Constant curvature models in sub-Riemannian geometry

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1. Cartan geometries
   - Subriemannian prolongation
   - Underlying parabolic geometries
2. Cohomologies
3. Free 2-step distributions
4. Constant curvature spaces

Subriemannian geometry

Definition

Subriemannian geometry (M, D, S) on a manifold M is given by a distribution D, and (positive definite) metric S on D.

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Sheaf $\mathcal{D}^{-1} = \mathcal{D}$ of vector fields valued in D generates the filtration by sheafs

$$\mathcal{D}^j = \{[X, Y], X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \dots .$$

We say that D is a bracket generating distribution if for some k, $\mathcal{D}^k$ is the sheaf of all vector fields on M.

Bracket generating distribution D defines the filtration of subspaces

$$T_xM = D_x^k \supset \cdots \supset D_x^{-1}$$

at each point $x \in M$.
The associated graded tangent space

$$\mathrm{gr}\, T_xM = T_xM/D_x^{k+1} \oplus \cdots \oplus D_x^{-1}$$

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Definition

(M, D, S) is a *sub-Riemannian geometry with constant symbol* if D is bracket generating, and the nilpotent algebra $\operatorname{gr} T_xM$, together with the metric, is isomorphic to a fixed graded Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$$

with a fixed metric σ on $\mathfrak{g}_{-1}$.

Prolongation of subriemannian geometries

Let $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$ be the Lie algebra of the Lie group G_0 of all automorphisms of the graded nilpotent algebra $\mathfrak{g}_-$ preserving the metric σ on $\mathfrak{g}_{-1}$.
The action of the derivations from $\mathfrak{g}_0$ on $\mathfrak{g}_-$ extends the Lie algebra structure on $\mathfrak{g}_-$ to the Lie algebra

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Observation 1

The Tanaka prolongation of $\mathfrak{g}$ is finite.[a]

[a] Corollary 2 of Theorem 11.1 in *Tanaka, N.*, On differential systems, graded Lie algebras and pseudo-groups, *J. Math. Koyto Univ., 10, 1 (1970), 1-82.*

Observation 2

Already the first prolongation is trivial.[a] Thus $\mathfrak{g}$ is the full prolongation of $\mathfrak{g}_-$.

[a] Yatsui, T., *On pseudo-product graded Lie algebras*, Hokkaido Math. J., 17 (1988), 333-343.

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Theorem

For each subriemannian manifold (M, D, S) with constant symbol, there is the unique Cartan connection $(\mathcal{G} \to M, \omega)$ of type $(\mathfrak{g}, G_0)$ with the curvature function $\kappa : \mathcal{G} \to \mathfrak{g} \otimes \Lambda^2 \mathfrak{g}_-^*$ satisfying $\partial^* \kappa = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection κ_H, i.e. the component with $\partial \kappa_H = 0$ as well.[a]

[a] Morimoto, T., *Cartan connection associated with a subriemannian structure*, Differential Geometry and its Applications 26 (2008), 75-78.

The distribution D on M itself often defines a nice finite type filtered geometry which enjoys a canonical Cartan connection, too. Many of them belong to the class of the parabolic geometries, for which the full Tanaka prolongation of $\mathfrak{g}_-$ is a semisimple Lie algebra

$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

and $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is the opposite nilpotent radical to the parabolic subalgebra $\mathfrak{p} = \bar{\mathfrak{g}}_0 \oplus \cdots \oplus \bar{\mathfrak{g}}_k \subset \bar{\mathfrak{g}}$, with $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$.

Fix one such graded semisimple $\bar{g}$ and consider the frame bundle $\mathcal{G}_0 \to M$ of gr TM giving a parabolic geometry. Often the structure group G_0 of $\mathcal{G}_0$ is the full group of graded automorphisms of $\mathfrak{g}_-$.[1] Adding a metric S on D, we have got two (related) Cartan connections there.

[1] See Čap, A., Slovák, J., Parabolic Geometries I, Background and General Theory, AMS, Math. Surveys and Monographs 154, x+628pp. for details.

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Theorem

Consider a bracket generating distribution D on M with the constant symbol equal to the negative part of a graded semisimple Lie algebra $\bar{\mathfrak{g}}$ and the corresponding frame bundle $\mathcal{G}_0 \to M$ of gr TM. Then there is the unique Cartan connection $(\bar{\mathcal{G}} \to M, \omega)$ of type $(\bar{\mathfrak{g}}, P)$ with the curvature function $\bar{\kappa} : \bar{\mathcal{G}} \to \bar{\mathfrak{g}} \otimes \Lambda^2 \mathfrak{g}_-^*$ satisfying $\partial^* \bar{\kappa} = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection $\bar{\kappa}_H$, i.e. the component with $\partial \bar{\kappa}_H = 0$ as well.

[1] See Čap, A., Slovák, J., Parabolic Geometries I, Background and General Theory, AMS, Math. Surveys and Monographs 154, x+628pp. for details.

Consider a parabolic geometry (M, D) equipped by the metric S on D, assume (M, D, S) has got constant symbol.
Thus we have got:

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$
$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

This is an instance of a $\mathfrak{g}_-$-submodule W of $\mathfrak{g}_-$-module V.

The short exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

induces the short exact sequence of differential complexes

$$0 \longrightarrow C^\bullet(\mathfrak{g}_-, W) \xrightarrow{i} C^\bullet(\mathfrak{g}_-, V) \xrightarrow{\pi} C^\bullet(\mathfrak{g}_-, V/W) \longrightarrow 0$$

and thus the long exact sequence in cohomologies

$$\begin{array}{l} \longrightarrow H^n(\mathfrak{g}_-, W) \xrightarrow{i} H^n(\mathfrak{g}_-, V) \xrightarrow{\pi} H^n(\mathfrak{g}_-, V/W) \xrightarrow{\delta} \\ \longrightarrow H^{n+1}(\mathfrak{g}_-, W) \xrightarrow{i} H^{n+1}(\mathfrak{g}_-, V) \xrightarrow{\pi} H^{n+1}(\mathfrak{g}_-, V/W) \longrightarrow \end{array}$$

The differentials ∂ respect our gradings, thus we get grading on the cohomology spaces, too. Clearly, we may consider the sequences for the individual homogeneities separately.

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Notice, the filtration is induced by the distribution D and we declare its symbol to be equal to the Lie algebra $\mathfrak{g}_-$ at all points. Thus, all the curvature homogeneities ≤ 0 vanish.

Theorem

Let $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_- \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_+$ be a graded Lie algebra and $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$ be a non-positively graded Lie algebra such that $\mathfrak{g}_- = \bar{\mathfrak{g}}_-$ and $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$. The cohomology $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ as a $\mathfrak{g}_0$-submodule is isomorphic to a direct sum of 2 parts:

1. $H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})/H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$,
2. $\ker \pi \subset H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$,

where $\pi\colon H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ is the projection induced by the projection in cochains $\pi\colon C^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to C^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$.

Proof.

The first rows of the long exact sequence are

$$\begin{tikzcd}
\cdots \arrow[r] & H^1_+(\mathfrak{g}_-,\bar{\mathfrak{g}}) \arrow[r, "\pi_1"] & H^1_+(\mathfrak{g}_-,\bar{\mathfrak{g}}/\mathfrak{g}) \arrow[dll, "\delta"'] \\
H^2_+(\mathfrak{g}_-,\mathfrak{g}) \arrow[r, "i_2"] & H^2_+(\mathfrak{g}_-,\bar{\mathfrak{g}}) \arrow[r, "\pi_2"] & H^2_+(\mathfrak{g}_-,\bar{\mathfrak{g}}/\mathfrak{g}) \arrow[r] & {}
\end{tikzcd}$$

Notice the connecting homomorphism δ is essentially given by ∂.
The first part of $H^2_+(\mathfrak{g}_-,\mathfrak{g})$ is $H^1_+(\mathfrak{g}_-,\bar{\mathfrak{g}}/\mathfrak{g})$, which is mapped by δ injectively into $H^2_+(\mathfrak{g}_-,\mathfrak{g})$. The second part is
$\operatorname{im} i_2 \colon H^2_+(\mathfrak{g}_-,\mathfrak{g}) \to H^2_+(\mathfrak{g}_-,\bar{\mathfrak{g}})$.
Exactness of the sequence implies $\operatorname{im} i_2 = \ker \pi_2$. $\square$

Remark 1

Notice, $\bar{\mathfrak{g}}_0$ equals to all derivations on the graded algebra $\mathfrak{g}_-$ if and only if all the non-negative homogeneities $H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$ vanish. If $H^1_0(\mathfrak{g}_-, \bar{\mathfrak{g}}) \neq 0$, then we need further reduction of the algebra of all derivations to $\bar{\mathfrak{g}}_0$. This is the case just for the following cases:

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- the length of the grading is $k = 1$;
- in all the contact cases (we remind that every complex simple Lie algebra admits a unique contact grading);
- $\bar{\mathfrak{g}}$ is of type A_l with $l \geq 3$ and the grading corresponds to $\{\alpha_1, \alpha_i\} \subset \Delta^+$, $2 \leq i \leq l$, where Δ^+ is the the set of simple roots;
- $\bar{\mathfrak{g}}$ is of type C_l with $l \geq 2$ and the grading corresponds to $\{\alpha_1, \alpha_l\} \subset \Delta^+$,

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Thus we see that for semisimple $\bar{\mathfrak{g}}$ in almost all cases H^1-part of $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ is equivalent to $H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$.

Remark 2

The projection π_2 is zero whenever the representing cochains are valued in $\mathfrak{g}$. $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$ is well known. Only five cases allow for curvature components valued in $\bar{\mathfrak{g}}_{\geq 0}$, except for the length $k = 1$ and contact cases:

- the path geometries with $\dim D \geq 3$ (type A_l, grading is given by $\{\alpha_1, \alpha_2\} \subset \Delta^+$, $l \geq 3$);
- the real forms corresponding to quaternionic contact geometries (type C_l, grading is given by $\{\alpha_2\} \subset \Delta^+$, $l \geq 3$);
- the Borel B_2 case;
- the lowest dimensional free 2-step distribution $\dim D = 3$, $\dim M = 6$ (type B_3, grading is given by $\{\alpha_3\} \subset \Delta^+$);
- one more C_l case corresponding to the first two simple roots $\{\alpha_1, \alpha_2\} \subset \Delta^+$, i.e. in the complex version this geometry includes the common correspondence space for the projective contact and the quaternionic contact geometries.

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- one more C_l case corresponding to the first two simple roots $\{\alpha_1, \alpha_2\} \subset \Delta^+$, i.e. in the complex version this geometry includes the common correspondence space for the projective contact and the quaternionic contact geometries.

In all other cases, the kernel of π_2 is the entire $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$.

Write $\bar{\mathfrak{g}}^i = \mathfrak{g} + \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_i$, $i \geq -1$, in particular $\bar{\mathfrak{g}}^{-1} = \mathfrak{g}$. All of them are $\mathfrak{g}_-$-submodules in $\bar{\mathfrak{g}}$.

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Further technical observations

For all $i \geq 0$ we have got $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ as

$$\ker \partial \colon \operatorname{Hom}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}^i/\bar{\mathfrak{g}}^{i-1})/\partial(\bar{\mathfrak{g}}_{i+1}) \to H^2_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^{i-1}/\mathfrak{g}).$$

In particular,

1. Cohomologies $H^1_{>k+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ are zero where k is the highest homogeneity in $\bar{\mathfrak{g}}$.
2. In homogeneity 1 we have
$$H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \operatorname{Hom}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_0/\mathfrak{g}_0)/\partial(\bar{\mathfrak{g}}_1).$$
3. In homogeneity 2 we have
$$\ker \partial \colon \operatorname{Hom}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_1)/\partial(\bar{\mathfrak{g}}_2) \to (\wedge^2 \mathfrak{g}^*_{-1}/\partial(\bar{\mathfrak{g}}^*_{-2})) \otimes (\bar{\mathfrak{g}}_0/\mathfrak{g}_0).$$

Theorem

Write E for the grading element of $\bar{\mathfrak{g}}$, $E \notin \mathfrak{g}_0$. Then

1. *Cohomologies $H^1_{>k+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ are zero.*
2. *In homogeneity 1 we have*

$$H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \mathfrak{g}^*_{-1} \otimes (\bar{\mathfrak{g}}_0/(\mathfrak{g}_0 \oplus \mathbb{R}E)).$$

   In particular, if $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{R}E$ then $H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = 0$;

3. *In homogeneity 2 we have*

$$H^1_2(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) \subseteq \mathrm{Sym}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_1)$$

   where $\mathrm{Sym}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_1) \subset \mathrm{Hom}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_1)$ denotes the subspace of symmetric tensors with respect to the Killing form. Moreover if $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{R}E$ or $\partial(\mathfrak{g}^*_{-2}) = \wedge^2 \mathfrak{g}^*_{-1}$ then

$$H^1_2(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}}_1).$$

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Let M be a manifold of dimension $n(n+1)$. We say that distribution D of dimension n is a free (step 2) distribution on M if $D+[D,D]=TM$.
This is a nice parabolic geometry, of type $(\bar{\mathfrak{g}}, \bar{P})$ with the Lie algebras of the form

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} A & X & Y \\ -Z^t & 0 & -X^t \\ T & Z & -A^t \end{pmatrix} \right\}, \quad \bar{\mathfrak{p}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ -Z^t & 0 & 0 \\ T & Z & -A^t \end{pmatrix} \right\},$$

where $A, Y, T \in \mathrm{Mat}_n(\mathbb{R})$, $X, Z \in \mathbb{R}^n$, $Y + Y^t = T + T^t = 0$.

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where $A, Y, T \in \mathrm{Mat}_n(\mathbb{R})$, $X, Z \in \mathbb{R}^n$, $Y + Y^t = T + T^t = 0$.
We introduce the obvious basis $e^{[ij]}$, e^j, e^i_j, e_j, $e_{[ij]}$ in $\bar{\mathfrak{g}}$.
The commutation relations are given by:

$$[e^{[ij]}, e_{[jk]}] = -e^i_k - \delta^i_k e^j_j = \begin{cases} -e^i_k, & k \neq i \\ -e^i_i - e^j_j, & k = i \end{cases}$$

The metric S defines a reduction of $\bar{P}$-principle bundle $\bar{\mathcal{G}}$ to $G_0 = SO_n(\mathbb{R})$-principle bundle $\mathcal{G}$ of orthogonal frames. The sub-Riemannian structure in the background can be given in terms of orthonormal frame $X_1, \dots, X_n$ on D.
We define $X_{[ij]} = -[X_i, X_j]$. Due to the fact that D is a free distribution the graded symbol of $\{X_i, X_{[jk]}\}$ is given by e_i, $e_{[jk]}$ with the same relations as in $\bar{\mathfrak{g}}$.
The infinitesimal model is given by

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \langle e_{[ij]} \rangle \oplus \langle e_k \rangle \oplus \langle a^i_j \rangle.$$

Theorem

The $H^2(\mathfrak{g}_-,\bar{\mathfrak{g}})$ part of $H^2(\mathfrak{g}_-,\mathfrak{g})$ is the entire $H^2(\mathfrak{g}_-,\bar{\mathfrak{g}})$, i.e. the subspace of totally trace-free elements in

$$\mathrm{Hom}(\mathfrak{g}_{-1}\wedge\mathfrak{g}_{-2},\mathfrak{g}_{-2}).$$

The $H^1(\mathfrak{g}_-,\bar{\mathfrak{g}}/\mathfrak{g})$ part of $H^2(\mathfrak{g}_-,\mathfrak{g})$ consists of 2 subspaces:

- *in degree 1 it is generated by symmetric and traceless in (i,j) tensors*

$$\alpha^k_{(ij)}=\Big(e_j\otimes e_i^*+e_i\otimes e_j^*+\sum_t\big(e_{[jt]}\otimes e^*_{[it]}+e_{[it]}\otimes e^*_{[jt]}\big)\Big)\wedge e_k^*$$

- *in degree 2 it is generated by symmetric in (p,q) tensors*

$$\alpha_{(pq)}=\sum_t e_t\otimes(e^*_{[tp]}\wedge e^*_q+e^*_{[tq]}\wedge e^*_p)+\sum_{t,r}e_{[tr]}\otimes e^*_{[tp]}\wedge e^*_{[qr]}.$$

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Thus we aim at finding all submodule.

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Theorem

Assume $n \geq 4$. The only constant curvature models for free step 2 sub-Riemannian geometries are defined on $SO(n+1)$ and $SO(n,1)$ with orthonormal frame given by the elements of $\mathfrak{so}_{n+1}$ of the form

$$\begin{pmatrix} 0 & A_i^t \\ -A_i & 0_n \end{pmatrix},$$

and by the elements of $\mathfrak{so}_{n,1}$ of the form

$$\begin{pmatrix} 0 & A_i^t \\ A_i & 0_n \end{pmatrix},$$

where the only non-zero element in A_i is on the place i.

We have to check the individual invariants components of the harmonic curvature for the trivial submodules in the $\mathfrak{so}_n$ decomposition.

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While there are no such trivial submodules in the totally tracefree part of $\mathrm{Hom}(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2}, \mathfrak{g}_{-2})$, and in the homogeneity one traceless in (i,j) tensors

$$\alpha^k_{(ij)} = \left(e_j \otimes e_i^* + e_i \otimes e_j^* + \sum_t (e_{[jt]} \otimes e^*_{[it]} + e_{[it]} \otimes e^*_{[jt]})\right) \wedge e_k^*,$$

there is just one such module in

$$\alpha_{(pq)} = \sum_t e_t \otimes (e^*_{[tp]} \wedge e^*_q + e^*_{[tq]} \wedge e^*_p) + \sum_{t,r} e_{[tr]} \otimes e^*_{[tp]} \wedge e^*_{[qr]}.$$

.
The models with positive and negative curvature are just those in the theorem.