Weyl vs. Conformal Invariance in QFT

arXiv:1702.07079

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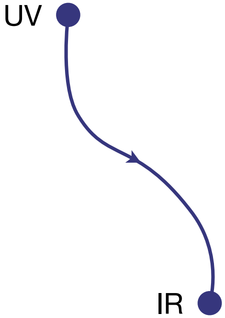
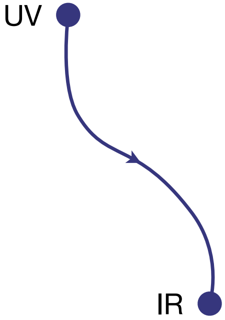
Renormalization Group Fixed Points



- high energy limit UV complete QFT
- low energy limit any QFT
- $\beta(\lambda) = \dfrac{\partial \lambda}{\partial \ln \mu} = 0$

Renormalization Group Fixed Points



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scale

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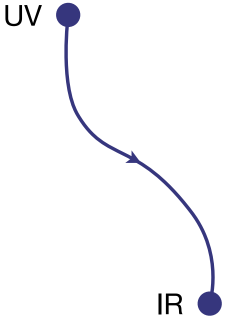

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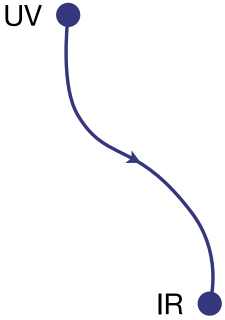

- high energy limit UV complete QFT
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scale

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Weyl

Scale vs. Conformal vs. Weyl

- scale
- conformal
- Weyl

proven in $d = 2$

perturbative proof, non-perturbative arguments in $d = 4$

perturbative proof in $d = 6$

Scale vs. Conformal vs. Weyl

- scale

proven in $d = 2$

perturbative proof, non-perturbative arguments in $d = 4$

perturbative proof in $d = 6$

- conformal

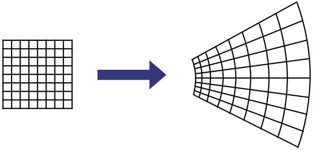
proof in $d = 2$

nothing in higher dimensions (until now)

- Weyl

Conformal vs. Weyl Transformations

Conformal



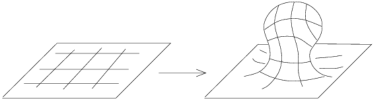
coordinate transformation

$$x \to f(x)$$

$$g_{\mu\nu} \to e^{2\hat{\sigma}(x)} g_{\mu\nu}$$

$$\nabla_\mu \nabla_\nu \hat{\sigma} = 0$$

Weyl



metric transformation[1]

$$g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$$

$$\sigma = \text{arbitrary}$$

[1] taken from D. Tong, arXiv:0908.0333

Invariance

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad \delta S = \int d^d x \sqrt{-g}\, \sigma\, T$$

Conformal	Weyl
$\nabla_\mu \nabla_\nu \hat{\sigma} = 0$	$\sigma = \text{arbitrary}$
$\Downarrow$	$\Downarrow$
$T = \nabla_\mu \nabla_\nu L^{\mu\nu}$	$T = 0$

Invariance

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad \delta S = \int d^d x \sqrt{-g}\, \sigma\, T$$

Conformal

Weyl

$$\nabla_\mu \nabla_\nu \hat{\sigma} = 0 \Rightarrow T = \nabla_\mu \nabla_\nu L^{\mu\nu}$$

$$\sigma = \text{arbitrary} \Rightarrow T = 0$$

Improvement:

$$\Delta S = \int d^d x \sqrt{-g}\left(\xi R L + \xi' R_{\mu\nu} L^{\mu\nu}\right)$$

Invariance

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad \delta S = \int d^d x \sqrt{-g}\, \sigma\, T$$

Conformal

$$\nabla_\mu \nabla_\nu \hat{\sigma} = 0$$

$T = 0$ (flat space)

Weyl

$\sigma =$ arbitrary

$T = 0$ (curved space)

Invariance

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad \delta S = \int d^d x \sqrt{-g}\,\sigma\, T$$

Conformal

$$\nabla_\mu \nabla_\nu \hat{\sigma} = 0$$

$$\Downarrow$$

$\boxed{T = 0 \text{ (flat space)}}$

Weyl

$$\sigma = \text{arbitrary}$$

$$\Downarrow$$

$\boxed{T = 0 \text{ (curved space)}}$

?

Counterexamples (Karananas, Monin arXiv:1510.08042)

$$\mathcal{L} = \frac{1}{2}\phi\Box^k\phi$$

- always conformal in flat space
- cannot be coupled to $g_{\mu\nu}$ in Weyl inv't way for even d, $d < 2k$
  - $k = 1 \rightarrow$ always okay
  - $k = 2 \rightarrow$ doesn't work for $d = 2$
  - $k = 3 \rightarrow$ doesn't work for $d = 2,\ 4$

Counterexamples (Karananas, Monin arXiv:1510.08042)

$$\mathcal{L} = \frac{1}{2}\phi\Box^k\phi$$

- always conformal in flat space
- cannot be coupled to $g_{\mu\nu}$ in Weyl inv't way for even d, $d < 2k$

Unitarity bounds on operator dimensions:

$$\Delta_\phi \geq \frac{d-2}{2} \quad \text{vs.} \quad \Delta_\phi = \frac{d-2k}{2}$$

- models violate unitarity unless $k = 1$

Method

- $T = 0$ in flat space
- $T = ?$ in curved space $\rightarrow$ rule out
  - $\propto R, R_{\mu\nu}, \cdots$
  - unitarity: $\Delta \geq \Delta_{min}$

$$T = RX + R_{\mu\nu}Y^{\mu\nu} + \nabla^{\mu}RZ_{\mu} + \cdots$$

$$\Delta_X \geq \frac{d-2}{2}, \qquad \Delta_Y \geq d \text{ or } d-2, \qquad \Delta_Z \geq d-1$$

Method

$$
\begin{aligned}
T = \ & RX && d < 6 \\
& + R\Box Y_1 + R^{\mu\nu}\nabla_\mu\nabla_\nu Y_2 + \nabla^\mu R \nabla_\mu Y_3 + \Box R Y_4 && \\
& \quad + R^2 Y_5 + R^2_{\mu\nu} Y_6 + R^2_{\mu\nu\rho\sigma} Y_7 && d < 10 \\
& + \cdots && d \geq 10
\end{aligned}
$$

$X, \ Y_i$ are primary scalars

$d < 6$

$$T = RX$$

- can define X using: $\Delta S = \int d^d x \sqrt{-g}\rho_X X$
- $T = RX$: operator statement
  - actually redundancy in definition of sources

$$\left(\delta g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \delta\rho_X \frac{\delta}{\delta \rho_X}\right) W_{eff}\left[g_{\mu\nu}, \rho_X\right] = 0$$

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad \delta\rho_X = \sigma R$$

$$\Rightarrow T = RX$$

$d < 6$

- if: $$\delta_\sigma W_{eff} \equiv \left(\delta g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \delta\rho_X \frac{\delta}{\delta\rho_X}\right) W_{eff} = 0$$
- then we should also have

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, W_{eff} = 0$$

- we get: $$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, g_{\mu\nu} = 0$$ $$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, \rho_X = 2(d-1)\underbrace{(\sigma_1\Box\sigma_2 - \sigma_2\Box\sigma_1)}_{\text{arbitrary}}$$

$\Rightarrow X = 0$

$T = 0 \rightarrow$ Weyl Invariant! $(d < 6)$

Conclusions

- shown for $d < 6$:
  - conformal in flat space + unitarity = Weyl in curved space
- in paper: go up to $d = 10$
- in paper: look at operator transformations
  - similar arguments: generally transform canonically
  - can have “anomalous” transformations:

$$\delta_\sigma \mathcal{O} = -\sigma \Delta \mathcal{O} + \sigma W^2_{\mu\nu\rho\sigma} A$$

  - future: rule these out

Thank you!