V^B = \nabla_a V^B \mp S_a^B C V^C \mp N^B \mathbb{L}_a C V^C,$$

where $S_{aBC} = \bar{X}_{BC}^c F_{ac},$ (\bar{X}_{BC}^c \text{ an invariant bundle injector}), and

$$F_{ab} = \frac{1}{n-2} \left( W_{acbd} n^c n^d + \mathcal{L}^2_{ab} - \frac{|\mathcal{L}|^2}{2(n-1)} g_{ab} \right).$$

Recall $\mathbb{L}_a C = \nabla_a N_C$. This shows that $F_{ab}$ is a conformal invariant of hypersurfaces. It is the so-called **Fialkow tensor**.
The above results and tools provide the first steps in an invariant calculus for conformal hypersurfaces that is somewhat analogous to the local invariant calculus for Riemannian hypersurfaces. In particular combining these with usual tractor calculus it is easy to proliferate hypersurface conformal invariants and conformally invariant operators. E.g.:

**Families of boundary operators** along a conformal hypersurface.

**Background:** In Riemannian geometry the Neumann operator is $n^a \nabla_a$. Higher transverse order transverse boundary operators similarly given: $n^a n^b \nabla_a \nabla_b$ etc.

**Conformal Robin op:** The tools above allow an immediate analogue. Recall $\delta_1 := n^a \nabla_a - w H g$, is the conformal Cherrier-Robin operator – it gives a conformal boundary Robin operator for the conformal Laplacian. This is recovered by

$$(n + 2w - 2)\delta_1 = N^A D_A \text{ along } \Sigma^{n-1} \text{ in } M^n.$$
Families of boundary/symmetry breaking operators

Higher order analogues are important for PDE boundary problems, and the construction of higher order conformal Dirichlet-Neumann operators. Juhl, Kobyashi et al have sought continuous families of such, which in the flat case they describe as symmetry breaking operators and interpret as intertwinors of the spherical principal series representations of the conformal group. Here is an immediate construction of such:

Lemma (G.-Peterson)

Given a conformal hypersurface embedding $\Sigma \hookrightarrow (M, c)$,

$$\delta_{j+1} := N^A_1 N^A_2 \cdots N^A_j \delta_1 D_A_1 D_A_2 \cdots D_A_j$$

constructs a family of natural conformally invariant hypersurface operators $\delta_K : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w - K]|_{\Sigma}$ along $\Sigma$.

This observation can be refined significantly (G.-Peterson). The key is identify and treat the special weights $w$. 

Rod Gover. background: G-, Waldon, arXiv:1506.02723, and Compactification and boundary calc
It would appear from the formula

$$\delta_K := N^A_1 N^A_2 \cdots N^A_{K-1} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_{K-1}} \text{ along } \Sigma$$

that the operator has “high” transverse order and is always at least of transverse order 1. But e.g.: (where $\bar{n} = \text{dim}(\Sigma)$ etc)

$$\delta_2 f = - (\bar{\Delta} - \frac{\bar{n} - 2}{4(\bar{n} - 1)} \bar{S}c)f + \frac{\bar{n} - 2}{4(\bar{n} - 1)} \hat{L}^{ab} \hat{L}_{ab} f, \text{ for } f \in E \left[ 1 - \frac{\bar{n}}{2} \right].$$

This is the intrinsic to $\Sigma$ Yamabe operator of $(\Sigma, c_\Sigma)$ (plus the conformal invariant $\hat{L}^{ab} \hat{L}_{ab}$). So:

\[ \text{at this weight } \delta_2 \text{ has transverse order } 0. \]

At the interior Yamabe weight $1 - \frac{n}{2}$ we have instead

$$\delta_2 = - (\Delta - \frac{n - 2}{4(n - 1)} \text{Sc}) \text{ along } \Sigma.$$  

- i.e. the interior Yamabe operator.
We return now to **conformally compact geometries** \((M, c, I)\). Recall the **scale tractor** \(I\) is given \(I = (\sigma, \nabla \sigma, -\frac{1}{d}(\Delta \sigma + J\sigma))\). We will consider in particular \((M, c, I)\) which near the conformal infinity are **asymptotically of constant nonzero scalar curvature**. By imposing a constant dilation we may assume that \(I^2\) approaches \(\pm 1\).

The \(\sigma\), equivalently scale tractor \(I\), strongly links the geometry of \(\Sigma = Z(\sigma)\) to the ambient by a beautiful agreement of \(I\) and the normal tractor:

**Proposition**

Let \((M^d, c, I)\) be an almost pseudo-Riemannian structure with scale singularity set \(\Sigma \neq \emptyset\) and \(I^2 = \pm 1 + \sigma^2 f\) for some smooth (weight \(-2\)) density \(f\). Then \(\Sigma\) is a smoothly embedded hypersurface and, with \(N\) denoting the normal tractor for \(\Sigma\), we have \(N = I|_\Sigma\).
Proof.

For simplicity assume the case $l^2 = \pm 1$ (so $f = 0$ and the structure is ASC). As usual let us write $\sigma := h(X, l)$. Along $Z(\sigma)$

$$l_A = \frac{1}{d} D_A\sigma = g \begin{pmatrix} 0 & \nabla_a \sigma \\ \nabla_b \sigma & -\frac{1}{d} \Delta \sigma \end{pmatrix} \Rightarrow g^{ab}(\nabla_a \sigma)\nabla_b \sigma = \pm 1$$

so $n_a := \nabla_a \sigma$ is the unit conormal and a computation gives

$$\frac{1}{d} \Delta \sigma = -\frac{1}{d-1} g^{ab} L g_{ab} = -H g.$$

Corollary

Let $(M^d, c, l)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $l^2|_\Sigma = \pm 1$, and $\nabla_a l_B = \sigma f_{aB}$ for some smooth (weight $-1$) tractor valued 1-form $f_{aB}$. Then $\Sigma$ is a totally umbilic hypersurface.
Agreement of tractor connections

If we assume the stronger asymptotics: \( l^2|_{\Sigma} = \pm 1 \), and \( \nabla_a l_B = \sigma^2 f_{aB} \) Then along \( \Sigma \), \( l_B \) is parallel to the given order, and so the tractor curvature satisfies

\[
\kappa_{ab}^C \ T^D = \kappa_{ab}^C \ T^N = 0 \quad \text{along} \quad \Sigma.
\]

This implies

\[
W_{ab}^c \ T^d n^d = 0, \quad \text{along} \quad \Sigma = Z(\sigma)
\]

\[
\therefore \quad \text{Fialkow } F_{ab} = \frac{1}{n-2} \left( W_{acbd} n^c n^d + \hat{l}_2^D - \frac{\hat{l}^2}{2(n-1)g_{ab}} \right) \text{ vanishes, } &
\]

Theorem

Let \((M^{d\ge4}, c, l)\) be an almost pseudo-Riemannian structure with scale singularity set \( \Sigma \neq \emptyset \), and that is asymptotically Einstein in the sense that \( l^2|_{\Sigma} = \pm 1 \), and \( \nabla_a l_B = \sigma^2 f_{aB} \). Then the tractor connection of \((M, c)\) preserves the intrinsic tractor bundle of \( \Sigma \), where the latter is viewed as a subbundle of the ambient tractors: \( T_{\Sigma} \subset T \). Furthermore the restriction of the parallel transport of \( \nabla_T \) coincides with the intrinsic tractor parallel transport of \( \nabla_{T_{\Sigma}=\overline{T}} \).
Any almost pseudo-Riemannian manifold with non-zero generalised scalar curvature (i.e. $I^2$ nowhere zero) has zero locus $\Sigma = \mathcal{Z}(\sigma)$ a smoothly embedded hypersurface.

If $g_\pm = \sigma^{-2} g$ is (asymptotically) Einstein in that $\nabla I$ vanishes (sufficiently quickly) along $\Sigma$ then for $\Sigma$ the conformal invariants $\mathcal{L}_{ab}$ and $\mathcal{F}_{ab}$ both vanish everywhere along $\Sigma$. This:

- gives agreement of the ambient and intrinsic tractor connections;
- excludes interesting embeddings of $\Sigma$ — i.e. such $\Sigma$ are not useful for studying general hypersurface geometry.

This is the classical Poincaré-Einstein setting. From the work of Fefferman-Graham and others we know that if $g_\pm = \sigma^{-2} g$ is (asymptotically) Einstein (to sufficiently high order) then the conformal geometry of $(\Sigma, \bar{c})$ actually formally determines the full geometry of $(M, g_\pm)$, at least up to order approximately $d - 1$.

Thus a powerful link intrinsic geometry of $(\Sigma, \bar{c})$ to $(M, g_\pm)$ — little freedom at all in the embedding of $\Sigma$. 

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Rod Gover. background: G-.-, Waldon, arXiv:1506.02723, and Compactification and boundary calc
Part V: Boundary calculus

Part VI: The Loewner-Nirenberg problem and higher Willmore
Recall: By embedding a conformal manifold \((\Sigma, \tilde{c})\) as the boundary at infinity of a Poincaré-Einstein manifold \((M, g_+)\), the Fefferman-Graham programme led to powerful tools: New approaches to the construction of conformal invariants (of \((\Sigma, \tilde{c})\)), The GJMS operators and \(Q\)-curvature, scattering theory, \(\cdots\), Applications to AdS/CFT conjecture, \(\cdots\).

From the early discussion we will see that there is an analogous program in for conformally embedded hypersurfaces. In part the generalises the FG program but in other ways it is a different program.

We work on \(M^{d=n+1}\).
Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold \((M, c, I)\) there is a canonical differential operator by coupling \(I^A\) to \(D_A\), namely

\[ I \cdot D := I^A D_A. \]

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

\[ I \cdot D : \mathcal{E}^\Phi[w] \rightarrow \mathcal{E}^\Phi[w - 1]. \]

It will be useful to define the weight operator \(w\): if \(\beta \in \Gamma(B[w_0])\) we have

\[ w \beta = w_0 \beta. \]

Then on \(\mathcal{E}^\Phi[w]\) we have

\[ I \cdot D \overset{g}{=} \left( -\frac{1}{d}(\Delta \sigma + J\sigma) \quad \nabla^a \sigma \quad \sigma \right) \begin{pmatrix} w(d + 2w - 2) \\ \nabla_a(d + 2w - 2) \\ -(\Delta + Jw) \end{pmatrix} = -\sigma \Delta + (d + 2w - 2)[(\nabla^a \sigma) \nabla_a - \frac{w}{d} (\Delta \sigma)] - \frac{2w}{d} (d + w - 1) \sigma J. \]
The canonical degenerate Laplacian

Now on $\mathcal{M} \setminus \mathcal{Z}(\sigma)$ in the metric $g_{\pm} = \sigma^{-2}g$, with densities trivialised accordingly, we have

$$I \cdot D \, g_{\pm} \equiv \mp \left( \Delta g_{\pm} + \frac{2w(d + w - 1)}{d} j_{g_{\pm}} \right).$$

In particular if $g_{\pm}$ satisfies $J^g_{\pm} = \mp \frac{d}{2}$ (i.e. $Sc^g_{\pm} = \mp d(d - 1)$ or equivalently $l^2 = \pm 1$) then, relabeling $d + w - 1 =: s$ and $d - 1 =: n$, we have

$$I \cdot D \, g_{\pm} \equiv \mp \left( \Delta g_{\pm} \mp s(n - s) \right).$$

so solutions are eigenvectors of the Laplacian (and $s$ is called the spectral parameter) as in scattering theory.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I \cdot D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC in the sense that $l^2 = \pm 1 + \sigma f$, for some smooth $f$, then along $\Sigma$

$$I \cdot D = (d + 2w - 2)\delta_n,$$

$$\delta_1 \overset{g}{=} n^a \nabla^a g - w H g = \text{conformal Robin}$$

Thus $I \cdot D$ is a degenerate Laplacian, natural to $(\mathcal{M}, c, l)$. 

Rod Gover.  background: G-., Waldon, arXiv:1506.02723, and Compactification and boundary calc
(\(M, c\)) be a conformal structure of dimension \(d \geq 3\), \(\sigma \in \Gamma(\mathcal{E}[1])\) and \(I_A = \frac{1}{d} D_A \sigma\) (as usual). Then a direct computation gives

**Lemma**

*Acting on any section of a weighted tractor bundle we have*

\[
[I \cdot D, \sigma] = I^2(d + 2w),
\]

*where \(w\) is the weight operator.*

Thus with **only the restriction that generalised scalar curvature is non-vanishing** we have:

**Proposition (G.-Waldron)**

*Suppose that \((M, c, \sigma)\) is such that \(I^2\) is nowhere vanishing. Setting \(x := \sigma\), \(y := -\frac{1}{I^2} I \cdot D\), and \(h := d + 2w\) we obtain the commutation relations*

\[
[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,
\]

*of standard \(\mathfrak{sl}(2)\)-algebra generators.*
Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathbb{Z}(\sigma)$

$$P_k : \mathcal{E}^{\Phi}[\frac{k-n}{2}] \to \mathcal{E}^{\Phi}[\frac{-k-n}{2}] \quad \text{given by} \quad P_k := \left(-\frac{1}{l^2} lD\right)^k \quad (2)$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^{\Phi}[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^{\Phi}[\frac{-k-n}{2}]|_\Sigma$. For $k$ even this takes the form

$$P_k = \overline{\Delta}^k + \text{lower order terms.} \quad (3)$$

Proof.

From the $sl(2)$-identities we have $[x, y^k] = y^{k-1}k(h - k + 1)$. Thus on $\mathcal{E}^{\Phi}[\frac{k-n}{2}]$

$$P_k(f + \sigma h) = y^k(f + xh) = P_k f + \sigma \tilde{P}_k h.$$ 

So $P_k$ is tangential. Expanding the $l\cdot D$s yields (3).
Suppose on a conformally compact manifold $M_+$ (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf := \left( \Delta^{g_+} + \frac{2w(d + w - 1)}{d} J^{g_+} \right) f = 0.$$ 

E.g. this is what is studied in the usual Poincaré-Einstein scattering program.

Then one needs to fix suitable boundary conditions. E.g. in the case of Riemannian signature one wants some elliptic boundary problem. Since the boundary $\partial M_+$ is at infinity, with $g_+$ singular along $\partial M_+$, this is non-trivial.

But if we view $f$ as the trivialisation of a density of weight $w$ then $Pf \overset{g_+}{=} I \cdot Df$ and $I \cdot D$ is well defined on all of $\overline{M}$ (and its smooth extension to $M$ beyond $\partial M_+$). Thus it is natural to study the $I \cdot D$ problem. We do this formally.

First we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data.
Asymptotic solutions of the first kind

**Problem**

Given $f|_{\Sigma}$, and an arbitrary extension $f_0$ of this to $\mathcal{E}^\Phi[w_0]$ over $M$, find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over $M$), $i = 1, 2, \cdots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $l \cdot Df = O(\sigma^\ell)$, off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$l \cdot Df = 0 \iff -\frac{1}{l^2} l \cdot Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h \rangle$.

**Set** $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1}k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^{\ell}(\ell + 1)(h + \ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = (h_0 - 2(\ell + 1))f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^{\ell}(\ell + 1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \quad (4)$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\ell \neq h_0 - 2$ we can solve

$$yf^{(\ell+1)} = O(x^{\ell+1})$$

and this **uniquely determines** $f_{\ell+1}|_{\Sigma}$. 

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Rod Gover. background: G-, Waldon, arXiv:1506.02723, and Compactification and boundary calc
So we can solve to all orders provided we do not hit \( \ell = h_0 - 2 \) i.e. provided \( w_0 \notin \left\{ \frac{k-n}{2} : k \in \mathbb{Z}_{\geq 1} \right\} \). Otherwise (4) shows that

\[
\ell = h_0 - 2 \quad \Rightarrow \quad yf^{(\ell)} = y(f^{(\ell)} + x^{\ell+1}f_{\ell+1}), \quad \text{modulo } O(x^{\ell+1}),
\]

regardless of \( f_{\ell+1} \). It follows that the map \( f_0 \mapsto x^{-\ell}yf^{(\ell)} \) is tangential and \( x^{-\ell}yf^{(\ell)}|_{\Sigma} \) is the obstruction to solving \( yf^{(\ell+1)} = O(x^{\ell+1}) \). Then by a simple induction this is seen to be a non-zero multiple of \( y^{\ell+1}f_0|_{\Sigma} \):

**Proposition**

If \( \ell = h_0 - 2 \) then the smooth extension is (in general) obstructed by \( P_{\ell+1}f_0|_{\Sigma} \), where \( P_{\ell+1} = \left( -\frac{1}{12} l \cdot Df \right)^{\ell+1} \) is the tangential operator on densities of weight \( w_0 \) given by Theorem 27.

If \( \ell = h_0 - 2 \) then the extension can be continued with log terms. If \( \overline{M} \) is almost Einstein to sufficiently high order then:

- the odd order \( P_{\ell+1} \) vanish identically; and
- the even order \( P_{\ell+1} \) are the GJMS operators on \( (\partial M_+, \bar{c}) \).
Now we consider the more general type of solution:

**Problem**

Given $\bar{f}_0|_\Sigma \in \Gamma \mathcal{E}_\Phi[w_0 - \alpha]|_\Sigma$ and an arbitrary extension $\bar{f}_0$ of this to $\Gamma \mathcal{E}_\Phi[w_0 - \alpha]$ over $\overline{M}$, find $\bar{f}_i \in \mathcal{E}_\Phi[w_0 - \alpha - i]$ (over $\overline{M}$), $i = 1, 2, \cdots$, so that

$$\bar{f} := \sigma^\alpha (\bar{f}_0 + \sigma \bar{f}_1 + \sigma^2 \bar{f}_2 + \cdots + O(\sigma^{\ell+1}))$$

solves $I \cdot D\bar{f} = O(\sigma^{\ell+\alpha})$, off $\partial M_+$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Now $\alpha$, if not integral, this Problem takes us outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. But it is straightforward to show that for any $\alpha \in \mathbb{R}$:

$$[x^\alpha, y] = x^{\alpha - 1}\alpha (h + \alpha - 1).$$
It follows immediately from (6) that \( I \cdot D\overline{f} = 0 \) has:

- no solution if \( \alpha \notin \{ 0, h_0 - 1 \} \), where \( h\overline{f} = h_0 \overline{f} \); and
- if \( \alpha = h_0 - 1 \) and \( \overline{f} = \sigma^\alpha f \) then

\[
I \cdot D\overline{f} = \sigma^\alpha I \cdot Df
\]

So \( \overline{f} \) is a solution iff \( f \) is!

So in this way second solutions arise from first and vv.

For \( w_0 \notin \{ \frac{k-n}{2} : k \in \mathbb{Z}_{\geq 1} \} \), and writing \( F = f \), \( G = \sigma^{-\alpha} \overline{f} \) we can combine these to a general solution

\[
F + \sigma^{h_0-1} G = F + \sigma^{n+2w_0} G
\]

or, trivialising the densities on \( M_+ \) using the generalised scale \( \sigma \):

\[
f = \sigma^{n-s} F + \sigma^{s} G = \sigma^{-w_0} (F + \sigma^{h_0-1} G)
\]

where \( s := w_0 + n \). Which is the form of solution used in the scattering theory (of Graham-Zworski, Mazzeo-Melrose, · · · ). The \( \mathfrak{sl}(2) \) approach above solves the asymptotics of \( F \) and \( G \).
The Poincaré-Einstein construction is a tool for studying a conformal manifold \((\Sigma, \bar{c})\) \textbf{holographically}. That is for obtaining the invariants and invariant operators of \((\Sigma, \bar{c})\) in terms (pseudo-)Riemannian objects on the manifold \(M_+\) of 1 greater dimension that has \(\Sigma = \partial M_+\).

\textbf{Conversely} the scattering theory of \((M, g_+)\) can be understood in terms of non-local conformal operators on the boundary \((\Sigma, \bar{c})\).

\textbf{But} requiring \(g_+\) to be Einstein (even asymp. near \(\partial M_+)\) is \textbf{highly restrictive}. It means that the conformal manifold with boundary \((\overline{M}, c)\) has \(\Sigma = \partial M_+\) totally umbilic, Fialkow vanishes, etcetera.

Here we seek to set up the analogous program for \((\overline{M}, c)\) a \textbf{general manifold with boundary}.

Thus, given \((\overline{M} = M_+ \cup \partial M_+, c)\) we \textbf{need a way to determine a distinguished metric} \(g_+ \in c|_{M_+}\) on \(M_+\) so that \((M_+, g_+)\) is conformally compact.
Recall a **conformal compactification** of a complete Riemannian manifold $(M^{n+1}, g_+)$ is a manifold $\overline{M}$ with boundary $\partial M$ s.t.:

- $\exists \bar{g}$ on $\overline{M}$, with $g_+ = r^{-2}\bar{g}$, where $r$ a **defining function** for $\partial M$: $\partial M = \mathcal{Z}(r) \& \, dr_p \neq 0 \forall p \in \partial M$.

$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [\bar{g}|_{\partial M}])$.

**Question/variant:** Given $\bar{g}$ (or really $c = [\bar{g}]$) can we find a defining function $r \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$$\text{Sc}(r^{-2}\bar{g}) = -n(n+1)?$$

**NB:** This satisfied for Poincaré-Einstein cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.
Can we solve \( \text{Sc}(r^{-2} \bar{g}) = -n(n + 1) \)? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

**Theorem (Andersson, Chruściel, & Friedrich)**

\[
\text{Sc}(r^{-2} \bar{g}) = -n(n + 1) + r^{n+1} B_n.
\]

Furthermore (they show)

\[
B_2 = \delta \cdot \delta \cdot \hat{L} + \text{lower order}
\]

is a conformal invariant of \( \Sigma^2 = \partial M \).

**Theorem.** [G. + Waldron] For \( n \geq 2 \) \( B_n \) is a conformal invariant of \( \Sigma = \partial M \), and \( B_2 = \text{Willmore Invariant} = \overline{\Delta} + \text{lower order} \).

- For \( n \) even the invariant \( B_n \) is higher order analogue of \( B_2 = B \).

**NB.** The existence of such a higher analogue was not previously obvious as the weight and leading order of \( B_n \) means standard tractor/ambient metric approaches fail.
Recasting the problem and holography

Recall the constant scalar curvature condition in terms of scale. A conformal manifold has a canonical conformal metric \( g \in S^2 T^* M \). A metric \( g_+ \in c \) is equivalent to a scale:

\[
g_+ = \sigma^{-2} g \iff \sigma \in \Gamma(\mathcal{E}_+[1]).
\]

Via the Thomas-D operator \( \bar{D} = \frac{1}{n+1} D \) the scale is equivalent to the scale tractor \( I_A := \bar{D}_A \sigma \), and

**Lemma**

\( \text{Sc}(g_+) = -n(n+1) \iff l^2 := h(l, l) = 1 \)

So we come to a “conformal Eikonal equation” \( (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 \), where \( \sigma \) a defining density for \( \Sigma \). **NB:**

- If we could solve uniquely then \( \Sigma \hookrightarrow (M, c) \) determines \( g \in c \).

Then invariants of conf. compact \( (M, g_+) \) would be invariants of \( \Sigma \).
The conformal Eikonal equation

Thus to solve the singular Yamabe problem formally we come to the following **non-linear** problem:

**Problem:** For a conformal manifold \((M, c)\) and an embedding \(\iota: \Sigma \rightarrow M\) solve

\[
I_A I^A = (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 + O(\sigma^\ell)
\]

for \(\ell\) as high as possible, and \(\sigma\) a \(\Sigma\) defining density.

A key observation is that the **linearisation** of \(I^A I_A = 1\) is \(I^A D_A \dot{\sigma} = 0\) – the \(I \cdot D\) problem on \(\mathcal{E}[1]\). Thus \(\exists\) hope that the \(\mathfrak{sl}(2)\) generated by \(x := \sigma, y := -\frac{1}{t^2} I^A D_A\) will again be useful.

Recall from the standard \(\mathfrak{sl}(2)\) identities we have

\[
[l \cdot D, \sigma^{k+1}] = l^2 \sigma^k (k + 1)(n + k + 1 + 2w),
\]

and this allows an inductive solution (using also other tractor identities) that mimics the linear case!
Lemma

Suppose that \( \sigma \in \Gamma(E[1]) \) defines \( \Sigma = \partial M_+ \) in \( (\overline{M}, c) \) and
\[
l_\sigma^2 = 1 + \sigma^k A_k \quad \text{where} \quad A_k \in \Gamma(E[-k])
\]
is smooth on \( M \), and \( k \geq 1 \), then
- if \( k \neq (n + 1) \) then \( \exists \ f_k \in \Gamma(E[-k]) \) s.t. \( \sigma' := \sigma + \sigma^{k+1} f_k \) satisfies \( l_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1} \), where \( A_{k+1} \) smooth;
- if \( k = (n + 1) \) then: \( l_{\sigma'}^2 = l_\sigma^2 + O(\sigma^{n+2}) \).

Proof.

Squaring with the tractor metric, using the \( sl(2) \), etc
\[
(\tilde{D}\sigma')^2 = (\tilde{D}\sigma + \tilde{D}(\sigma^{k+1} f_k))^2
\]
\[
= l_\sigma^2 + \frac{2}{n+1} l_\sigma \cdot D(\sigma^{k+1} f_k) + (\tilde{D}(\sigma^{k+1} f_k))^2
\]
\[
= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1} (k+1)(n+1-k) f_k + O(\sigma^{k+1}).
\]

Rod Gover. background: G-, Waldon, arXiv:1506.02723, and Compactification and boundary calc
This applies formally off any hypersurface in a Riemannian conformal manifold \((M, c)\) (and even more generally) so we have:

**Theorem (G.-, Waldron arXiv:1506.02723)**

For \(\Sigma^n\) embedded in \((M^{n+1}, c)\) there is a distinguished defining density \(\bar{\sigma}\), unique modulo \(+O(\sigma^{n+2})\), s.t.

\[
I_\bar{\sigma}^2 = 1 + \bar{\sigma}^{n+1} B_{\bar{\sigma}}.
\]

Moreover:

\[
B := B_{\bar{\sigma}}|\Sigma \in \Gamma(E\Sigma[-n - 1])
\]

is determined by \((M, c, \Sigma)\) and is a natural conformal invariant.

For \(n\) even \(B = 0\) generalises the Willmore equation in that:

\[
B = \bar{\Delta}^n H + \text{lower order terms};
\]

while for \(n\) odd \(B\) has no linear leading term.
The construction can be used to obtain other submanifold invariants: Our Theorem above shows that:

\[ (M, c, \Sigma) \text{ determines } \bar{\sigma} \text{ modulo } + O(\sigma^{n+2}). \]

Suppose that \( \mathcal{I} \) is any coupled conformal invariant of \((M, c, \bar{\sigma})\) involving only the jet \( j^{n+1}\bar{\sigma} \). Then along \( \Sigma \)

\[ \mathcal{I}|_{\Sigma} \text{ is a conformal invariant of } (M, c, \Sigma). \]

This **holographic** approach fails at order \( n + 2 \) precisely because of the existence of the **obstruction invariant** \( B \). This is precisely an analogue of the use Fefferman-Graham’s Poincaré and ambient metric constructions to find conformal invariants – that fails at order \( n + 1 \) because of **Bach** \( B_{ab} \) in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.
Extrinsically coupled GJMS operators

Recall on any almost Riemannian manifold \((M, c, I)\) we had:

**Theorem**

Let \(\mathcal{E}^\Phi\) be any tractor bundle and \(k \in \mathbb{Z}_{\geq 1}\). Then, for each \(k \in \mathbb{Z}_{\geq 1}\), along \(\Sigma = \mathcal{Z}(\sigma)\)

\[
P^\sigma_k : \mathcal{E}^\Phi\left[\frac{k - n}{2}\right] \rightarrow \mathcal{E}^\Phi\left[\frac{-k - n}{2}\right]
\]

is a tangential differential operator, and so determines a canonical differential operator \(P^\sigma_k : \mathcal{E}^\Phi\left[\frac{k - n}{2}\right]|_\Sigma \rightarrow \mathcal{E}^\Phi\left[\frac{-k - n}{2}\right]|_\Sigma\). For \(k\) even this takes the form

\[
P_k = \Delta^k + \text{lower order terms}.
\]

Because \((M, c, \Sigma)\) determines \(\bar{\sigma}\) modulo \(O(\sigma^{n+2})\), we have:

**Theorem**

For \(k \leq n = d - 1\) the operators \(P_k\) are determined canonically by the data \((M, c, \Sigma)\).
Higher Willmore energies

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\text{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + A \log \epsilon + V_{\text{ren}} + O(\epsilon).$$

**Theorem (Graham 2016: arXiv:1606.00069)**

If $g_+ = \bar{\sigma}^{-2}g$, i.e. it is the approximate solution of the sing. Yamabe problem then $A$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta A}{\delta \Sigma} = \frac{d(d-2)}{2} \beta_n$$

So the anomaly term in the renormalised volume expansion provides an energy with functional gradient the obstruction density, in other words an energy generalising the Willmore energy.
Extrinsic $Q$-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

**Theorem (G.- Waldron arXiv:1603.07367)**

$$\mathcal{A} = \frac{1}{(d - 1)!(d - 2)!} \int_\Sigma Q$$

where, with $\tau \in \Gamma E_+[1]$ a scale giving the boundary metric, $Q := (-I \cdot D)^n \log \tau$.

- $Q$ here is an **extrinically coupled** $Q$-curvature meaning e.g.

  $$Q\hat{g}_\Sigma = e^{-nf}(Qg + P_nf) \quad \text{where} \quad \hat{g}_\Sigma = e^{2f}g_\Sigma$$

  and for $n$ even

  $$P_n = \Delta^\frac{n}{2}_\Sigma + \text{lower order terms}; \quad P_n \text{ FSA, and } P_n 1 = 0,$$

  is an **extrinically coupled** GJMS type operator. $Q$ and $P_n$ are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.
Idea of proof

Use a Heaviside function $\theta$ to “cut off” an integral over all $\overline{M}$

$$\text{Vol}_\epsilon = \int_{\overline{M}} \frac{dV g_\tau}{\sigma^d} \theta\left(\frac{\sigma}{\tau} - \epsilon\right).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{d-1-k}}{d\epsilon^{d-1-k}} \left(\epsilon^d \frac{d}{d\epsilon} \text{Vol}_\epsilon\right)\bigg|_{\epsilon=0},$$

So

$$v_k \sim \int_{\overline{M}} \frac{\delta^{d-1-k}(\sigma)}{\tau^k} \quad \text{and} \quad A \sim \int_{\overline{M}} \delta^{d-2}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_{\Sigma} \frac{1}{\tau^k} \quad \text{and} \quad A \sim \int_{\Sigma} (I \cdot D)^{d-1} \log \tau$$
Compactification and boundary calc
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