On the cup-length of the oriented Grassmann manifold and the characteristic rank of its canonical bundle

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Introduction

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We may suppose that $k \leq n-k$ for both of them.

Introduction

Definition

For a path connected space X, we may define its $\mathbb{Z}_2$–cup-length $\mathrm{cup}_{\mathbb{Z}_2}(X)$ as the greatest number r such that there exist cohomology classes $x_1, \ldots, x_r \in H^*(X; \mathbb{Z}_2)$ in positive dimensions such that $x_1 \cdots x_r \neq 0$.

Since we are interested only in cohomology with $\mathbb{Z}_2$ coefficients, we will abbreviate $H^j(X; \mathbb{Z}_2)$ to $H^j(X)$ and $\mathrm{cup}_{\mathbb{Z}_2}(X)$ to $\mathrm{cup}(X)$ henceforth.

Cohomology ring of $G_{n,k}$

The cohomology ring of the Grassmann manifold $G_{n,k}$ is

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, w_2, \dots, w_k]/I_{n,k},$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \dots, \bar{w}_n$, where each $\bar{w}_i$ denotes the i–dimensional component of the formal power series

$$1+(w_1+w_2+\cdots+w_k)+(w_1+w_2+\cdots+w_k)^2+(w_1+w_2+\cdots+w_k)^3+\cdots .$$

Each indeterminate w_i is a representative of the ith Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical k–plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

Introduction

However, for the oriented Grassmann manifold $\widetilde{G}_{n,k}$ the cohomology ring is in general unknown, apart from cases $k = 1$ (spheres) and $k = 2$.
For $k = 3$ there is following conjecture[1]

Fukaya's conjecture

For $t \geq 3$

$$\mathrm{cup}(\widetilde{G}_{n,3}) = \begin{cases} 2^t - 3, & \textit{if } 2^t - 1 \leq n \leq 2^t + 2^{t-1} - 3, \\ 2^t - 3 + s, & \textit{if } n = 2^t + 2^{t-1} - 2 + s, 0 \leq s \leq 2, \\ 2^t + 2^{t-1} + \dots & \textit{if } n = 2^t + 2^{t-1} + \cdots + 2^j + 1 + s, \\ +2^{j+1} + 2^{j-1} + s, & 0 \leq s \leq 2^{j-1} - 1 \end{cases}$$

[1] T. Fukaya: *Gröbner bases of oriented Grassmann manifolds*, HHA **10** (2008), 195-209.

Characteristic rank

The notion of characteristic rank of a manifold was introduced by Korbaš[2] and later generalized to the characteristic rank of a vector bundle by Naolekar and Thakur[3]

Definition

Let X be a connected, finite CW–complex and ξ a vector bundle over X. The *characteristic rank* of the vector bundle ξ, $\mathrm{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

[2] J. Korbaš: *The cup-length of the oriented Grassmannians vs a new bound for zero-cobordant manifolds*, Bull. Belg. Math. Soc., **17**, 2010, 69-81.

[3] A. C. Naolekar, A. S. Thakur: *Note on the characteristic rank of vector bundles*, Math. Slovaca **64** (2014), 1525-1540.

Characteristic rank

Theorem

Let X be a connected closed smooth d-manifold and let ξ be a vector bundle over X, such that there exists j, $j \leq \mathrm{charrank}(\xi)$, such that every monomial $w_{i_1}(\xi)\dots w_{i_r}(\xi)$ for $0 \leq i_t \leq j$ of degree d is zero. Then

$$\mathrm{cup}(X) \leq 1 + \frac{d-j-1}{r_X},$$

where r_X is the smallest positive integer, such that $\widetilde{H}^{r_X}(X;\mathbb{Z}_2) \neq 0$.

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Oriented Grassmann manifold $\widetilde{G}_{n,k}$ and its canonical bundle $\widetilde{\gamma}_{n,k}$ satisfy the assumptions of the theorem with $j = \mathrm{charrank}(\widetilde{\gamma}_{n,k})$ and $r_X = 2$.

Characteristic rank of $\widetilde{\gamma}_{n,k}$

There is a covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n,k) \neq (2,1)$. To this 2–fold covering, there is an associated line bundle ξ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k}) \to$$

where $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel–Whitney class $w_1(\xi) = w_1(\gamma_{n,k})$.

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where $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel–Whitney class $w_1(\xi) = w_1(\gamma_{n,k})$.
The image $\mathrm{Im}(p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$–vector space $H^j(\widetilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel–Whitney characteristic classes of $\widetilde{\gamma}_{n,k}$.

Characteristic rank of $\widetilde{\gamma}_{n,k}$

This implies that the characteristic rank of $\widetilde{\gamma}_{n,k}$ is equal to the greatest integer q, such that the homomorphism $p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k})$ is surjective for all j, $0 \le j \le q$, or equivalently, that the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective for all j, $0 \le j \le q$.

Homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$

Recall that

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/(\bar{w}_{n-k+1}, \ldots, \bar{w}_n).$$

Since in the quotient ring there are no relations in dimensions $n-k$ or lower, we always have $\mathrm{charrank}(\widetilde{\gamma}_{n,k}) \geq n-k-1$.

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Let $g_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ be the reduction of the polynomial $\bar{w}_i$ modulo w_1.
Denote $g_i(\gamma_{n,k})$ the corresponding cohomology class in $H^i(G_{n,k})$.

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Denote $g_i(\gamma_{n,k})$ the corresponding cohomology class in $H^i(G_{n,k})$.
For $i \in \{n-k+1, n-k+2, \ldots, n\}$ the cohomology classes $g_i(\gamma_{n,k})$ lie in the image of $w_1 : H^{i-1}(G_{n,k}) \longrightarrow H^i(G_{n,k})$.

Homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$

By estimating the dimension of the image of the homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ we obtain the following proposition.[4]

Proposition

For a non-negative integer x, we associate with $H^{n-k+x+1}(G_{n,k})$ $(2 \leq k \leq n-k)$ the set

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}.$$

If $x \leq n-k-1$ and the set $N_x(G_{n,k})$ is linearly independent, then

$$w_1 \colon H^{n-k+x}(G_{n,k}) \longrightarrow H^{n-k+x+1}(G_{n,k})$$

is a monomorphism.

[4] J. Korbaš, T. Rusin: *On the cohomology of oriented Grassmann manifolds.* Homology, Homotopy Appl., vol.18(2), 2016, 71–84.

The polynomials g_i for the Grassmann manifold $G_{n,3}$

From now on, we will consider the case $k = 3$ and the polynomials $g_i \in \mathbb{Z}_2[w_2, w_3]$ associated with $G_{n,3}$.
We have

$$\begin{aligned} g_1 &= 0, \\ g_2 &= w_2, \\ g_3 &= w_3, \\ g_4 &= w_2^2, \\ g_5 &= 0, \\ &\vdots \\ g_i &= w_2 g_{i-2} + w_3 g_{i-3} \end{aligned}$$

The polynomials g_i for the Grassmann manifold $G_{n,3}$

The recurrence formula can be generalized since we are working over $\mathbb{Z}_2$.

Recurrence formula

$g_i = w_2^{2^s} g_{i-2.2^s} + w_3^{2^s} g_{i-3.2^s}$

This leads to following lemma.[5]

Lemma

For the polynomials g_i associated with $G_{n,3}$ we have $g_i = 0$ iff $i = 2^t - 3$ for some $t \geq 2$.

[5] J. Korbaš: *The characteristic rank and cup-length in oriented Grassmann manifolds*, Osaka J. Math. **52** (2015), 1163-1172.

Some properties of the polynomials g_i

Lemma

Let $t \geq 3$. Then $g_{2^t-4} = g_{2^{t-1}-2}^2$.

Lemma

For $t \geq 3$ the polynomials g_{2^t-2} and $g_{2^{t-1}-2}$ are coprime.

Lemma

For $t \geq 2$ the polynomials g_{2^t-1} and g_{2^t-2} are coprime.

The results

For $t \geq 4$ we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \begin{cases} \geq 2^t - 5, & \text{if } 2^{t-1} + 2^{t-2} \leq n \leq 2^t - 4, \quad ^1 \\ = 2^t - 5, & \text{if } 2^t - 3 \leq n \leq 2^t - 1, \quad ^2 \\ = 2^t - 2, & \text{if } n = 2^t, \quad ^2 \\ = 2^t + 1, & \text{if } n = 2^t + 1, \quad ^3 \\ = 2^t + 4, & \text{if } n = 2^t + 2, \quad ^3 \\ = 2^t + 7, & \text{if } n = 2^t + 3, \quad ^3 \\ \geq n + 4, & \text{if } 2^t + 4 \leq n < 2^t + 2^{t-1}, \quad ^3 \end{cases}$$

[1] T. Rusin: *A note on the characteristic rank of oriented Grassmann manifolds*. Topol. Appl. 216 (2017), 48-58.

[2] J. Korbaš: *The characteristic rank and cup-length in oriented Grassmann manifolds*, Osaka J. Math. **52** (2015), 1163-1172.

[3] J. Korbaš, T. Rusin: *On the cohomology of oriented Grassmann manifolds*. Homology, Homotopy Appl., vol.18(2), 2016, 71–84.

Conclusion

As a consequence, for the cup-length we have the following.

Theorem

For $t \geq 4$ we have

$$\begin{aligned} \mathrm{cup}(\widetilde{G}_{2^t+2^{t-1}+1,3}) &= 2^t + 2^{t-2}, & \mathrm{charrank}(\widetilde{\gamma}_{2^t+2^{t-1}+1,3}) &= 2^{t+1} - 5, \\ \mathrm{cup}(\widetilde{G}_{2^t+2^{t-1}+2,3}) &= 2^t + 2^{t-2} + 1, & \mathrm{charrank}(\widetilde{\gamma}_{2^t+2^{t-1}+2,3}) &= 2^{t+1} - 5. \end{aligned}$$

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Theorem

For $t \geq 4$ we have

$$\begin{gathered} \mathrm{cup}(\widetilde{G}_{2^t-1,3}) = 2^t - 3, \\ \vdots \\ \mathrm{cup}(\widetilde{G}_{2^t+2,3}) = 2^t - 3, \\ \mathrm{cup}(\widetilde{G}_{2^t+3,3}) = 2^t - 3. \end{gathered}$$

Conclusion

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For any $k \geq 4$ and t sufficiently large so that it satisfies $2^{t-1} \geq 3k + 2$ we have

$$\mathrm{cup}(\widetilde{G}_{2^t+k,3}) = 2^t - 3.$$

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Thank you.