Recent developments on conformal symmetry breaking operators

joint work with Bent Ørsted and Petr Somberg, available on ArXiv: https://arxiv.org/abs/1711.01546

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The title

We present a new way of defining conformal symmetry breaking differential operators

$$D_N(\lambda): C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}),$$

which are the residues of

Conformal symmetry breaking operators

=

2-**parameter families of integral operators**

$$\begin{aligned} A^{\pm}_{\lambda,\nu}(x', x_n) &: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}) \\ (A^{\pm}_{\lambda,\nu}(x', x_n) f)(y') &:= (K^{\pm}_{\lambda,\nu}(x', x_n) * f)(y', 0) \end{aligned}$$

for any $f \in C^\infty(\mathbb{R}^n)$.

As application we find a recursive structure for $D_N(\lambda)$.

1. Introduction
2. Notation and set-up
3. Results
4. Further comments

My way to that subject

In summer 2016 I visited P. Somberg in Prague. He asked me about the origin of a (non-constant coefficient) second-order differential operator $Q(\lambda)$, [PS15], shifting the polynomial degree of Gegenbauer polynomials $C_N(\lambda)$ up.

He got no help from me, BUT he got my attension.

Generating all Gegenbauer polynomials out of the 1 was a facinating fact to me, because Gegenbauer polynomials correspond via F-method to conformal symmetry breaking differential operators $D_N(\lambda)$, [J09, KØSS15, KS15], (CSBDO) which I was studying in those times.

CSBO correspond to compositions of Fourier transformed $Q(\lambda)$'s.

My way to that subject

Back in Aarhus:

$$C_N(\lambda) \simeq \text{residues of a hypergeoemtric function } {}_2F_1$$

$$K^{\pm}_{\lambda,\nu} \simeq \text{Fourier transform of } {}_2F_1$$

Here $K^{\pm}_{\lambda,\nu}$ are distributional kernels studied [KS15]. Those kernels correspond to conformal symmetry breaking operators (integral operators with residues given by $D_N(\lambda)$). This is how Bent Ørsted came on board, as expert on integral operators.
WANTED: Extend the action of the Fourier transform of $Q(\lambda)$ to $K^{\pm}_{\lambda,\nu}$. With success:

$$Q(\lambda)C_N(\lambda) \simeq C_{N+1}(\lambda) \to Q^{\mathcal{F}}(\lambda)K^{\pm}_{\lambda,\nu} \simeq K^{\mp}_{\lambda,\nu+1}$$

My way to that subject

BUT:

- $Q^{\mathcal{F}}(\lambda)$ is a third-order differential operator with non-constant coefficients. It has no conceptional definition, since it was found by hand.
- Bent asked: Can we find the meromorphic extension of $K^{\pm}_{\lambda,\nu}$ which is holomorphically well-defined with respect to $\lambda, \nu \in \mathbb{C}$ such that $\Re(\lambda - \nu) > 0$ and $\Re(\lambda + \nu) > n - 1$. **Answer: NO**

So let us do something else, but similar:

Distributional kernels

We consider $\mathbb{R}^n$ with the Euclidian metric. A point $x \in \mathbb{R}^n$ is decomposed as $x = (x', x_n)$ for $x' \in \mathbb{R}^{n-1}$. We define

$$K^+_{\lambda,\nu}(x', x_n) := |x_n|^{\lambda+\nu-n}(|x'|^2 + x_n^2)^{-\nu},$$
$$K^-_{\lambda,\nu}(x', x_n) := x_n K^+_{\lambda-1,\nu}(x', x_n).$$

These are the distributional kernels which by convolution and restriction to $x_n = 0$ define conformal symmetry operators. Their residues are the conformal symmetry breaking operators

$$D_N(\lambda) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}).$$

They are polynomial of degree N in Δ' and ∂_n, and coefficients are given by Gegenbauer coefficients.

Why to study such operators?

- They are the flat version of so-called residue families $D_N^{res}(h; \lambda)$. A family of differential operators attached to any Riemannian manifold (M, h), [J09]
- Residue families carry at special family-parameter the GJMS-operators $P_{2N}(h)$, [GJMS92], on (M, h).
- Residue families recover Branson's Q-curvature, [B93].
- Last but not least residue families lead to the holographic Laplacian, [J13], which itself has a very interesting spectral theory.

The shift operator

We define a non-constant coefficient differential operator

$$P(\lambda) := x_n\Delta - (2\lambda - n - 2)\partial_n$$

when acting on $C^\infty(\mathbb{R}^n)$. It is of second-order for $x_n \neq 0$ while it is of first order for $x_n = 0$.

This operator is not new at all:

- $x_n P(\lambda)$ is, up to an additive constant, a Casimir operator, [MØZ16].
- $P(\lambda)$ correspond to an operator introduced by Graham-Zworski, [GZ01].
- $P(\lambda)$ corresponds to a version of degenerate Laplacian, [GW15].
- $P(\lambda)$ is the conjugation of $x_n\cdot$ by appropriate Knapp-Stein intertwining operators on $\mathbb{R}^n$, [C17].

The shift property

The operators $P(\lambda)$ and $M_{x_n} := x_n\cdot$ shift the λ-parameter in the distributional kernels $K^{\pm}_{\lambda,\nu}(x',x_n)$ and change the $\pm$-parity, i.e.,

$$P(\lambda)K^{\pm}_{\lambda,\nu}(x',x_n) = (\lambda+\nu-n)(\nu-\lambda+1)K^{\mp}_{\lambda-1,\nu}(x',x_n),$$
$$M_{x_n}K^{\pm}_{\lambda,\nu}(x',x_n) = K^{\mp}_{\lambda+1,\nu}(x',x_n).$$

This results fits the philosophy of Gelfand-Bernstein-Sato; They studied the problem of finding the meromorphic continuation of a given distribution defined in some region in the complex plane.

CSBDO vs. shift-family

We introduce the family

$$P_N(\lambda) := P(\lambda - N + 1) \circ \cdots \circ P(\lambda)$$

of differential operators on $C^\infty(\mathbb{R}^n)$. This a of order $2N$ away from $x_n = 0$ and of order N along $x_n = 0$. Then it holds:

$$P_N(\lambda)_{|x_n=0} \simeq D_N(n - \lambda).$$

This gives a new structural formula for CSBDO's.

Recursive structure for CSBDO

By definition and previous result it holds

$$D_N(n-\lambda) \simeq D_{N-1}(n-\lambda+1) \circ P(\lambda).$$

The spinor and differential form case

The operator $P(\lambda)$ has analogs in the spinor and differential form case with applications to conformal symmetry breaking operators for spinors and differential forms.

The curved case

We extend the notion of shift operator $P(\lambda)$ to the curved setting. It has impact into new structural results concerning residue families. Furthermore, it allows to study a set of second-order differential operators attached to any Riemannian manifold. Those second-order operators are the coefficients of the holographic Laplacian introduced in [J13].

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