# The wrapped Fukaya category of a Weinstein manifold is generated by the cocores of the critical Weinstein handles

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(joint with B. Chantraine, G. Dimitroglou Rizell, and P. Ghiggini, arXiv:1712.09126)

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- There is a version of Fukaya categories for symplectic manifolds cylindrical at infinity. They are called wrapped Fukaya categories $\mathcal{W}Fuk(W)$. They were introduced by Abouzaid and Seidel, and $\mathcal{W}Fuk(M)$ replaces $\mathcal{F}uk(M)$ in the homological mirror symmetry conjecture for such manifolds.

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- $f$ has critical points of degree $\leq n$. Handles which correspond to the critical points of index $< n$ are called *subcritical*, and the handles which correspond to the critical points of index $n$ are called *critical*.

# Cores, cocores and Lagrangian skeleton

- We can present $W$ as a union of a subcritical domain $W^{subcrit}$, which consists of subcritical handles, and a union of critical handles $H_i$, i.e. $W = W^{subcrit} \cup H_1 \cup \cdots \cup H_m$.

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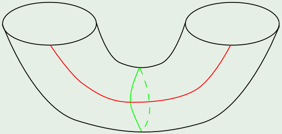
**Example $(T^*S^1, pdq)$**

$$W^{sk} = C = 0_{S^1}$$

$$D = T^*_q S^1$$



# Main Result

**Theorem (Chantraine-Dimlitroglou Rizell-Ghiggini-G)**

*If $(W, \theta, f)$ is a 2n-dimensional Weinstein manifold of finite type, then its wrapped Fukaya category $\mathcal{W}Fuk(W, \theta)$ is generated by the Lagrangian cocore planes of the index n critical points of $f$.*

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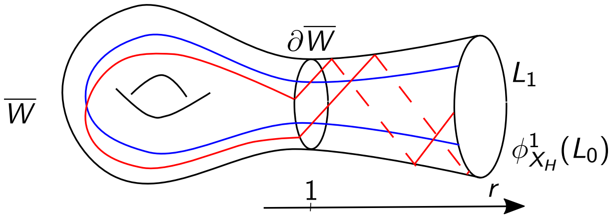
- Ganatra-Pardon-Shende, announced strategy to get split-generation using localization properties of arboreal Lagrangian skeleta.

- Consider a Weinstein manifold $(W,\theta)$ and two exact Lagrangian submanifolds $L_0$ and $L_1$ such that $L_i|_{(\partial\overline{W}\times(1,\infty),d(e^r\alpha))} = \Lambda_i \times (1,\infty)$, $\Lambda_i$ is a Legendrian submanifold of $(\partial\overline{W},\alpha := \theta|_{\partial\overline{W}})$, $i = 0,1$. $L_i$s are called exact Lagrangian fillings.
- Take a Hamiltonian function $H \in C^\infty([0,1]\times W,\mathbb{R})$ which leads to the Hamiltonian vector field defined by $i_{X_H}d\theta = dH$ and $H = r^2$ on $\partial\overline{W}\times(1,\infty)$, where $r$ is a coordinate on $(1,\infty)$.



# Wrapped Floer theory II

- $CW(L_0, L_1)$ is a $\mathbb{Z}_2$-vector space generated by the time 1 trajectories of the flow of $X_H$ which start on $L_0$ and end on $L_1$. Take two such trajectories $\gamma_-, \gamma_+$.

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- Define $\mathcal{M}^J(\gamma_+, \gamma_-)$. Consider $u : \mathbb{R} \times [0,1] \to W$, where $s$ is a coordinate on $\mathbb{R}$ and $t$ is a coordinate on $[0,1]$, such that
  - $\frac{\partial u}{\partial s} + J(t,u)(\frac{\partial u}{\partial t} - X_H(t,u)) = 0$,
  - $u(s,0) \in L_0$, $u(s,1) \in L_1$,
  - $E(u) = \int u^* d\theta < \infty$,
  - when $s \to \pm\infty$, the strip is asymptotic to $\gamma_\pm$.

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- Define $< \partial\gamma_+, \gamma_- >= \#\mathcal{M}^J(\gamma_+, \gamma_-)/\mathbb{R}$ for $\dim \mathcal{M}^J(\gamma_+, \gamma_-) = 1$.
- $H(CW(L_0, L_1), \partial)$ is called wrapped Floer homology, and it is a Hamiltonian isotopy invariant.

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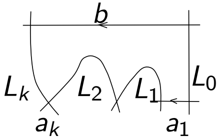
Wrapped Fukaya category is an $\mathcal{A}_\infty$ category, where $\mu_k : CW(L_{k-1}, L_k) \otimes \cdots \otimes CW(L_0, L_1) \to CW(L_0, L_k)$ such that $< \mu_k(a_k \otimes \cdots \otimes a_1), b >$ is given by the count of pseudoholomorphic curves



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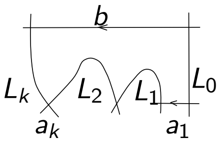


Recall the $\mathcal{A}_\infty$ equations:

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^{j+deg(p_1)+\cdots+deg(p_j)} \mu_{k+1-l}(a_k \otimes \cdots \otimes a_{j+l+1} \otimes$$

$$\mu_l(a_{j+l} \otimes \cdots \otimes a_{j+1}) \otimes a_j \otimes \cdots \otimes a_1) = 0$$

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## Definition

- The objects $G_1, \ldots, G_r$ are said to generate the $\mathcal{A}_\infty$ -category $A$ if, in $Tw(A)$, every object of $A$ is quasi-isomorphic to a twisted complex built from copies of $G_1, \ldots, G_r$. ( In other terms, every object of $A$ can be obtained from $G_1, \ldots, G_r$ by taking iterated mapping cones.)
- The objects $G_1, \ldots, G_r$ are said to split-generate $A$ if every object of $A$ is quasi-isomorphic to a direct summand in a twisted complex built from copies of $G_1, ..., G_r$.

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- Each resolution/surgery corresponds to taking the mapping cone. After all resolutions (that should be done carefully following certain iterative procedure) we get Floer homologically trivial object, and then the result follows from homological algebra.

As a corollary, we get the following result, which is a conjecture of Paul Seidel.

**Corollary**

*The open-closed map*

$$\mathcal{OC}\colon HH_*(\mathcal{W}F(W,\theta),\mathcal{W}F(W,\theta)) \to SH^*(W)$$

*is an isomorphism. Here $HH_*$ denotes Hochschild homology and $SH^*$ denotes symplectic cohomology.*

Thank you for your attention!