

# Independence of gauge-fixing in a higher-order BV formalism

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# Independence of gauge-fixing in a higher-order BV formalism

- 1 Higher-Order BV Formalism
- 2 Independence of Gauge-Fixing

# $\Delta$ Operator

Diff. Op.

$$\Delta(\hbar, z, \partial)$$

$$\partial_A \equiv \frac{\partial}{\partial z^A}$$

$$[\partial_A, z^B] = \delta_A^B$$

Nilpotent

$$\Delta^2 = 0$$

Grassmann-odd

$$\epsilon(\Delta) = 1$$

## Planck number grading

$$\text{PI}(\hbar) = 1$$

$$\text{PI}(z) = 0$$

$$\text{PI}(\partial) = \begin{cases} 0 & \text{if } \partial \text{ acts inside } \Delta \\ -1 & \text{if } \partial \text{ acts outside } \Delta \end{cases}$$

### Super Additivity

$$\text{PI}(FG) \geq \text{PI}(F) + \text{PI}(G)$$

$$\text{PI}([F, G]) \geq \text{PI}(F) + \text{PI}(G) + 1$$

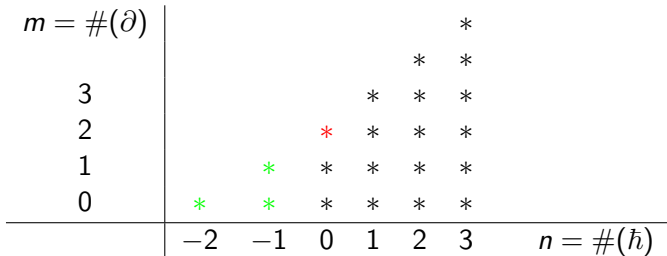
### Triangular form

$$\text{PI}(\Delta) \geq -2$$

## Higher-Order $\Delta$ Operator, triangular form

$$\Delta = \sum_{n=-2}^{\infty} \sum_{m=0}^{n+2} \left(\frac{\hbar}{i}\right)^n \Delta_{n,m}$$

$$\Delta_{n,m} = \Delta_{n,m}^{A_1 \dots A_m}(z) \vec{\partial}_{A_m} \dots \vec{\partial}_{A_1}$$



$\Delta_{0,2}$  original standard 2nd-order BV op.

# Motivation

1. Closed SFT, higher antibrackets,  $L_\infty, \dots$
- 2.

Operator  
Formalism



Path Int.  
Formalism

## Example: (Oversimplified) Point particle in curved space

### Op. Formalism

$$\langle x_f | \exp \left\{ -\frac{i}{\hbar} \hat{H} \Delta t \right\} | x_i \rangle = \langle x_f, t_f | x_i, t_i \rangle$$

### Path Int. Formalism

$\sim$

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} [dx] [dp] \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left( p_A \dot{x}^A - H_{\text{cl}} \right) \right\}$$



## Hamiltonian: Point particle in curved space

Operator

DeWitt (1957)

$$\hat{H} = -\frac{\hbar^2}{2} \left( \Delta_{\sqrt{g}} - \frac{R}{4} \right) + \mathcal{O}(\hbar^3)$$

Op

Beltrami  
Laplacian

Scalar  
Curvature

$\longleftrightarrow$

Function

$$H_{\text{cl}} = \frac{1}{2} p_A g^{AB} p_B$$

Fct

# $W$ - $X$ Formalism

## Path Int./Partition Fct.

$$Z_X = \int d\mu \quad \overset{1}{w} \quad \overset{2}{x}$$

### 1. Path Int. Measure

$$d\mu = \rho[dz][d\lambda]$$

## Fields & Antifields

$$z^A = \{\phi^\alpha, \phi_\alpha^*\}$$

## Lagr. Mult. for Gauge-Fixing

$$\lambda^\alpha$$

### 2. Gauge-generating QME

$$w \equiv e^{\frac{i}{\hbar}W} \quad (\Delta w) = 0 \quad \text{PI}(W) \geq 0$$

### 3. Gauge-Fixing QME

$$x \equiv e^{\frac{i}{\hbar}X} \quad (\Delta^T x) = 0 \quad \text{PI}(X) > 0$$

## Transposed Operator (Int. by parts)

Transposed Operator  $F^T$

$$\int d\mu (F^T f) g = (-1)^{fF} \int d\mu f (Fg)$$

$$(F + G)^T = F^T + G^T$$

$$(z^A)^T = z^A$$

$$(FG)^T = (-1)^{FG} G^T F^T$$

$$(\partial_A)^T = -\rho^{-1} \vec{\partial}_{A\rho}$$

## Transposed Operator (Int. by parts)

Affine Leibniz rule

$$\partial_A^T(fg) = (\partial_A^T f)g - (-1)^{A_f} f(\partial_A g)$$

Transposed  $\Delta$  operator also nilpotent

$$(\Delta^T)^2 = 0$$

## Original 2nd-Order BV (Int. out $\lambda$ & $\phi^*$ )

Path int. measure density

$$\rho = 1$$

Darboux Coord.

$$\Delta = (-1)^\alpha \frac{\delta}{\delta\phi^\alpha} \frac{\delta}{\delta\phi_\alpha^*} = \Delta^T$$

Gauge-Fixing Action

$$X = \left( \phi_\alpha^* - \frac{\delta\psi}{\delta\phi^\alpha} \right) \lambda^\alpha$$

CME

$$(X, X) = 0$$

$\mathcal{O}(\hbar)$

$$(\Delta^T X) = 0$$

QME

$$(\Delta^T \exp \left\{ \frac{i}{\hbar} X \right\}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}(X, X) = i\hbar(\Delta^T X)$$

# Original 2nd-Order BV

(1981)

Path Int./Partition Fct.

$$Z_\psi = \int [d\phi] \exp \left\{ \frac{i}{\hbar} W(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}) \right\}$$

QME

$$(\Delta \exp \left\{ \frac{i}{\hbar} W \right\}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}(W, W) = i\hbar(\Delta W)$$

# Independence of gauge-fixing in a higher-order BV formalism

- 1 Higher-Order BV Formalism
- 2 Independence of Gauge-Fixing

## Finite Deformation of Solution to QME

$$\begin{aligned}
 x &\longrightarrow x' = \left( e^{[\Delta^T, \Psi]} x \right) \\
 &= x + \left( [\Delta^T, \Psi] x \right) + \frac{1}{2} \left( [\Delta^T, \Psi] [\Delta^T, \Psi] x \right) + \dots
 \end{aligned}$$

$$(\Delta^T x) = 0 \longrightarrow (\Delta^T x') = 0$$

Deformation generating operator

$$\Psi(\hbar, z, \partial) \qquad \text{Pl}(\Psi) \geq 0 \qquad \varepsilon(\Psi) = 1$$



## Infinitesimal Deformation of Solution to QME

$\Delta^T$ -closed

$$(\Delta^T \delta x) = 0$$

$\Delta^T$ -exact

$$\delta x = ([\Delta^T, \Psi]x) = (\Delta^T \Psi x)$$

Assumption

No  $\Delta^T$ -cohomology in pertinent sector

## Independence of Gauge-Fixing: Int. by parts

Int. by parts

Batalin, Damgaard & KB (1996)

$$\begin{aligned}\delta Z &\equiv Z_{X+\delta X} - Z_X = \int d\mu w \delta x \\ &= \int d\mu w (\Delta^T \Psi_X) = \int d\mu (\Delta w) (\Psi_X) = 0\end{aligned}$$

Question

Can we construct proof using change of int. var. instead?

# Homotopy operator $\overset{\rightarrow A}{h}(\Delta)$

## $\Delta$ operator

$$\Delta(\partial, z) = \sum_{m=0}^{\infty} \Delta_m(\partial, z)$$

## Anti-normal order

$$\Delta_m(\partial, z) = \overset{\rightarrow}{\partial}_{A_m} \cdots \overset{\rightarrow}{\partial}_{A_1} \Delta_m^{A_1 \cdots A_m}(z)$$

## Homotopy operator

$$\overset{\rightarrow A}{h}(\Delta_m) := \begin{cases} -\frac{1}{m}[z^A, \Delta_m] & \text{for } m \geq 1, \\ 0 & \text{for } m = 0. \end{cases}$$

$\rightsquigarrow$  Extended by linearity

## Homotopy Formula

$$(-1)^A \overset{\rightarrow}{\partial}_A \overset{\rightarrow}{h} (\Delta(\partial, z)) = \Delta(\partial, z) - \Delta(0, z)$$

$$\overset{\rightarrow}{h} \overset{A \rightarrow B}{h} (\Delta) = (-1)^{AB} \overset{\rightarrow}{h} \overset{B \rightarrow A}{h} (\Delta)$$

## Bilinear Homotopy Operator $B^A(f, \Delta)$

$$\begin{aligned}
 (-1)^{Af} B^A(f, \Delta) &= f : \frac{1}{1 - \overleftarrow{\partial}_B \overrightarrow{h}} : \overrightarrow{h}^A(\Delta) 1 \\
 &\equiv f : \sum_{n=0}^{\infty} \left( \overleftarrow{\partial}_B \overrightarrow{h} \right)^n : \overrightarrow{h}^A(\Delta) 1 \\
 &= f \underbrace{\overrightarrow{h}^A(\Delta) 1}_{n=0} + \underbrace{(f \overleftarrow{\partial}_B) \overrightarrow{h} \overrightarrow{h}^A(\Delta) 1}_{n=1} \\
 &\quad + \underbrace{(f \overleftarrow{\partial}_B \overleftarrow{\partial}_C) \overrightarrow{h} \overrightarrow{h} \overrightarrow{h}^A(\Delta) 1}_{n=2} + \dots
 \end{aligned}$$

## Bilinear Homotopy Formula

$$(-1)^A \left( \partial_A^T B^A(f, \Delta) \right) = (-1)^{f\Delta} (\Delta^T f) - f(\Delta 1)$$

## Infinitesimal change of Int. Variables

$$\delta z^A = \frac{1}{w_X} B^A(\Psi_X, \vec{\Delta} w)$$

$\Psi$  infinitesimal & Grassmann-odd op.

## Divergence

$$\begin{aligned} \operatorname{div}_{\rho w x} \delta z &= \frac{(-1)^A}{\rho w x} (\vec{\partial}_{A \rho x w} \delta z^A) = -\frac{(-1)^A}{w x} (\vec{\partial}_A^T B^A(\Psi_X, \vec{\Delta} w)) \\ &= \frac{1}{w x} \left\{ (\Psi_X) \underbrace{(\Delta w)}_{=0} + w(\Delta^T \Psi_X) \right\} = \frac{\delta x}{x} \end{aligned}$$



# Independence of Gauge-Fixing

Change of Int. Variables  $z'^A = z^A + \delta z^A$

$$\begin{aligned} 0 &= \int [d\lambda][dz'] \rho[z'] w[z'] x[z'] - \int [d\lambda][dz] \rho[z] w[z] x[z] \\ &= \int [d\lambda][dz] (-1)^A (\vec{\partial}_A \rho x w \delta z^A) = \int d\mu w x \operatorname{div}_{\rho x w} \delta z \\ &= \int d\mu w \delta x = Z_{x+\delta x} - Z_x \equiv \delta Z \end{aligned}$$

## Conclusions

- The formal higher-order BV path int/partition fct  $Z_X$  does not depend on the gauge-fixing condition  $X$ .
- Proof either via int. by parts or change of int. variables.
- It is possible to generalize to BRST/anti-BRST  $Sp(2)$ -symmetric theories.
- Recently, finite BRST transformations have been considered by Batalin, Lavrov, Tyutin & KB.

## Classical 2nd-order case (from now on)

### $\Delta$ Operator

$$\Delta = \Delta_\rho + V + \nu$$

Odd Laplacian + Odd vector field + Odd scalar curvature

### Nilpotent

$$\Delta^2 = 0$$

### Transposed

$$\Delta^T = \Delta_\rho - V + \nu$$

# QME

$$\frac{1}{2}(W, W) + \frac{\hbar}{i} ((\Delta_\rho + V) W) + \left(\frac{\hbar}{i}\right)^2 \nu = 0$$

$$\frac{1}{2}(X, X) + \frac{\hbar}{i} ((\Delta_\rho - V) X) + \left(\frac{\hbar}{i}\right)^2 \nu = 0$$

## Q BRST

$$\sigma_W f = (W, f) + \frac{\hbar}{i} ((\Delta_\rho + V) f)$$

$$\sigma_X f = (X, f) + \frac{\hbar}{i} ((\Delta_\rho - V) f)$$

$$\sigma_W^2 = 0$$

$$\sigma_X^2 = 0$$

## Infinitesimal Change of Int. Var.

$$2 \delta z^A = \frac{i}{\hbar} \psi (\sigma_W z^A - \sigma_X z^A) - (\psi, z^A)$$

$\psi$  infinitesimal & Grassmann-odd fct.

## The Odd Scalar $\nu_\rho$

### Odd Scalar in Antisymplectic Geometry

(KB 2006)

$$\nu_\rho := \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}$$

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### Terms built from $E$ and $\rho$

$$\nu_\rho^{(0)} := \frac{1}{\sqrt{\rho}}(\Delta_1 \sqrt{\rho})$$



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$$\nu^{(1)} := (-1)^{\varepsilon_A} \left( \frac{\overrightarrow{\partial}^\ell}{\partial z^A} E^{AB} \frac{\overleftarrow{\partial}^r}{\partial z^B} \right) (-1)^{\varepsilon_B}$$

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$$\nu^{(2)} := -(-1)^{\varepsilon_B} \left( \frac{\overrightarrow{\partial}^\ell}{\partial z^A} E_{BC} \right) (z^C, (z^B, z^A))$$

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$$\nu^{(2)} := -(-1)^{\varepsilon_B} \left( \frac{\overrightarrow{\partial}^\ell}{\partial z^A} E_{BC} \right) (z^C, (z^B, z^A))$$

$$= (-1)^{\varepsilon_A \varepsilon_D} \left( \frac{\overrightarrow{\partial}^\ell}{\partial z^D} E^{AB} \right) E_{BC} \left( E^{CD} \frac{\overleftarrow{\partial}^r}{\partial z^A} \right)$$

## Odd scalar curvature

- $\nu_\rho$  is constructed from antisymplectic structure  $E^{AB}$  and  $\rho$ .
- $\nu_\rho$  transforms as a scalar.
- $\nu_\rho = 0$  can be viewed as a compatibility condition between  $E^{AB}$  and  $\rho$ .
- $\nu_\rho = -\frac{R}{8}$  is an odd scalar curvature for any tangent space connection, that is
  - 1 torsionfree,
  - 2 antisymplectic,
  - 3 compatible with  $\rho$ .

## References

- Igor A. Batalin & KB, *Gauge Independence in a Higher-Order Lagrangian Formalism via Change of Variables in the Path Integral*, Phys. Lett. **B742** (2015) 23; arXiv:1408.5121; DOI:10.1016/j.physletb.2015.01.005
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