Models of 2–nondegenerate CR manifolds

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$(\mathfrak{g}_{-} = \cdots \oplus \mathfrak{g}_{-i} \oplus \cdots \oplus \mathfrak{g}_{-1}, I)$ a non–degenerate CR algebra
(non–degenerate part of the symbol)
$(\mathfrak{k}, I)$... a complex vector space (degenerate part of the symbol)

Second order symbol – Idea

We want to add more algebraic data to the symbol $(\mathfrak{g}_- \oplus \mathfrak{k}, I)$. The properties $[\mathcal{K}, \mathcal{D}] \subset \mathcal{D}$, $[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}$ suggest to map $\mathfrak{k}$ inside $\mathfrak{gl}(\mathfrak{g}_{-1})$.

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The conjugation by I defines an involutive automorphism of the Lie algebra $\mathfrak{gl}(\mathfrak{g}_{-1})$.

$$\mathfrak{gl}(\mathfrak{g}_{-1})^+ := \{X \in \mathfrak{gl}(\mathfrak{g}_{-1}) | X(I(Y)) = I(X(Y)) \text{ for all Y} \in \mathfrak{g}_{-1}\}$$

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The infinitesimal CR automorphisms vanishing at x live in $\mathfrak{gl}(\mathfrak{g}_{-1})^+$. This suggests to map $\mathfrak{k}$ inside $\mathfrak{gl}(\mathfrak{g}_{-1})^-$.

Second order symbol – Realization

Observation (Freeman series):
The bracket of vector fields of $\mathcal{D} \otimes \mathbb{C} = \mathcal{D}^{10} \oplus \mathcal{D}^{01}$ and $\mathcal{K} \otimes \mathbb{C} = \mathcal{K}^{10} \oplus \mathcal{K}^{01}$ has tensorial parts:

$$l_2 : \mathcal{K}^{10} \otimes \mathcal{D}^{01}/\mathcal{K}^{01} \to \mathcal{D}^{10}/\mathcal{K}^{10},\ l_{-2} : \mathcal{K}^{01} \otimes \mathcal{D}^{10}/\mathcal{K}^{10} \to \mathcal{D}^{01}/\mathcal{K}^{01}$$

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$$\mathrm{ad}(X)(Y) := \frac{1}{8}(l_2(X - iI(X), Y + iI(Y)) + l_{-2}(X + iI(X), Y - iI(Y)))$$

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Theorem

The map ad *is a well–define map* $\mathfrak{k} \to \mathfrak{gl}(\mathfrak{g}_{-1})^-$ *that satisfies*

$$\mathrm{ad}(I(X))(Y) = I(\mathrm{ad}(X)(Y)),\ [\mathrm{ad}(X)(Y), Z] + [Y, \mathrm{ad}(X)(Z)] = 0$$

for all $X \in \mathfrak{k}$ *and* $Y, Z \in \mathfrak{g}_{-1}$.

Second order symbol

Definition

We call the triple $(\mathfrak{g}_-, I, \mathrm{ad}(\mathfrak{k}))$ the second order symbol of the CR manifold $(M, \mathcal{D}, I)$ at the regular point x.
We say that $(M, \mathcal{D}, I)$ is 2–nondegenerate at x if the map ad is injective.

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We say that the second order symbol is weakly regular if $\mathrm{ad}(\mathfrak{k}) \subset \mathfrak{der}(\mathfrak{g}_-)$.
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holds in $\mathfrak{der}(\mathfrak{g}_-)$.

If $\mathfrak{g}_{-3} = 0$, then the symbol is always weakly regular, otherwise, the symbol does not have to be weakly regular.

Characterization of all possible (weakly) regular second order symbols

Definition

Let $(\mathfrak{g}_-, I)$ be a non–degenerate fundamental CR algebra. Suppose the subspace $\mathfrak{h}$ of $\mathfrak{der}(\mathfrak{g}_-)^-$ consisting of complex anti–linear grading preserving derivations of $\mathfrak{g}_-$ that are acting trivially on $\mathfrak{g}_{-2}$ is non–trivial, i.e., $0 \neq \mathfrak{h} \subset \mathfrak{der}(\mathfrak{g}_-)^-$. Suppose $\mathfrak{k} \subset \mathfrak{h}$ is preserved by the left multiplication by I. Then we say that a triple $(\mathfrak{g}_-, I, \mathfrak{k})$ is a weakly regular fundamental CR algebra of second order. If $[[\mathfrak{k}, \mathfrak{k}], \mathfrak{k}] \subset \mathfrak{k}$ holds in $\mathfrak{der}(\mathfrak{g}_-)$ then we call it a regular fundamental CR algebra of second order.

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Theorem

Each (weakly) regular second order symbol is a (weakly) regular fundamental CR algebra of second order. Each (weakly) regular fundamental CR algebra of second order is a (weakly) regular second order symbol of a 2–nondegenerate CR manifold.

Construction of models in the regular case

Conjugation by I restricts to an involutive automorphism of the Lie algebra $\mathfrak{l} := \mathfrak{k} \oplus [\mathfrak{k}, \mathfrak{k}]$. There is the corresponding (complex) symmetric space L/L^+ for $L \subset Aut(\mathfrak{g}_-)$.

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Theorem

The triple $(\exp(\mathfrak{g}_-) \rtimes L/L^+, \exp(\mathfrak{g}_-) \rtimes L \times_{L^+} (\mathfrak{g}_{-1} \oplus \mathfrak{k}), I)$ *is a homogeneous 2–nondegenerate CR manifold with the regular second order symbol isomorphic to* $(\mathfrak{g}_-, I, \mathfrak{k})$ *at every point.*

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There is a local embedding $\phi : \mathfrak{g}_- \oplus \mathfrak{k} \to (\mathfrak{g}_- \otimes \mathbb{C})/\mathfrak{g}_{-1}^{01} \oplus \mathfrak{k}^{10}$ given by

$$\phi(X+Y) = \exp^{-1}(\exp(X)\exp(\frac{1}{2}(Y - iI(Y)))\exp(-\frac{1}{2}(X_{-1} + iI(X_{-1}))))$$

for $X \in \mathfrak{g}_-$ and $Y \in \mathfrak{k}$. This provides polynomial/rational defining equations that are homogeneous w.r.t. the weighted order.

Construction of models in the weakly regular case

If $[[\mathfrak{k},\mathfrak{k}],\mathfrak{k}] \not\subset \mathfrak{k}$, then one would need to modify the Lie brackets (by some curvature/torsion) to get a homogeneous model.
Instead consider $\tilde{\mathfrak{k}} := \mathfrak{k} + [[\mathfrak{k},\mathfrak{k}],\mathfrak{k}] + \dots$ and the corresponding regular fundamental CR algebra of second order $(\mathfrak{g}_-, I, \tilde{\mathfrak{k}})$ with model $(\exp(\mathfrak{g}_-) \rtimes L/L^+, \exp(\mathfrak{g}_-) \rtimes L \times_{L^+} (\mathfrak{g}_{-1} \oplus \tilde{\mathfrak{k}}), I)$.

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The CR submanifold of the manifold $\exp(\mathfrak{g}_-) \rtimes L/L^+$ given by points $\exp(X)\exp(Y)L^+$ for all $X \in \mathfrak{g}_{-1}$ and all $Y \in \mathfrak{k}$ (for exp in L/L^+) is a 2–nondegenerate CR manifold with (weakly) regular second order symbol isomorphic to $(\mathfrak{g}_-, I, \mathfrak{k})$ at every point.

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Theorem

Each second order symbol is a fundamental CR algebra of second order. Each fundamental CR algebra of second order is a second order symbol of a 2–nondegenerate CR manifold.

Construction of models in general case

There is a "free" fundamental CR algebra of second order of the form $(\mathfrak{f}_- = \cdots \oplus \mathfrak{f}_{-i} \oplus \cdots \oplus \mathfrak{f}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, I)$. Therefore $(\mathfrak{f}_-, I, \mathfrak{k})$ is a weakly regular second order symbol. This symbol has polynomial model provided by the (local) embedding
$\phi : \mathfrak{f}_- \oplus \mathfrak{k} \to (\mathfrak{f}_- \otimes \mathbb{C})/\mathfrak{g}_{-1}^{01} \oplus \mathfrak{k}^{10}$ given by
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for $X \in \mathfrak{f}_-$ and $Y \in \mathfrak{k}$.

There is an (graded) ideal $\mathfrak{q} \subset \mathfrak{f}_-$ such that $\mathfrak{f}_-/\mathfrak{q} = \mathfrak{g}_-$. For any complement $\mathfrak{m}$ of $\mathfrak{q}$ in $\mathfrak{f}_-$, there is a 2–non–degenerate CR submanifold $\phi(\mathfrak{m} \oplus \mathfrak{k})$ in $(\mathfrak{q} \otimes \mathbb{C})\backslash(\mathfrak{f}_- \otimes \mathbb{C})/\mathfrak{g}^{01}_{-1} \oplus \mathfrak{k}^{10}$.
In general, $\phi(\mathfrak{m} \oplus \mathfrak{k})$ depends on the choice of $\mathfrak{m}$ and only the second order symbol at 0 is isomorphic to $(\mathfrak{g}_-, I, \mathfrak{k})$.