BGG sequences for infinite-dimensional representations

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Recurrent theme

Rewrite $\mathcal{D}s = 0$ as first order system $\nabla\widetilde{s} = 0$.

...covariant constancy, unfolded form, ...

Algebraic (sub)problem

In the homogeneous case we have $M = G/P$ and the associated homogeneous bundle $\mathcal{V} = G \times_P \mathbb{V} \to G/P$ and the construction gives *invariant differential operators* between sections of such bundles

$$\mathcal{D}\colon \mathcal{C}^\infty(G/P, \mathcal{V}) \to \mathcal{C}^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

Any (linear) differential operator $\mathcal{D}$ of order k is given by a (linear) bundle map from the k-th jet prolongation

$$D\colon \mathcal{C}^\infty(G/P, \mathcal{J}^k\mathcal{V}) \to \mathcal{C}^\infty(G/P, \mathcal{W})$$

From invariance this is equivalent to homomorphism $J^k\mathbb{V} \to \mathbb{W}$, where $J^k\mathbb{V}$ denotes the algebraic jet prolongation of $\mathbb{V}$.
Passing to dual maps and taking the limit $k \to \infty$ we get

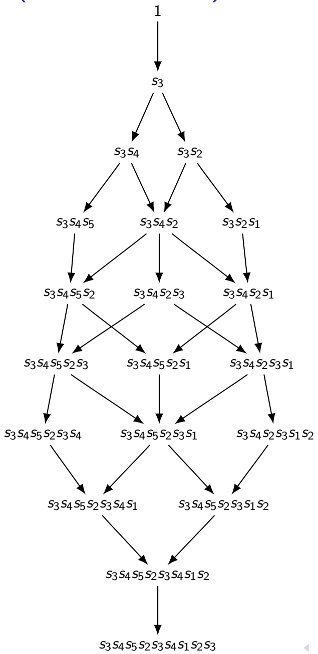
$$\mathrm{Hom}_{\mathfrak{p}}\left(\mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right)$$

BGG resolution

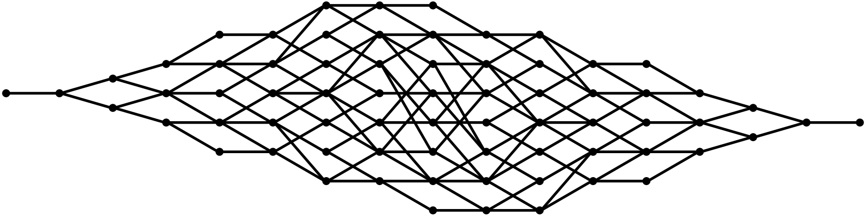
For G semisimple, P parabolic subgroup and $\mathbb{W} = \mathbb{F}_\lambda$ finite-dimensional irreducible P-module we get (generalized / parabolic) Verma modules $M_\lambda = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{F}_\lambda$.
If the highest weight λ determines even finite-dimensional G-module $\mathbb{V}_\lambda$ then we have (projective) BGG resolution

$$\cdots \to \bigoplus_{w \in W^2} M_{w \cdot \lambda} \to \bigoplus_{w \in W^1} M_{w \cdot \lambda} \to M_\lambda \to \mathbb{V}_\lambda \to 0$$

The BGG graph for $(A_5, A_2 \times A_2)$



The BGG graph of type $(A_7, A_3 \times A_3)$



$$\cdots \to \bigoplus_{w \in W^2} M_{w\cdot\lambda} \to \bigoplus_{w \in W^1} M_{w\cdot\lambda} \to M_\lambda \to \mathbb{V}_\lambda \to 0$$

(Curved) BGG sequences – [ČSS01; CD01] I

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type $(\mathfrak{g}, P)$ and let $\mathbb{W}$ be a finite dimensional $(\mathfrak{g}, P)$-module. Then there is a naturally defined sequence

$$\mathcal{C}^\infty(H_0(W)) \stackrel{\mathcal{D}_0}{\to} \mathcal{C}^\infty(H_1(W)) \stackrel{\mathcal{D}_1}{\to} \mathcal{C}^\infty(H_2(W)) \stackrel{\mathcal{D}_2}{\to} \cdots$$

of linear differential operators such that the kernel of the first operator is isomorphic to the parabolic twistors associated to W and the symbols of the differential operators depend only on $(\mathfrak{g}, P, \mathbb{W})$ not (M, ω). If M is flat then this is sequence is locally exact and hence computes the cohomology of M with coefficients in the locally constant sheaf of parallel sections of W.

(Curved) BGG sequences – [ČSS01; CD01] II

Suppose further that $\mathbb{W}_1$, $\mathbb{W}_2$ and $\mathbb{W}_3$ are three finite dimensional $(\mathfrak{g}, P)$-modules with a nontrivial $(\mathfrak{g}, P)$-equivariant linear map $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$ (for instance $\mathbb{W}_3 = \mathbb{W}_1 \otimes \mathbb{W}_2$). Then there are nontrivial bilinear differential pairings

$$\begin{array}{rcl} \mathcal{C}^\infty(H_k(W_1)) \times \mathcal{C}^\infty(H_\ell(W_2)) & \to & \mathcal{C}^\infty(H_{k+\ell}(W_3)) \\ (\alpha, \beta) & \mapsto & \alpha \sqcup_\omega \beta \end{array}$$

whose symbols depend only on $(\mathfrak{g}, P, \mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3)$ and which have the following properties if M is flat: for $k = \ell = 0$ the pairing extends the given pairing of twistors $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$, while more generally the following Leibniz rule holds

$$\mathcal{D}_{k+\ell}(\alpha \sqcup_\omega \beta) = (\mathcal{D}_k \alpha) \sqcup_\omega \beta + (-1)^k \alpha \sqcup_\omega (\mathcal{D}_\ell \beta),$$

and hence the pairing descends to a cup product in cohomology.

(Curved) BGG sequences – [ČSS01; CD01] III

$$\mathcal{D}_{k+\ell}(\alpha \sqcup \beta) = \mathcal{D}_k \alpha \sqcup \beta + (-1)^k \alpha \sqcup \mathcal{D}_\ell \beta - \langle \mathcal{K}, \alpha, \beta \rangle + \langle \alpha, \mathcal{K}, \beta \rangle - \langle \alpha, \beta, \mathcal{K} \rangle$$

$$\begin{aligned} \mathcal{D}_{k+\ell+m-1}\langle \alpha, \beta, \gamma \rangle = {} & (\alpha \sqcup \beta) \sqcup \gamma - \alpha \sqcup (\beta \sqcup \gamma) \\ & - \langle \mathcal{D}_k \alpha, \beta, \gamma \rangle - (-1)^k \langle \alpha, \mathcal{D}_\ell \beta, \gamma \rangle \\ & - (-1)^{k+\ell} \langle \alpha, \beta, \mathcal{D}_m \gamma \rangle \\ & + \langle \mathcal{K}, \alpha, \beta, \gamma \rangle - \langle \alpha, \mathcal{K}, \beta, \gamma \rangle \\ & + \langle \alpha, \beta, \mathcal{K}, \gamma \rangle - \langle \alpha, \beta, \gamma, \mathcal{K} \rangle \end{aligned}$$

(Curved) BGG sequences – [ČSS01; CD01] IV

$$\Box = (\partial^* + \partial)^2 = \partial^* \partial + \partial \partial^*$$
$$\Lambda^* \mathfrak{p}_+ \otimes \mathbb{W} = \operatorname{im} \partial^* \oplus \ker \Box \oplus \operatorname{im} \partial$$

$$\operatorname{proj}_{\ker \Box} = \operatorname{Id} - \Box^{-1} \Box = \operatorname{Id} - \Box^{-1} \partial^* \partial - \partial \Box^{-1} \partial^*$$

$$\partial \rightsquigarrow d^\omega \approx \nabla + \partial + \kappa$$

$$\Box_\omega = \partial^* d^\omega + d^\omega \partial^*$$
$$Q = \Box_\omega^{-1} \partial^*$$
$$\Pi = \operatorname{Id} - Q d^\omega - d^\omega Q$$

Hermitian symmetric spaces

- Compact symmetric space H/K is called Hermitian symmetric space if it admits H-invariant complex structure
- $H/K \simeq G/P$ where G is a complexification of H and P is a parabolic subgroup with abelian radical
- Levi part of P is complexification of the maximal compact subgroup K

$$I_{m,n} = \begin{pmatrix} \mathrm{Id}_m & 0 \\ 0 & -\mathrm{Id}_n \end{pmatrix} \quad J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \quad X^{\dagger} = \overline{X}^t$$

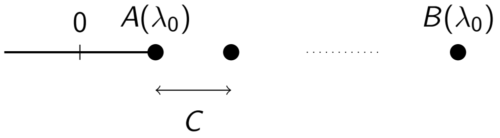
[DES91]

Unitarizable highest weight modules

β ... maximal non-compact root

any weight $\lambda \in \mathfrak{h}^*$ can be written uniquely as $\lambda = \lambda_0 + z\zeta$ where $\langle \zeta, \beta^\vee \rangle = 1$ & $\langle \lambda_0 + \rho, \beta \rangle = 0$

set of $z \in \mathbb{C}$ for which the simple factor of Verma module $L(\lambda)$ is unitarizable:



$A(\lambda_0)$, $B(\lambda_0)$ and $C(\lambda_0)$ are real numbers expressible in terms of certain root systems $Q(\lambda_0)$ and $R(\lambda_0)$ associated to λ_0

$\mathfrak{g}$	$\mathrm{SU}(p,q)$	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{SO}^*(2n)$	$\mathrm{SO}(2,2n-2)$	$\mathrm{SO}(2,2n-1)$
C	1	$\frac{1}{2}$	2	$n-2$	$n-\frac{3}{2}$

BGG resolutions for unitarizable highest weight modules

First operators known on the big affine cell of the homogeneous space and given by translation principle on the cone C – [DES91]. The whole resolution given by

Theorem

For unitarizable highest weight modules $L(\lambda)$ and for $i \in \mathbb{N}$ we have

$$H^i(\mathfrak{p}_+, L(\lambda)) \simeq \bigoplus_{w \in W_\lambda^{c,i}} F(\overline{w(\lambda+\rho)} - \rho)$$

where $\overline{\lambda}$ is the unique Φ_c^+-dominant element in the W_c orbit of λ and $W_\lambda^{c,i} = \{w \in W_\lambda \,|\, w\rho$ is $\Phi_{\lambda,c}^+$-dominant and $l_\lambda(w) = i\}$.

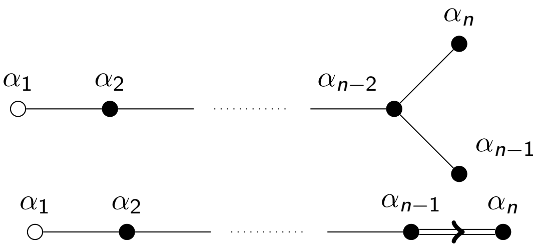
Conformal cases

$$G = SO(n+2, \mathbb{C})$$

$$\mathfrak{p} = \text{stabilizer of a line}$$

$$H = SO(2, n) = \{g \in GL(n, \mathbb{R}) \mid g^t I_{2,n} g = I_{2,n}\}$$

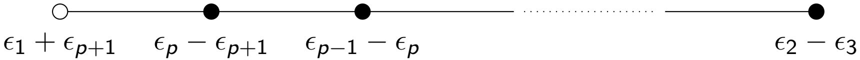
$$K = S(O(2) \times O(n))$$



$(A - a_n - 2n + p)\omega_1 + (a_{p+1} + 1)\omega_{p+1} + \sum_{i=p+2}^{n} a_i\omega_i$

$$G = SO(2n+1, \mathbb{C})$$
$$A = -2(a_{p+1} + \cdots + a_{n-1})$$
$$z \in M_{p,n}(\mathbb{C}),\ T \in \mathbb{C}^n$$
$$I = (1, 3, \ldots, 2p-3),\ J = (2, 4, \ldots, 2p-2)$$

$$\mathcal{D} = \mathcal{F}_T\left[\det(zT^t | z_I - \sqrt{-1}z_J)\right]$$



$$(A - a_n - 2n + p,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1} + 1,\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_n - 2n + p - 1,\ 0,\ 0, \ldots,\ 0,\ 1,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_n - 2n + p - 2,\ 0,\ 0, \ldots,\ 1,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\vdots$$

$$(A - a_n - 2n + 2,\ 0,\ 1, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_n - 2n + 1,\ 1,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_n - 2n + 1,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$(3/2 - n)\omega_1$

$$G = SO(2n+1, \mathbb{C})$$

$$\mathcal{D} = \Delta$$

$(1/2 - n)\omega_1 + \omega_n$

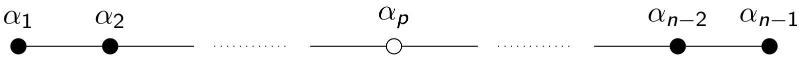
$$G = SO(2n+1, \mathbb{C})$$

$$\mathcal{D} = \not{D}$$

A_n

$$G = SL(p+q, \mathbb{C})$$
$$\mathfrak{p} = M_{p,q}(\mathbb{C})$$

$$H = SU(p,q) = \{g \in GL(p+q, \mathbb{C}) \mid g I_{p,q} g^{\dagger} = I_{p,q}\}$$
$$K = S(U(p) \times U(q))$$



$\omega_{p'} + \omega_{n-q'} - (n + l + 1 - p' - q')\omega_p$

$$G = SL(p+q, \mathbb{C})$$

$$1 \leq p' \leq p, \quad 1 \leq q' \leq q, \quad 1 \leq l \leq \min(p', q')$$

$$\mathcal{D} = \det(\partial_{i,j})_{i \in I, j \in J}$$

$\omega_{p'} + \omega_{n-q'} - (n+l+1-p'-q')\omega_p$

Resolutions built from subset of indices:

$q' - q + p' - l > 0$:

$M_\lambda = \{1, \dots, q'-q+p'-l, p'-l+1, p+1, \dots, m, n-q'+l, m+p+1, \dots, n\}$

$q' - q + p' - l \leq 0$:

$$M_\lambda = \{p' - l + 1, n - q' + l, m + p + 1, \dots, n\}$$

But there is a large family
$\omega_{p'} + \omega_{n-q'} - (n+l+1-p'-q')\omega_p + \lambda$ for $\lambda \in C$ where

$$C = \{a_{p'}\omega_{p'} + \cdots + a_p\omega_p + \cdots + a_{n-q'}\omega_{n-q'} \mid a_p = -a_{p'} - \cdots - a_{p-1} - a_{p+1} - \cdots - a_{n-q'}\}$$

C_n

$$G = Sp(n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) \mid g^t J g = J\}$$

$$\mathfrak{p} = \{A \in M_{n,n}(\mathbb{C}) \mid A^T = A\}$$

$$H = Sp(n, \mathbb{R}) = Sp(n, \mathbb{C}) \cap SU(n, n)$$

$$K = U(n)$$

$$\omega_q + \omega_r - (2 + n - \tfrac{1}{2}(r + q - l + 1))\omega_n$$

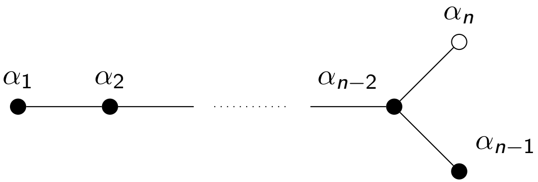
$$1 \leq q \leq r \leq n, \quad 1 \leq l \leq q$$

$$C = \{a_r\omega_r + \cdots + a_n\omega_n \mid a_n = -(a_r + \cdots + a_{n-1})\}$$

D_n

$$G = SO(2n, \mathbb{C})$$
$$\mathfrak{p} = \{A \in M_{n,n}(\mathbb{C}) \mid A^T = -A\}$$

$$H = SO^*(2n) = SU(n,n) \cap \{g \in GL(n, \mathbb{C}) \mid g^t J I_{n,n} g = J I_{n,n}\}$$
$$K = U(n)$$



Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$l(\lambda_a)$
$\omega_2 - (2n-2)\omega_n$	$-(2n-2)\omega_n$	$\mathrm{SU}(1,1)$	1
$\omega_p - 2(n-p+l)\omega_n$	$\omega_{p-2l} - 2(n-p+l)\omega_n$	$\mathrm{SO}^*(2p)$[1]	$1 \leq l \leq \left[\frac{p}{2}\right]$
$\omega_{n-1} - (1+2l)\omega_n$	$\omega_{n-1-2l} - 2(1+l)\omega_n$	$\mathrm{SO}^*(2n-2)$	$1 \leq l \leq \left[\frac{n-1}{2}\right]$
$-(2l-2)\omega_n$	$\omega_{n-2l} - 2l\omega_n$	$\mathrm{SO}^*(2n)$	$1 \leq l \leq \left[\frac{n}{2}\right]$
$\omega_1 + \omega_{q+1} - (2n-q)\omega_n$	$\omega_q - (2n-q)\omega_n$	$\mathrm{SU}(1,q)$[2]	1
$\omega_1 + \omega_{n-1} - (n+1)\omega_n$	$\omega_{n-2} - (n+2)\omega_n$	$\mathrm{SU}(1,n-2)$	1
$\omega_1 - (n-1)\omega_n$	$\omega_{n-1} - n\omega_n$	$\mathrm{SU}(1,n-1)$	1

[1] $3 \leq p \leq n-2$
[2] $2 \leq q \leq n-3$

Table: Vertices and root systems for $\mathrm{SO}^*(2n)$, $n \geq 4$

Let $a = (Q, R, l)$, $Q = R$. Then for $R = \mathrm{SO}^*(2p)$, $3 \leq p \leq n$

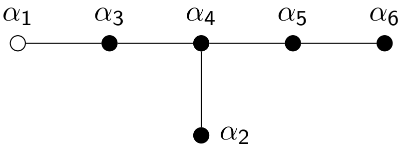
$$C_a = \{a_p\omega_p + \cdots + a_n\omega_n \,|\, a_n = -2a_p - \cdots - 2a_{n-2} - a_{n-1}\}$$

and for $R = \mathrm{SU}(1,q)$, $1 \leq q \leq n-1$

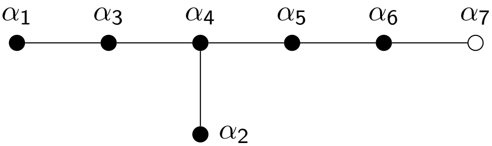
$$C_a = \{a_1\omega_1 + a_{q+1}\omega_{q+1} + \cdots + a_n\omega_n \,|\, a_n = -(a_1 + 2a_{q+1} + \cdots + 2a_{n-2} + a_{n-1}$$

Exceptional cases – see [EH04]

$$E_6^{-14}/U(1)Spin(10) \simeq E_6^{\mathbb{C}}/P_1$$

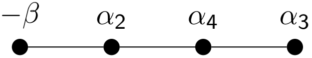


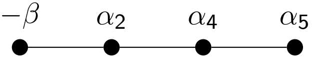
$$E_7^{-25}/U(1)E_6^{\mathrm{cpt}} \simeq E_7^{\mathbb{C}}/P_7$$



E_6

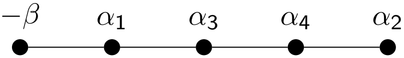
Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$I(\lambda_a)$
$-12\omega_1 + \omega_2$	$-12\omega_1$	$\mathrm{SU}(1,1)$	1
$-12\omega_1 + \omega_4$	$-12\omega_1 + \omega_2$	$\mathrm{SU}(1,2)$	1
$-12\omega_1 + \omega_3 + \omega_5$	$-12\omega_1 + \omega_4$	$\mathrm{SU}(1,3)$	1
$-9\omega_1 + \omega_5$[1]	$-10\omega_1 + \omega_3$	$\mathrm{SU}(1,4)$	1
$-10\omega_1 + \omega_3 + \omega_6$[2]	$-10\omega_1 + \omega_5$	$\mathrm{SU}(1,4)$	1
$-8\omega_1 + \omega_3$	$-8\omega_1 + \omega_6$	$\mathrm{SU}(1,5)$	1
$-5\omega_1 + \omega_6$	$-6\omega_1 + \omega_2$	$\mathrm{SO}(2,8)$	1
$-8\omega_1 + \omega_6$	$-9\omega_1$	$\mathrm{SO}(2,8)$	2
0	$-2\omega_1 + \omega_3$	$EIII$	1
$-3\omega_1$	$-5\omega_1 + \omega_6$	$EIII$	2

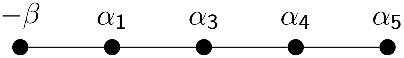
[1] Dynkin diagram 

[2] Dynkin diagram 

E_7

Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$I(\lambda_a)$
$\omega_1 - 18\omega_7$	$-18\omega_7$	$\mathrm{SU}(1,1)$	1
$\omega_3 - 18\omega_7$	$\omega_1 - 18\omega_7$	$\mathrm{SU}(1,2)$	1
$\omega_4 - 18\omega_7$	$\omega_3 - 18\omega_7$	$\mathrm{SU}(1,3)$	1
$\omega_2 + \omega_5 - 18\omega_7$	$\omega_4 - 18\omega_7$	$\mathrm{SU}(1,4)$	1
$\omega_5 - 15\omega_7$ [1]	$\omega_2 - 15\omega_7$	$\mathrm{SU}(1,5)$	1
$\omega_2 + \omega_6 - 16\omega_7$ [2]	$\omega_5 - 16\omega_7$	$\mathrm{SU}(1,5)$	1
$\omega_2 - 13\omega_7$	$\omega_6 - 14\omega_7$	$\mathrm{SU}(1,6)$	1
$\omega_6 - 10\omega_7$	$\omega_1 - 10\omega_7$	$\mathrm{SO}(2,10)$	1
$\omega_6 - 14\omega_7$	$-14\omega_7$	$\mathrm{SO}(2,10)$	2
0	$\omega_6 - 2\omega_7$	*EVII*	1
$-4\omega_7$	$\omega_1 - 6\omega_7$	*EVII*	2
$-8\omega_7$	$-10\omega_7$	*EVII*	3

[1] Dynkin diagram 

[2] Dynkin diagram 

Thank you for attention!

References I

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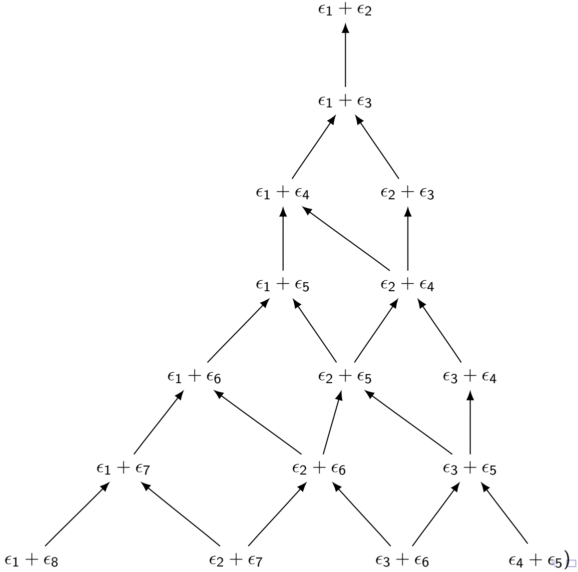
Example – scalar products with positive roots

$\mathrm{SO}^*(16)$:

$\lambda = (a_5 + 1)\,\omega_5 + a_6\omega_6 + a_7\omega_7 - (2\,a_5 + 2\,a_6 + a_7 + 8)\,\omega_8$



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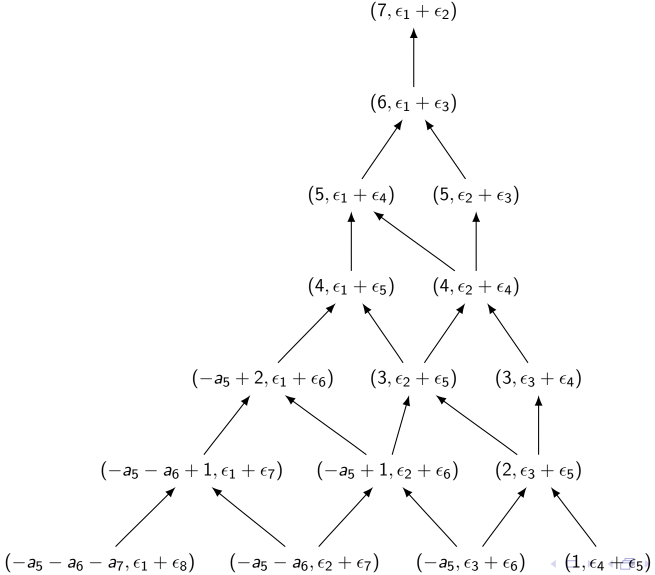

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