Properads and Homotopy algebras related to surfaces

Joint work with M. Doubek and B. Jurčo

Lada Peksová

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Generalization of operads

Category of directed corollas: DCor := Cor $\times$ Cor :

- the objects are pairs (C, D) of finite sets (the outputs and inputs)
- morphism $(\rho, \sigma) : (C, D) \to (C', D')$ is a pair of bijections $\rho : C \xrightarrow{\sim} C'$, $\sigma : D \xrightarrow{\sim} D'$.

Properad $\mathcal{P}$ consists of

- collection $\{\mathcal{P}(C,D) \mid (C,D) \in \mathsf{DCor}\}$ of dg vector spaces
- two collections of degree 0 morphisms

$$\{\mathcal{P}(\rho,\sigma) : \mathcal{P}(C,D) \to \mathcal{P}(C',D') \mid (\rho,\sigma) : (C,D) \to (C',D')\}$$

$$\{\, {}_B\overset{\eta}{\circ}{}_A : \mathcal{P}(C_1, D_1 \sqcup B) \otimes \mathcal{P}(C_2 \sqcup A, D_2) \to \mathcal{P}(C_1 \sqcup C_2, D_1 \sqcup D_2) \mid \eta : B \xrightarrow{\sim} A\}$$

satisfying the following axioms:

1. (Σ-bimodule)
$\mathcal{P}((1_C, 1_D)) = 1_{\mathcal{P}(C,D)}, \quad \mathcal{P}((\rho\rho', \sigma'\sigma)) = \mathcal{P}((\rho,\sigma))\,\mathcal{P}((\rho',\sigma'))$
2. (equivariance)
$(\mathcal{P}((\rho_1 \sqcup \rho_2|_{C_2}, \sigma_1|_{D_1} \sqcup \sigma_2))\ {}_B\overset{\eta}{\circ}{}_A = {}_{\sigma_1(B)}\overset{\rho_2\eta\sigma_1^{-1}}{\circ}{}_{\rho_2(A)}$
$(\mathcal{P}((\rho_1,\sigma_1)) \otimes \mathcal{P}((\rho_2,\sigma_2))$
3. (associativity) ${}_B\overset{\eta}{\circ}{}_A\ (1 \otimes {}_{B'}\overset{\eta'}{\circ}{}_{A'}) = {}_{B'}\overset{\eta'}{\circ}{}_{A'}\ ({}_B\overset{\eta}{\circ}{}_A \otimes 1)$

Additional grading by $\mathbb{N}_0$ - **genus** or the **Euler characteristic**

$$\chi = 2G + |C| + |D| - 2$$

$\Longrightarrow$ components $\mathcal{P}(C, D, \chi)$

We assume only **stable components**, i.e. $\chi > 0$

$$G = 0 \quad \Longrightarrow \quad |C| + |D| \geq 3$$
$$G = 1 \quad \Longrightarrow \quad |C| + |D| \geq 1$$

Example: (Closed) Frobenius properad $\mathcal{F}$:

$(C, D) \in$ DCor and $\chi > 0$ put $\mathcal{F}(C, D, \chi) = \mathbb{k}$
$\Longrightarrow$ has trivial differential and Σ-structure
$\Longrightarrow$ ${}_B\overset{\eta}{\circ}_A$ do not depend on sets A, B

Directed graph

- consists of vertices and half-edges
- two half-edges composing one edge have corresponding orientation
- no directed circuits

We can assign to the graph G a non-negative integer

$$G := \dim_{\mathbb{Q}} H_1(\mathsf{G}, \mathbb{Q}) + \sum_i G_i$$

The stable graph satisfies for every vertex V_i

$$2(G_i - 1) + |C_i| + |D_i| > 0$$

Cobar complex of properad $\mathcal{P}$

- Elements are iso class of G with "decoration" by element

$$(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n)$$

  where V_i are vertices of G and $P_i \in \mathcal{P}(C_i, D_i, G_i)^{\#}$

- The structure maps $({}_B\overset{\eta}{\circ}{}_A)_{C\mathcal{P}}$ work as grafting together $|A|$ pairs of half-edges with suitable orientation
- The differential $\partial_{C\mathcal{P}}$ on one vertex is given as a sum

$$\partial_{C\mathcal{P}} = d_{P\#} \otimes 1 + \sum_{\substack{C_1 \sqcup C_2 = C \\ D_1 \sqcup D_2 = D \\ 1 \leq |A| \leq G+1 \\ G_1 + G_2 + |A| - 1 = G \\ \eta}} \frac{1}{|A|!} ({}^{(C_1, D_1 \sqcup B, G_1)}{}_B\overset{\eta}{\circ}{}_A{}^{(C_2 \sqcup A, D_2, G_2)})_P^{\#} \otimes (\uparrow V \wedge \cdot)$$

For (V,d) dg vector space, $(C,D) \in$ DCor, $\chi > 0$ define

$$\mathcal{E}_V(C,D,\chi) := \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V)$$

where $\bigodot_D V$ denotes an unordered tensor product.

The Σ-**actions** and **compositions** - easily seen from canonical isomorphism

$$\bigodot_{c' \in C'} V_{c'} \otimes \bigodot_{c'' \in C''} V_{c''} \cong \bigodot_{c \in C' \sqcup C''} V_c$$

For $\bar{f} \in \mathrm{Hom}_{\Bbbk}(\bigotimes_D V, \bigotimes_C)$ corresponding to $f \in \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V)$

$$d(\bar{f}) = \sum_{i=0}^{m-1}(1^{\otimes i} \otimes d \otimes 1^{\otimes m-i-1})\bar{f} - (-1)^{|\bar{f}|}\sum_{i=0}^{n-1}\bar{f}(1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1})$$

The collection $\mathcal{E}_V = \{\mathcal{E}_V(C,D,\chi) | (C,D) \in \mathrm{DCor}, \chi > 0\}$ is an **endomorphism properad**

Algebra over properad for dg vector space V is a properad morphism $\alpha : \mathcal{P} \to \mathcal{E}_V$, i.e. it is a collection

$$\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi) \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathsf{DCor}, \chi > 0\}$$

satisfying

1. $\alpha \circ \mathcal{P}(\rho,\sigma) = \mathcal{E}_V(\rho,\sigma) \circ \alpha$
2. $\alpha \circ ({}_B\overset{\eta}{\circ}_A)_{\mathcal{P}} = ({}_B\overset{\eta}{\circ}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha)$

Theorem: Algebra over the cobar complex $C\mathcal{P}$

Algebra over $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection $\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi)^{\#} \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathsf{DCor}, \chi > 0\}$ of deg 1 linear maps s.t.

$$\mathcal{E}_V(\rho,\sigma) \circ \alpha(C,D,\chi) = \alpha(C',D',\chi) \circ \mathcal{P}(\rho^{-1},\sigma^{-1})^{\#}$$

$$d \circ \alpha = \alpha \circ d_{\mathcal{P}\#} + \sum \frac{1}{|A|!}({}_B\overset{\eta}{\circ}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha) \circ ({}_B\overset{\eta}{\circ}_A)^{\#}_{\mathcal{P}}$$

For simplicity assume $C = [m], D = [n]$

By isomorphism

$$\mathrm{Hom}_{\Sigma_m \times \Sigma_n}(\mathcal{P}([m],[n],\chi)^{\#}, \mathcal{E}_V([m],[n],\chi)) \xrightarrow{\cong} {}^{\Sigma_m}(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi))^{\Sigma_n}$$

$$\alpha \mapsto \sum_i p_i \otimes \alpha(p_i^{\#})$$

where $\{p_i\}$ is a basis of $\mathcal{P}([m],[n],\chi)$, we can rewrite algebra over $C\mathcal{P}$ as element

$$L \in \prod_{\substack{m,n \\ \chi > 0}} \kappa^{\chi}\, {}^{\Sigma_m}(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi))^{\Sigma_n}$$

satisfying **Master equation** $d(L) + L \circ L = 0$ with differential

$$d = d_{\mathcal{P}} \otimes 1_{\mathcal{E}_V} - 1_{\mathcal{P}} \otimes d_{\mathcal{E}_V}$$

But the invariants are isomorphic to coinvariants (with respect to the diagonal $\Sigma_m \times \Sigma_n$ action)

$$
{}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n} \cong \mathcal{P}([m],[n],\chi)\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} V^{\otimes m} \otimes ((V^{\#})^{\otimes n})
$$

Put $f_{p_i} := \alpha(p_i^{\#})$ and pick basis $\{a_i\}$ of V and the dual basis $\{\phi^i\}$.

$$
\sum_i p_i \otimes \alpha(p_i^{\#}) \mapsto \frac{1}{m!n!} \sum_{i,I,J} f^J_{p_i I}(p_i\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (a_J \otimes \phi^I))
$$

We get an isomorphism

$$
\prod_{\substack{m,n\\ \chi>0}} \kappa^{\chi}\, {}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n} \cong
$$

$$
\prod_{m,n,\chi} \kappa^{\chi} \left(\mathcal{P}([m],[n],\chi)\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (V^{\otimes m} \otimes (V^{\#})^{\otimes n}\right)
$$

The "transfered" differential is

$$\tilde{d}\left(p\,{}_{\Sigma_m}\otimes_{\Sigma_n}(a_J\otimes\phi^I)\right)=$$
$$= d_{\mathcal{P}}(p)\,{}_{\Sigma_m}\otimes_{\Sigma_n}(a_J\otimes\phi^I)-(-1)^{|p|}p\,{}_{\Sigma_m}\otimes_{\Sigma_n}d_{\mathcal{E}_V}(a_J\otimes\phi^I)$$

and the composition is

$$\left(p_1\,{}_{\Sigma_{m_1}}\otimes_{\Sigma_{n_1}}(a_{J_1}\otimes\phi^{I_1})\right)\tilde{\circ}\left(p_2\,{}_{\Sigma_{m_2}}\otimes_{\Sigma_{n_2}}(a_{J_2}\otimes\phi^{I_2})\right)=$$
$$=\sum_{M,N,\xi}\left(\left({}_N\overset{\xi}{\circ}{}_M\right)_{\mathcal{P}}(p_1\otimes p_2)\right)\,{}_{\Sigma_{m_1+m_2-|M|}}\otimes_{\Sigma_{n_1+n_2-|M|}}$$
$$\left(\left({}_N\overset{\xi}{\circ}{}_M\right)_{\mathcal{E}_V}(a_{J_1}\otimes\phi^{I_1})\otimes(a_{J_2}\otimes\phi^{I_2})\right)$$

If $m, n \geq 1$ we can introduce positional derivations

$$\frac{\partial^{(k)}}{\partial a_j}\left(a_{i_1}\dots a_{i_{m_2}}\right) = (-1)^{|a_j|(|a_{i_1}|+\dots|a_{i_{k-1}}|)}\,\delta_j^{i_k}\left(a_{i_1}\dots\widehat{a_{i_k}}\dots a_{i_{m_2}}\right)$$

for sets $J = \{j_1, \dots j_{|N|}\}$ and $K = \{k_1, \dots k_{|N|}\}$

$$\frac{\partial^{(K)}}{\partial a_J} = \frac{\partial^{(k_1)}}{\partial a_{j_1}}\dots\frac{\partial^{(k_{|N|})}}{\partial a_{j_{|N|}}}.$$

And interpret the "inputs" as partial derivations acting on "outputs"

$$p_1\,{}_{\Sigma_{m_1}}\otimes_{\Sigma_{n_1}}\left(a_{J_1}\otimes\phi^{I_1}\right) : p_2\,{}_{\Sigma_{m_2}}\otimes a_{J_2} \mapsto$$
$$\pm\sum_{M,N,\xi}\frac{\partial^{\xi(N)}}{\partial a_N}\left(a_M\right)\left({}_N\overset{\xi}{\circ}{}_M\right)_{\mathcal{P}}(p_1\otimes p_2)\,{}_{\Sigma_{m_1+m_2-|M|}}\otimes_{\Sigma_{n_1-|M|}}a_{J_1}a_{J_2-M}$$

Thank you for your attention!