Kontsevich integral for a non-even associator and cabling

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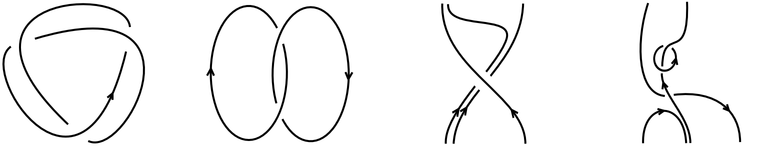
Joint work with Pavol Ševera

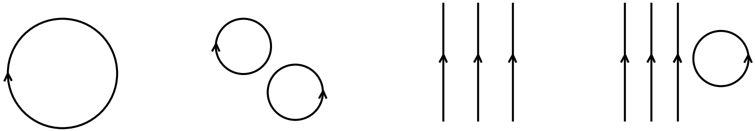
Kontsevich integral

Invariant of:



knots links braids tangles

valued in the vector space of *chord diagrams*. Chord is a pair of points on a 1-dim manifold, the *support*.

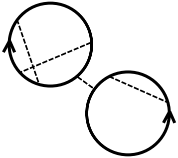


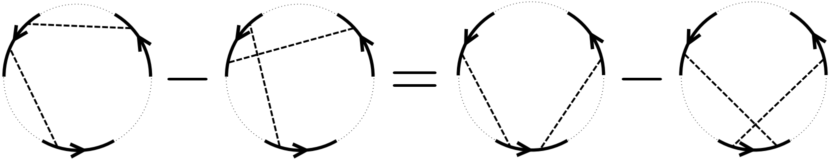
$\Longrightarrow$ support is a disjoint union of oriented circles S^1 and intervals I.

Chord diagrams

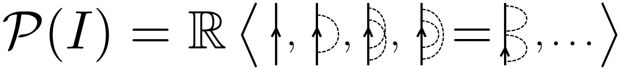
$\mathcal{P}(X)$ contains linear combination of *chord diagrams on X*:

- chord connects two points on X
- for n chords, all $2n$ points are distinct
- graded by the number of chords
- quotient by diffeomorphisms and the 4-term relation:





For X an interval:



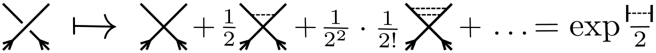
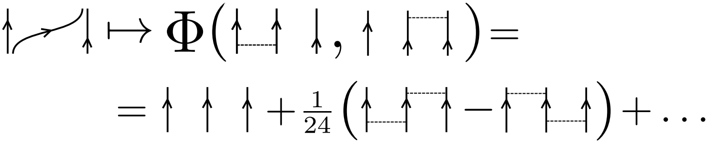
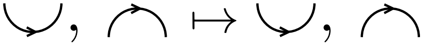
How to compute $\hat{Z}(T)$

à la Le, Murakami

Main ingredient is the **Drinfeld associator** Φ: element of free associative algebra $\mathbb{C}\langle\langle X, Y\rangle\rangle$ satisfying some consistency equations.

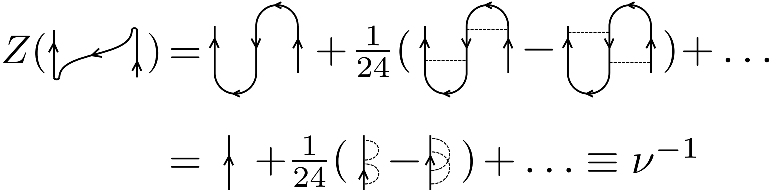
$$\Phi(X,Y) = 1 + \frac{1}{24}(XY - YX) + \dots$$

Unnormalized Kontsevich integral Z:

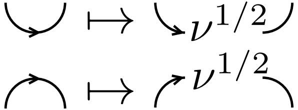
- 
- 
- No normalization: 

Normalization of $\cap$, $\cup$

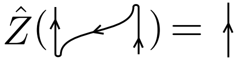
Why normalize?



Normalized $\hat{Z}$ is defined as before, but

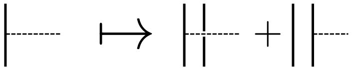


which implies



Last thing we need: $\hat{Z}($$)$

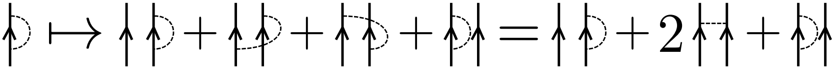
On a doubled component, use a *Leibniz rule*:

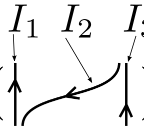


For k chord endpoints on a component $C \subset X$, sum over 2^k possible combinations.

$$\boxed{\text{This gives a map } \Delta_C : \mathcal{P}(X) \to \mathcal{P}(X^{(2,C)}).}$$

For example, $\Delta : \mathcal{P}(I) \to \mathcal{P}(I \sqcup I)$ acts as



At last: $\hat{Z}($$) = \Delta_{I_2}\Delta^2_{I_3}\, \hat{Z}($

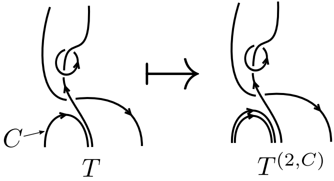
$)$

Recapitulation

- $Z, \hat{Z}$ is a map: tangles $\to$ chord diagrams $\mathcal{P}(T)$.
- Defined by values on $\times$, $\cap$, $\cup$, $|\diagup|$ and then stacked to form a tangle.
- For multiple strands in associator, use $\Delta_C : \mathcal{P}(X) \to \mathcal{P}(X^{(2,C)})$.

Cabling

We can double components of tangles as well:



$$\boxed{\textbf{Question}\text{: Is it true that } \hat{Z}(T^{(2,C)}) = \Delta_C \hat{Z}(T)}$$

Question: Is it true that $\hat{Z}(T^{(2,C)}) = \Delta_C \hat{Z}(T)$

- For $\vert\diagup\vert$, true by definition
- For $\times$, true (the *hexagon identity* for the associator)
- Is it true for $\cap$, i.e. $\hat{Z}(\Cap) = \Delta\hat{Z}(\cap) = \Delta\nu^{1/2}$

Theorem (Le & Murakami, 95)

$\hat{Z}(\Cap) = \Delta\nu^{1/2} = \hat{Z}(\Cup)$ if the associator is even, i.e. if it contains only terms with even numbers of chords $\Phi(X,Y) = \Phi(-X,-Y)$

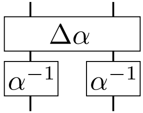
Thus, $\hat{Z}(T^{(2,C)}) = \Delta_C \hat{Z}(T)$. For knots and links, this is true for any associator, as $\hat{Z}$ does not depend on this associator.

Can we make this work for any associator? (already posed by Le & Murakami)

Cabling for any associator

- For general associator $\hat{Z}(\Cap) \neq \hat{Z}(\Cup) \implies$ normalize cap and cup differently
- Let us define $\hat{Z}(\cap) = \alpha$, $\hat{Z}(\cup) = \beta$, both in $\mathcal{P}(I)$
- We still need $\alpha\beta = \nu$: the snake equation $\hat{Z}($$) =$ 
- The other equation is $\hat{Z}(\Cap) = \Delta\alpha \iff Z(\Cap) = (\alpha^{-1} \otimes \alpha^{-1})\Delta\alpha$

$$(\alpha^{-1} \otimes \alpha^{-1})\Delta\alpha =$$



- Similar equation for $\Cup$ would follow from the previous two.

Proposition (J.P.)

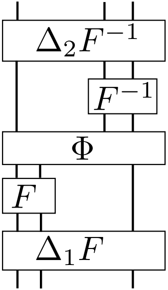
Such α and β do not exist.

Proof by *twisting* of associator Φ.

Associator twists

Let us take (symmetric, invertible) $F \in \mathcal{P}(I \sqcup I)$.

Then, for an associator Φ, define Φ^F as:



This Φ^F is again an associator (satisfies all the necessary equations).
Simple observation: if $F_t = (\alpha^{-1} \otimes \alpha^{-1})\Delta\alpha$, then $\Phi^{F_t} = \Phi$.

(the opposite implication: a theorem by Drinfeld, 1990)

$\implies$ twist Φ by $Z(\Cap)$.

Recall that we want to find α such that $Z(\Cap) = (\alpha^{-1} \otimes \alpha^{-1})\Delta\alpha$

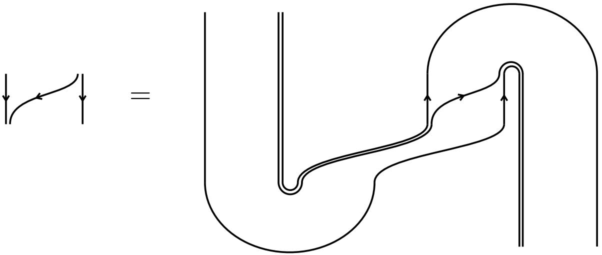
Proposition (J.P.)

The associator $\Phi(X,Y)$ *twisted by* $Z(\Cap)$ *is equal to*

$$\Phi(X,Y)^{Z(\Cap)} = \Phi(-X,-Y)\,.$$

Thus, if $\Phi(X,Y) \neq \Phi(-X,-Y)$*, there exists no* α *such that* $Z(\Cap) = (\alpha^{-1} \otimes \alpha^{-1})\Delta\alpha$*.*

Proof: use the *siphon identity*

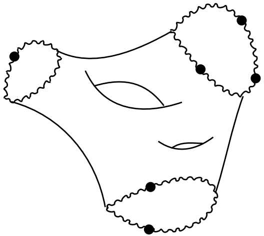


Moduli space of flat connections

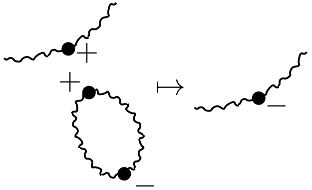
with Pavol Ševera

Moduli space of flat connections on a surface *with marked points*: gauge transformations are trivial in the points.

Algebra of functions: $\mathfrak{g}$-quasi-Poisson algebra.



Li-Bland & Ševera, 2013: Quantization of this moduli space: algebra in the category $U\mathfrak{g}{-}\mathrm{mod}^{\Phi}$, where $\Phi(X,Y)$ specifies the associative structure.



- minus: algebra in $U\mathfrak{g}{-}\mathrm{mod}^{\Phi(-X,-Y)}$.
- fusion (as in picture): monoidal functor $U\mathfrak{g}{-}\mathrm{mod}^{\Phi(X,Y)} \to U\mathfrak{g}{-}\mathrm{mod}^{\Phi(-X,-Y)}$.
- monoidal structure given by $Z(\Cap)$.