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On Clifford groups in quantum computing

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Outline

- Introduction
- Weyl-Heisenberg groups of single N-level systems
- Clifford quotient groups of single N-level systems
- Clifford quotient groups for multipartite systems
- Illustrative examples
- Concluding remarks

Introduction

In quantum mechanics of single N-level systems in Hilbert spaces of finite dimension N, the basic operators are the *generalized Pauli matrices*. They generate the *finite Weyl-Heisenberg group* as a subgroup of the unitary group $\mathrm{U}(N)$.

Its automorphism group within the unitary group $\mathrm{U}(N)$, i.e. the largest subgroup of $\mathrm{U}(N)$ having the Weyl-Heisenberg group as a normal subgroup, is in quantum information conventionally called the *Clifford group*.

Since this normalizer necessarily contains the continuous group $\mathrm{U}(1)$ of phase factors, some authors adopt an alternative definition of the Clifford group as the quotient of the normalizer with respect to $\mathrm{U}(1)$. We call it the *Clifford quotient group*.

I am going to show that our comprehensive work on symmetries of the Pauli gradings of quantum operator algebras involves just these Clifford quotient groups and, moreover, covers all single as well as composite finite quantum systems.

Weyl-Heisenberg groups of single N-level systems

In finite-dimensional quantum mechanics of a single N-level system the N-dimensional Hilbert space $\mathcal{H}_N = \mathbb{C}^N$ has an orthonormal basis $\mathcal{B} = \{|0\rangle, |1\rangle, \dots |N-1\rangle\}$. The basic unitary operators Q_N, P_N are defined by their action on the basis (Weyl, Schwinger)

$$\begin{aligned} Q_N|j\rangle &= \omega_N^j |j\rangle, \\ P_N|j\rangle &= |j-1 \pmod{N}\rangle, \end{aligned}$$

where $j = 0, 1, \dots, N-1$, $\omega_N = \exp(2\pi i/N)$.
This is the well-known *clock-and-shift representation* of the basic operators Q_N, P_N.

In the standard basis $\mathcal{B}$ the operators Q_N and P_N are represented by the generalized Pauli matrices

$$Q_N = \mathrm{diag}\left(1, \omega_N, \omega_N^2, \cdots, \omega_N^{N-1}\right)$$

and

$$P_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Their commutation relation

$$P_N Q_N = \omega_N Q_N P_N$$

expresses the minimal non-commutativity of the operators P_N and Q_N. Further, they are of the order N,

$$P_N^N = Q_N^N = I, \quad \omega_N^N = 1.$$

Weyl H 1931 *The Theory of Groups and Quantum Mechanics* (New York: Dover) pp 272–280

J. Schwinger, Unitary operator bases, *Proc. Nat. Acad. Sci. (U.S.A.)* 46, 570-579, (1960);
reprinted in J. Schwinger, *Quantum Kinematics and Dynamics*, Benjamin, New York (1970), pp 63–72

Šťovíček P and Tolar J 1984 Quantum mechanics in a discrete space-time *Rep. Math. Phys.* **20** 157–170

Elements of $\mathbb{Z}_N = \{0, 1, \ldots N-1\}$ label the vectors of the standard basis $\mathcal{B}$ with the physical interpretation that $|j\rangle$ is the (normalized) eigenvector of position at $j \in \mathbb{Z}_N$. In this sense the cyclic group $\mathbb{Z}_N$ plays the role of the *configuration space of an N-level quantum system*.

The Pauli group of order N^3 is generated by ω_N, Q_N and P_N

$$\Pi_N = \left\{ \omega_N^l Q_N^i P_N^j | l, i, j = 0, 1, 2, \ldots, N-1 \right\}.$$

Now the Clifford group as a matrix subgroup of $\mathrm{U}(N)$ should contain the *Weyl-Heisenberg group* as its subgroup. The phase factors emerging in the Clifford group lead to the necessity to define the Weyl-Heisenberg groups $H(N)$ in even dimensions N by doubling the Pauli group so that $H(N)$ for even N contains the Pauli group Π_N as a subgroup.

For this purpose the phase factor is introduced

$$\tau_N = -e^{\frac{\pi i}{N}}$$

such that $\tau_N^2 = \omega_N$. Then the Weyl-Heisenberg groups

$$H(N) = \Pi_N = \left\{ \tau_N^l Q_N^i P_N^j | l, i, j = 0, 1, \dots, N-1 \right\} \quad \text{for odd } N,$$

$$H(N) = \left\{ \tau_N^l Q_N^i P_N^j | i, j = 0, 1, \dots, N-1, l = 0, 1, \dots, 2N-1 \right\} \text{ for even } N$$

are of orders N^3, $2N^3$, respectively.

For example

$$H(2) = \langle \tau_2 I_2, Q_2, P_2 \rangle,$$

where $\tau_2 = -i$, $Q_2 = \sigma_z$, $P_2 = \sigma_x$, is of order 16.

The *center $Z(H(N))$ of the Weyl-Heisenberg group* is the set of all those elements of $H(N)$ which commute with all elements in $H(N)$.
For odd N:

$$Z(H(N)) = \{\omega_N, \omega_N^2, \cdots, \omega_N^N = 1\} = \{\tau_N, \tau_N^2, \cdots, \tau_N^N = 1\},$$

for even N:

$$Z(H(N)) = \{\tau_N, \tau_N^2, \cdots, \tau_N^{2N} = 1\}.$$

Since the center is a normal subgroup, one can go over to the *quotient group*

$$\mathcal{P}_N = H(N)/Z(H(N)),$$

usually identified with the *finite phase space* $\mathbb{Z}_N \times \mathbb{Z}_N$.

Here the cosets $\left\{\tau_N^l Q_N^i P_N^j | \forall l\right\}$, are labeled by points $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$ of the phase space, $i, j = 0, 1, \ldots, N-1$ and we denote them shortly by $Q^i P^j$ (without subscripts N). The correspondence

$$\Phi : H(N)/Z(H(N)) \to \mathbb{Z}_N \times \mathbb{Z}_N : Q^i P^j \mapsto (i, j),$$

is clearly an *isomorphism of Abelian groups*.

Clifford groups of single N-level quantum systems

The term Clifford group means the group of symmetries of the Weyl-Heisenberg group in the following sense:

Definition
The Clifford group comprises all unitary operators $X \in \mathsf{U}(N)$ for which the Ad*-action preserves the subgroup $H(N)$ in $\mathsf{U}(N)$, i.e. the Clifford group is the normalizer $N_{\mathsf{U}(N)}(H(N))$.*

Gottesman D 1998 Theory of fault-tolerant quantum computation *Phys. Rev. A* **57** 127–137

Appleby D M 2005 SIC-POVMs and the extended Clifford group *J. Math. Phys.* **46** 052107

In this sense the Clifford group consists of all those $X \in \mathsf{U}(N)$ such that their Ad-action leaves $H(N)$ invariant,

$$\mathsf{Ad}_X H(N) = XH(N)X^{-1} = H(N).$$

But $H(N)$ is generated by τ_N, Q_N and P_N, so the Clifford group consists of all $X \in \mathsf{U}(N)$ such that

$$\mathsf{Ad}_X Q_N = XQ_N X^{-1} \in H(N) \quad \text{and} \quad \mathsf{Ad}_X P_N = XP_N X^{-1} \in H(N).$$

Since the subgroup $H(N)$ is a normal subgroup of the normalizer, one can formally write a short exact sequence of group homomorphisms

$$1 \to H(N) \to N_{\mathsf{U}(N)}(H(N)) \to N_{\mathsf{U}(N)}(H(N))/H(N) \to 1.$$

But the full structure of the normalizer $N_{\mathsf{U}(N)}(H(N))$ for arbitrary N is complicated by phase factors and rather difficult to describe in general.

In order to get insight into the structure of the normalizer *up to arbitrary phase factors* we turn to the definition of the *Clifford quotient group* $C(N)$ as the quotient

$$C(N) = N_{\mathsf{U}(N)}(H(N))/\mathsf{U}(1).$$

Its elements are the cosets $\{e^{i\alpha}X\}$. The following lemma is crucial for our alternative view of the Clifford quotient group.

Lemma
Let $X, Y \in \mathsf{U}(N)$. Then the equality $\mathrm{Ad}_X A = \mathrm{Ad}_Y A$ holds for all $A \in H(N)$ if and only if $X = e^{i\alpha}Y$.

The proof of the converse implication is trivial. For the proof of the direct implication one writes down the first identity in the form $Y^{-1}XA = AY^{-1}X$, i.e. $Y^{-1}X$ commutes with all $A \in H(N)$. But the elements of $H(N)$ form an irreducible set, hence by Schur's lemma $Y^{-1}X$ is proportional to the unit matrix, $Y^{-1}X = e^{i\alpha}I_N$.$\square$

Hence instead of the cosets $\{e^{i\alpha}X\}$ one equivalently considers Ad-actions induced by the elements X of the normalizer.
In general, the mappings

$$\mathsf{Ad}_X \; : \; A \to XAX^{-1},$$

where $A \in \mathsf{GL}(N,\mathbb{C})$, are inner automorphisms of $\mathsf{GL}(N,\mathbb{C})$ induced by elements $X \in \mathsf{GL}(N,\mathbb{C})$.

In our case, if inner automorphisms Ad_X should transform unitary operators A in unitary operators $A' = XAX^{-1}$, then to each $X \in \mathsf{GL}(N,\mathbb{C})$ there exists a unitary operator $U \in \mathsf{U}(N)$ such that $\mathsf{Ad}_U = \mathsf{Ad}_X$, and U is unique up to a phase factor.
The subgroup of $\mathsf{Int}(\mathsf{GL}(N,\mathbb{C}))$ generated by unitary operators will be denoted

$$\mathcal{M}_N = \{\mathsf{Ad}_X | X \in \mathsf{U}(N)\}.$$

Now the Clifford quotient group $C(N)$ can be studied as a subgroup of $\mathcal{M}_N$ – the subgroup of those Ad-actions which preserve the Weyl-Heisenberg group $H(N)$.

But $C(N)$, consisting of the cosets $\{e^{i\alpha}X\}$ leaving $H(N)$ invariant, contains the cosets $\{e^{i\alpha}A\}$ of operators $A \in H(N)$ as a subgroup, and this subgroup is isomorphic to the group of Ad-actions Ad_A, $A \in H(N)$.

In this picture the subgroup of Ad-actions Ad_A, $A \in H(N)$, is generated by the commuting Ad-actions Ad_{Q_N} and Ad_{P_N},

$$\{\mathrm{Ad}_{Q_N^i P_N^j} | i,j = 0,1,\ldots,N-1\} \cong \mathbb{Z}_N \times \mathbb{Z}_N \cong \mathcal{P}_N.$$

Proposition

The Clifford quotient group $C(N)$ is isomorphic to the subgroup of those unitary inner automorphisms in $\mathcal{M}_N$ which preserve $\mathcal{P}_N$, i.e. $C(N)$ is the normalizer of $\mathcal{P}_N$ in $\mathcal{M}_N$,

$$C(N) \cong N_{\mathcal{M}_N}(\mathcal{P}_N).$$

The corresponding short exact sequence of homomorphisms of subgroups of $\mathcal{M}_N$

$$1 \to \mathcal{P}_N \to N_{\mathcal{M}_N}(\mathcal{P}_N) \to N_{\mathcal{M}_N}(\mathcal{P}_N)/\mathcal{P}_N \to 1$$

can be shown to be fully decoded.

Obviously, Ad-actions of elements of $\mathcal{P}_N$ leave $\mathcal{P}_N$ invariant. Then the elements of the quotient group $N_{\mathcal{M}_N}(\mathcal{P}_N)/\mathcal{P}_N$ are the cosets corresponding to possibly non-trivial transformations of $\mathcal{P}_N$ forming a symmetry (or Weyl) group. Let us consider the Ad-actions $\text{Ad}_X(A) = XAX^{-1}$, where $X \in U(N)$, on elements $A \in H(N)$, which induce permutations of cosets in $H(N)/Z(H(N)$. We consider them to be *equivalent* if, for each pair $(i,j) \in \mathbb{Z}_N \times \mathbb{Z}_N$, they define the same transformation of cosets in $H(N)/Z(H(N))$:

$$\text{Ad}_Y \sim \text{Ad}_X \quad \Leftrightarrow \quad YQ^iP^jY^{-1} = XQ^iP^jX^{-1}.$$

Since the group $\mathcal{P}_N$ has two generators, Ad_{Q_N} and Ad_{P_N}, corresponding to the cosets Q and P, then for each Ad_Y inducing a permutation of elements in $\mathcal{P}_N$ there must exist $a, b, c, d \in \mathbb{Z}_N$ such that

$$YQY^{-1} = Q^a P^b \quad \text{and} \quad YPY^{-1} = Q^c P^d,$$

i.e., to each equivalence class of Ad-actions Ad_Y a quadruple $(a, b, c, d) \in \mathbb{Z}_N$ is assigned.

In matrix notation

$$\mathrm{Ad}_Y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\mathrm{Ad}_Y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now inserting the relations

$$\mathsf{Ad}_Y Q_N = \mu Q_N^a P_N^b, \qquad \mathsf{Ad}_Y P_N = \nu Q_N^c P_N^d,$$

into the basic commutation condition

$$\mathsf{Ad}_Y(P_N Q_N) = \omega_N \mathsf{Ad}_Y(Q_N P_N),$$

we find $\omega_N^{ad-1} = \omega_N^{bc}$, i.e.

$$ad - bc = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1 \pmod{N}.$$

Theorem
For integer $N \geq 2$ there is an isomorphism Ψ between the set of equivalence classes of Ad*−actions* Ad_Y *which induce permutations of cosets, and the group* $\mathrm{SL}(2, \mathbb{Z}_N)$ *of 2×2 matrices with determinant equal to* $1 \pmod N$*,*

$$\Psi(\mathrm{Ad}_Y) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}_N. \tag{1}$$

The action of these automorphisms on $\mathcal{P}_N$ is given by left action of $\mathrm{SL}(2, \mathbb{Z}_N)$ *on elements $(i, j)^T$ of the phase space $\mathcal{P}_N = \mathbb{Z}_N \times \mathbb{Z}_N$,*

$$\mathrm{Ad}_Y \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}. \tag{2}$$

Havlíček M, Patera J, Pelantová E and Tolar J 2002 Automorphisms of the fine grading of $sl(n, \mathbb{C})$ associated with the generalized Pauli matrices *J. Math. Phys.* **43** 1083-1094; arXiv: math-ph/0311015

Returning to the short exact sequence of group homomorphisms

$$1 \to \mathcal{P}_N \to N_{\mathcal{M}}(\mathcal{P}_N) \to N_{\mathcal{M}}(\mathcal{P}_N)/\mathcal{P}_N \to 1,$$

it follows from the above results that the entering groups are isomorphic to

$$\begin{aligned} \mathcal{P}_N &\cong \mathbb{Z}_N \times \mathbb{Z}_N, \\ N_{\mathcal{M}_N}(\mathcal{P}_N)/\mathcal{P}_N &\cong \mathsf{SL}(2, \mathbb{Z}_N), \\ C(N) \cong N_{\mathcal{M}_N}(\mathcal{P}_N) &\cong (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathsf{SL}(2, \mathbb{Z}_N). \end{aligned}$$

Summarizing, the *Clifford quotient group* $C(N)$ is isomorphic to the normalizer of the Abelian subgroup $\mathcal{P}_N$ in the group of unitary inner automorphisms $\mathcal{M}_N$. Since it contains all inner automorphisms transforming the phase space into itself, it necessarily contains $\mathcal{P}_N$ as an Abelian semidirect factor. The symmetry (or Weyl) group is then isomorphic to the quotient group of the normalizer with respect to $\mathcal{P}_N$.

The generators of the normalizer $N_{\mathcal{M}_N}(\mathcal{P}_N)$ are Ad_{Q_N}, Ad_{P_N} as generators of $\mathcal{P}_N$, and the two generators Ad_{S_N}, Ad_{D_N} of $\mathrm{SL}(2, \mathbb{Z}_N)$.

The unitary Sylvester matrix S_N is the matrix of the discrete Fourier transformation (for $N = 2$ the Hadamard gate):

$$(S_N)_{jk} = \frac{\omega_N^{jk}}{\sqrt{N}}.$$

It acts on Q_N and P_N according to

$$S_N Q_N S_N^{-1} = P_N^{-1} \quad S_N P_N S_N^{-1} = Q_N,$$

The unitary phase operator D_N (for $N = 2$ the phase gate) is diagonal,

$$D_N = \text{diag}\ (d_0, d_1, \ldots, d_{N-1}),$$

where $d_j = \tau_N^{j(1-j)}$ if N is odd, $d_j = \tau_N^{j(N-j)}$ if N is even. It acts on Q_N and P_N according to

$$D_N Q_N D_N^{-1} = Q_N, \qquad D_N P_N D_N^{-1} = \alpha_N Q_N P_N$$

where $\alpha_N = 1$ for N odd and $\alpha_N = \tau_N^{N+1}$ for N even.

Given the prime decomposition of $N = \prod_{i=1}^{r} p_i^{k_i}$, the general formula for the number of elements of $\mathrm{SL}(2, \mathbb{Z}_N)$ is the following multiplicative function of number theory:

$$|\mathrm{SL}(2, \mathbb{Z}_N)| = N^3 \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^2}\right).$$

Example $N = 2$ (N.J.A. Sloane): The phase space consists of 4 elements $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. The group $\mathrm{SL}(2,\mathbb{Z}_2)$ has 6 elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and acts transitively on the orbit $\{(1,0),(0,1),(1,1)\}$. Unitary operators S_2 and D_2 are

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

This *finite Clifford group* is generated by S_2 and D_2, and has $24 \times 8 = 192$ elements, since $(S_2 D_2)^3 = \eta I_2$ is of order 8:

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \beta \\ \alpha & -\alpha\beta \end{pmatrix}, \eta^{\nu} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\eta = \exp(i\frac{\pi}{4})$, $\nu = 0, 1, \ldots, 7$, $\alpha, \beta \in \{1, i, -1, -i\}$.

Clifford quotient groups for multipartite systems

Our further results concern detailed description of groups of symmetries of finite Weyl-Heisenberg groups for *finitely composed quantum systems consisting of subsystems with arbitrary dimensions*. We have fully described these symmetries on the level of Ad-actions. In our notation the *symmetry (or Weyl) groups* are $\mathrm{Sp}_{[n_1,\ldots,n_k]}$, where the indices denote arbitrary dimensions of the constituent Hilbert spaces.

Korbelář M and Tolar J 2012 Symmetries of finite Heisenberg groups for multipartite systems *J. Phys. A: Math. Theor.* **45** 285305 (18pp); arXiv: 1210.0328 [quant-ph]

Korbelář M and Tolar J 2010 Symmetries of the finite Heisenberg group for composite systems *J. Phys. A: Math. Theor.* **43** 375302 (15pp); arXiv: 1006.0328 [quant-ph]

More in detail, let the Hilbert space of a composite system be the tensor product $\mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_k}$ of dimension $N = n_1 \ldots n_k$, where $n_1, \ldots, n_k \in \mathbb{N}$. For the *composite system*, quantum phase space is the Abelian subgroup of $\mathrm{Int}(\mathrm{GL}(N, \mathbb{C}))$ defined by

$$\mathcal{P}_{(n_1,\ldots,n_k)} = \{\mathrm{Ad}_{M_1 \otimes \cdots \otimes M_k} |\ M_i \in H(n_i), i = 1, \ldots, k\}.$$

The *Clifford quotient group*, or the normalizer of this Abelian subgroup in the group of unitary inner automorphisms of $\mathrm{GL}(N, \mathbb{C})$, contains all unitary inner automorphisms transforming the phase space into itself, hence necessarily *contains* $\mathcal{P}_{(n_1,\ldots,n_k)}$ *as an Abelian semidirect factor*. The symmetry (or Weyl) group is then given by the quotient group of the normalizer with respect to this Abelian subgroup.

The generating elements of $\mathcal{P}_{(n_1,\ldots,n_k)}$ are the inner automorphisms

$$e_j := \mathrm{Ad}_{A_j} \quad \text{for} \quad j = 1,\ldots,2k,$$

where (for $i = 1,\ldots,k$)

$$A_{2i-1} := I_{n_1\cdots n_{i-1}} \otimes P_{n_i} \otimes I_{n_{i+1}\cdots n_k}, A_{2i} := I_{n_1\cdots n_{i-1}} \otimes Q_{n_i} \otimes I_{n_{i+1}\cdots n_k}.$$

The normalizer of $\mathcal{P}_{(n_1,\ldots,n_k)}$ in $\mathrm{Int}(\mathrm{GL}(n_1 \cdots n_k, \mathbb{C}))$ will be denoted

$$\mathcal{N}(\mathcal{P}_{(n_1,\ldots,n_k)}) := N_{\mathrm{Int}(\mathrm{GL}(n_1\cdots n_k,\mathbb{C}))}(\mathcal{P}_{(n_1,\ldots,n_k)}),$$

We need also the normalizer of $\mathcal{P}_n$ in $\mathrm{Int}(\mathrm{GL}(n, \mathbb{C}))$,

$$\mathcal{N}(\mathcal{P}_n) := N_{\mathrm{Int}(\mathrm{GL}(n,\mathbb{C}))}(\mathcal{P}_n),$$

and

$$\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) \subseteq \mathcal{N}(\mathcal{P}_{(n_1,\ldots,n_k)}) \subset \mathrm{Int}(\mathrm{GL}(N, \mathbb{C})).$$

Now the *symmetry group* $\mathsf{Sp}_{[n_1,\ldots,n_k]}$ is defined in several steps.

First let $\mathcal{S}_{[n_1,\ldots,n_k]}$ be a set consisting of $k \times k$ matrices H of 2×2 blocks

$$H_{ij} = \frac{n_i}{\mathsf{gcd}(n_i, n_j)} A_{ij}$$

where $A_{ij} \in \mathsf{M}_2(\mathbb{Z}_{n_i})$ for $i, j = 1, \ldots, k$.

Then $\mathcal{S}_{[n_1,\ldots,n_k]}$ is (with the usual matrix multiplication) a monoid.

Next, for a matrix $H \in \mathcal{S}_{[n_1,\ldots,n_k]}$, we define its adjoint $H^* \in \mathcal{S}_{[n_1,\ldots,n_k]}$ by

$$(H^*)_{ij} = \frac{n_i}{\mathrm{gcd}(n_i, n_j)} A^T_{ji}.$$

Further, we need a skew-symmetric matrix

$$J = \mathrm{diag}(J_2, \ldots, J_2) \in \mathcal{S}_{[n_1,\ldots,n_k]}$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the *symmetry group* is defined as

$$\mathrm{Sp}_{[n_1,\ldots,n_k]} := \{H \in \mathcal{S}_{[n_1,\ldots,n_k]} | \quad H^* J H = J\}$$

and is a finite subgroup of the monoid $\mathcal{S}_{[n_1,\ldots,n_k]}$.

Our theorems state group isomorphism and generating elements:

Theorem 1

$\mathcal{N}(\mathcal{P}_{(n_1,\ldots,n_k)})/\mathcal{P}_{(n_1,\ldots,n_k)} \cong \mathsf{Sp}_{[n_1,\ldots,n_k]}$.

Theorem 2

The normalizer $\mathcal{N}(\mathcal{P}_{(n_1,\ldots,n_k)})$ is generated by

$$\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) \quad \textit{and} \quad \{\mathrm{Ad}_{R_{ij}}\},$$

where (for $1 \le i < j \le k$)

$$R_{ij} = I_{n_1 \cdots n_{i-1}} \otimes \mathrm{diag}(I_{n_{i+1} \cdots n_j}, T_{ij}, \ldots, T_{ij}^{n_i - 1}) \otimes I_{n_{j+1} \cdots n_k}$$

and

$$T_{ij} = I_{n_{i+1} \cdots n_{j-1}} \otimes Q_{n_j}^{\frac{n_j}{\mathrm{gcd}(n_i, n_j)}}.$$

Corollary

If $n_1 = \cdots = n_k = n$, i.e. $N = n^k$, the symmetry group is $\mathrm{Sp}_{[n,\dots,n]} \cong \mathrm{Sp}_{2k}(\mathbb{Z}_n)$.

These cases are of particular interest, since they uncover *symplectic symmetry* of k-partite systems composed of subsystems with the same dimensions. This circumstance was found, to our knowledge, first by PST 2006 for $k = 2$ under additional assumption that $n = p$ is prime, leading to $\mathrm{Sp}(4, \mathrm{F}_p)$ over the field F_p.

Pelantová E, Svobodová M and Tremblay J 2006 Fine grading of $sl(p^2, \mathbb{C})$ generated by tensor product of generalized Pauli matrices and its symmetries *J. Math. Phys.* **47** 5341–5357

Han G 2010 The symmetries of the fine gradings of sl($n^k, \mathbb{C}$) associated with direct product of Pauli groups *J. Math. Phys.* **51** 092104 (15 pages)

Illustrative examples

For simplicity let a bipartite system be created by coupling two single multi-level subsystems with arbitrary dimensions n, m, i.e.

$$G = \mathbb{Z}_n \times \mathbb{Z}_m \quad \text{and} \quad \mathcal{H} = \mathcal{H}_n \otimes \mathcal{H}_m.$$

The corresponding finite Weyl-Heisenberg group is embedded in $\mathrm{GL}(N, \mathbb{C})$, $N = nm$. Via inner automorphisms it induces an Abelian subgroup in $\mathrm{Int}(\mathrm{GL}(N, \mathbb{C}))$.

The Clifford group or the normalizer of this Abelian subgroup in the group of inner automorphisms of $\mathrm{GL}(N, \mathbb{C})$ contains all inner automorphisms transforming the phase space into itself, hence necessarily contains $\mathcal{P}_{(n,m)}$ as an Abelian semidirect factor.

The symmetry group is then given by the quotient group of the normalizer with respect to this Abelian subgroup.

Special attention should be paid to the independent building blocks of finite quantum systems - quantal degrees of freedom. In special cases the symmetry groups are reducible to standard types $\mathrm{SL}(2,\mathbb{Z}_n)$, $\mathrm{Sp}(2k,\mathbb{Z}_n)$, but in general these standard types do not exhaust the obtained class of symmetry groups, e.g. $\mathrm{Sp}_{[p^k,p^l,\dots]}$.

The case of $n=m$, $N=n^2$, corresponds to the symmetry group

$$\mathrm{Sp}_{[n,n]} \cong \mathrm{Sp}(4,\mathbb{Z}_n).$$

If $N=nm$, n,m coprime, the symmetry group is

$$\mathrm{Sp}_{[n,m]} \cong \mathrm{SL}(2,\mathbb{Z}_n)\times \mathrm{SL}(2,\mathbb{Z}_m) \cong \mathrm{SL}(2,\mathbb{Z}_{nm}).$$

Further, if $d = gcd(n, m)$, $n = ad$, $m = bd$, the finite configuration space can be further decomposed under the condition that a, b are both coprime to d,

$$G = \mathbb{Z}_n \times Z_m = Z_{ad} \times Z_{bd} \cong (Z_a \times Z_d) \times (Z_b \times Z_d).$$

Thus the symmetry group is reduced to the direct product

$$\mathsf{Sp}_{[n,m]} \cong \mathsf{Sp}_{[a,b]} \times \mathsf{Sp}(4, \mathbb{Z}_d)$$

For instance, if $n = 15$, $m = 12$, then $d = 3$ is coprime to both $a = 5$ and $b = 4$, and also a and b are coprime, hence the symmetry group is reduced to the standard types SL and Sp,

$$\mathsf{Sp}_{[n,m]} \cong \mathsf{SL}(2, \mathbb{Z}_a) \times \mathsf{SL}(2, \mathbb{Z}_b) \times \mathsf{Sp}(4, \mathbb{Z}_d).$$

It is clear that the general situation may be more complicated. For instance, let $n = 180$ and $m = 150$. Then $d = 30$, $a = 180/30 = 6$ and $b = 150/30 = 5$, hence a divides d and also b divides d. In this case reduction to standard groups Sp and SL is not possible. One has to break down the composite system consisting of two single systems into its elementary building blocks. We decompose each of the finite configuration spaces

$$\mathbb{Z}_{180} \times Z_{150} = (\mathbb{Z}_{2^2} \times Z_{3^2} \times Z_5) \times (\mathbb{Z}_2 \times Z_3 \times Z_{5^2}),$$

and take notice of coprime factors 2.2^2, 3.3^2 and 5.5^2 leading to the factorization of the symmetry group in agreement with the elementary divisor decomposition

$$\mathrm{Sp}_{[180,150]} \cong \mathrm{Sp}_{[2,2^2]} \times \mathrm{Sp}_{[3,3^2]} \times \mathrm{Sp}_{[5,5^2]}.$$

One sees that symmetry groups like $\mathrm{Sp}_{[p^k,p^l]}$ with indices given by different powers of the same prime p deserve to be added to the standard types Sp and SL.

Concluding remarks

The term Clifford group was introduced in 1998 by D. Gottesmann in his investigation of quantum error-correcting codes. The simplest Clifford group in multi-qubit quantum computation is generated by a restricted set of unitary Clifford gates – the Hadamard, $\pi/4$-phase and controlled-X gates. This restriction is known to lead to the

Theorem (Gottesman-Knill)
The Clifford model of quantum computation can be efficiently simulated on a classical computer.

However, this fact does not diminish the importance of the Clifford model, since it may serve as a suitable starting point for full-fledged quantum algorithms.

Thank you for your attention