Killing equations on Riemannian spaces of constant curvature

Petr Zima
(joint with P. Somberg)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

38th Winter School Geometry and Physics
January 13–20, 2018

Introduction

Killing equations are certain natural systems of partial differential equations defined on a (pseudo-)Riemannian or Spin manifold.

Different types

- Killing vectors are infinitesimal generators of isometries.
- Symmetric Killing tensors and Killing forms are direct generalizations.
- Killing spinors arise on Spin manifolds of constant scalar curvature.
- Killing spinor-valued forms are a combination of K. spinors and forms.

Introduction

Killing equations are certain natural systems of partial differential equations defined on a (pseudo-)Riemannian or Spin manifold.

Different types

- Killing vectors are infinitesimal generators of isometries.
- Symmetric Killing tensors and Killing forms are direct generalizations.
- Killing spinors arise on Spin manifolds of constant scalar curvature.
- Killing spinor-valued forms are a combination of K. spinors and forms.

Properties and applications

- Overdetermined systems of PDEs
- Intrinsic relationship to the *curvature*
- Related to some special additional geometric structures: Sasakian, nearly Kähler, nearly parallel G_2-manifolds
- Integrals of motion for the geodesic equation
- In physics: general relativity, super-gravity, super-symmetry, regarded as *"hidden symmetries"*

Prolongation procedure

Prolongation procedure transforms a wide class of overdetermined systems of PDEs into the equation for a *parallel section*.
It works roughly in the following steps:

1. Introduce new indeterminates so that the first derivative of all indeterminates is completely determined by algebraic terms.
2. Absorb the algebraic terms into a covariant derivative.

Typically it produces *tractor bundles* with a modified tractor connection.

Prolongation procedure

Prolongation procedure transforms a wide class of overdetermined systems of PDEs into the equation for a *parallel section*.
It works roughly in the following steps:

1. Introduce new indeterminates so that the first derivative of all indeterminates is completely determined by algebraic terms.
2. Absorb the algebraic terms into a covariant derivative.

Typically it produces *tractor bundles* with a modified tractor connection.

Elementary approach

In simple cases the prolongation can be deduced ad hoc by repeated differentiations and projections on suitable components.

- The *curvature* appears from skew-symmetrizing a second derivative.

Prolongation procedure

Prolongation procedure transforms a wide class of overdetermined systems of PDEs into the equation for a *parallel section*.
It works roughly in the following steps:

1. Introduce new indeterminates so that the first derivative of all indeterminates is completely determined by algebraic terms.
2. Absorb the algebraic terms into a covariant derivative.

Typically it produces *tractor bundles* with a modified tractor connection.

Elementary approach

In simple cases the prolongation can be deduced ad hoc by repeated differentiations and projections on suitable components.

- The *curvature* appears from skew-symmetrizing a second derivative.

General construction

There is a systematic construction based on the Lie algebra cohomology.

- Main assumption is that the highest weight component of the derivative (the *"Twistor operator"*) is prescribed to vanish.
- The basic construction can be further refined in order to gain desired invariance properties. (Branson, Čap, Eastwood and Gover 2006)

Integrability conditions

Let M be a connected smooth manifold,

- $E \longrightarrow M$ a finite-dimensional smooth vector bundle over M,
- and ∇ an *arbitrary* linear connection in E.

We consider the equation for a **parallel section** $\Phi \in \Gamma(E)$,

$$\nabla_X \Phi = 0, \qquad \forall X \in \mathfrak{T}(M). \tag{E}$$

Integrability conditions

Let M be a connected smooth manifold,

- $E \longrightarrow M$ a finite-dimensional smooth vector bundle over M,
- and ∇ an *arbitrary* linear connection in E.

We consider the equation for a **parallel section** $\Phi \in \Gamma(E)$,

$$\nabla_X \Phi = 0, \qquad \forall X \in \mathcal{T}(M). \tag{E}$$

The *integrability conditions* of (E) say simply that Φ has to be annihilated by the **curvature** $\mathcal{R}$ of ∇ and its derivatives:

$$\mathcal{R}_{X,Y}\,\Phi = 0, \tag{I$_0$}$$

$$(\nabla_{Z_k}(\ldots(\nabla_{Z_1}(\mathcal{R}_{X,Y}))\ldots))\,\Phi = 0, \qquad \forall X, Y, Z_1, \ldots, Z_k \in \mathcal{X}(M). \tag{I$_k$}$$

Integrability conditions

Let M be a connected smooth manifold,

- $E \longrightarrow M$ a finite-dimensional smooth vector bundle over M,
- and ∇ an *arbitrary* linear connection in E.

We consider the equation for a **parallel section** $\Phi \in \Gamma(E)$,

$$\nabla_X \Phi = 0, \qquad \forall X \in \mathfrak{T}(M). \tag{E}$$

The *integrability conditions* of (E) say simply that Φ has to be annihilated by the **curvature** $\mathcal{R}$ of ∇ and its derivatives:

$$\mathcal{R}_{X,Y}\,\Phi = 0, \tag{I$_0$}$$

$$(\nabla_{Z_k}(\ldots(\nabla_{Z_1}(\mathcal{R}_{X,Y}))\ldots))\,\Phi = 0, \qquad \forall X, Y, Z_1, \ldots, Z_k \in \mathfrak{X}(M). \tag{I$_k$}$$

The conditions are tensorial in Φ so we can evaluate them just pointwise and define subspaces consisting of admissible values in the fibers E_x,

$$S_x = \left\{\phi \text{ such that } (\mathrm{I}_0) \text{ and } (\mathrm{I}_k) \text{ hold at } x\right\} \subseteq E_x. \tag{S}$$

Note that $\dim S_x$ is not generally constant but only upper semi-continuous.

Existence results

Under a simple regularity assumption on S_x the integrability conditions are in fact sufficient for the existence of at least a local solution.

Theorem

If $\dim S_x$ *is constant on some neighborhood of* $x_0 \in M$ *then for each* $\phi \in S_{x_0}$ *exists a unique local solution* Φ *of* (E) *such that* $\Phi(x_0) = \phi$.

In detail, the hypothesis ensures that S_x form a smooth vector subbundle $S \subseteq E$ and ∇ consequently restricts to a *flat connection* in S.

Existence results

Under a simple regularity assumption on S_x the integrability conditions are in fact sufficient for the existence of at least a local solution.

Theorem

If $\dim S_x$ *is constant on some neighborhood of* $x_0 \in M$ *then for each* $\phi \in S_{x_0}$ *exists a unique local solution* Φ *of* (E) *such that* $\Phi(x_0) = \phi$.

In detail, the hypothesis ensures that S_x form a smooth vector subbundle $S \subseteq E$ and ∇ consequently restricts to a *flat connection* in S.

Proposition

If (M, E, ∇) *are real-analytic then* $\dim S_x$ *is constant on* M.

Existence results

Under a simple regularity assumption on S_x the integrability conditions are in fact sufficient for the existence of at least a local solution.

Theorem

If $\dim S_x$ *is constant on some neighborhood of* $x_0 \in M$ *then for each* $\phi \in S_{x_0}$ *exists a unique local solution* Φ *of* (E) *such that* $\Phi(x_0) = \phi$.

In detail, the hypothesis ensures that S_x form a smooth vector subbundle $S \subseteq E$ and ∇ consequently restricts to a *flat connection* in S.

Proposition

If (M, E, ∇) *are real-analytic then* $\dim S_x$ *is constant on* M.

In general, there is no upper bound on the order k of differentiation in (I_k) which needs to be considered, even in the real-analytic case. Let us define

$$S_x^l = \left\{\phi \text{ such that } (\mathrm{I}_0) \text{ and } (\mathrm{I}_k) \text{ hold at } x \text{ for } k \le l\right\} \subseteq E_x. \tag{S^l}$$

With assumption on S_x^l we can determine S_x in a finite number of steps.

Proposition

If $\dim S_x^l$ *is constant on some neighborhood of* $x_0 \in M$ *and* $S_{x_0}^l = S_{x_0}^{l-1}$ *then* $S_x = S_x^l = S_x^{l-1}$ *on a (possibly smaller) neighborhood of* x_0.

Homogeneous spaces

Suppose that $M = G/H$ is a simply connected homogeneous space,

- $E \longrightarrow M$ a finite-dimensional *homogeneous* vector bundle over M,
- and ∇ an *invariant* linear connection in E.

Homogeneous spaces

Suppose that $M = G/H$ is a simply connected homogeneous space,

- $E \longrightarrow M$ a finite-dimensional *homogeneous* vector bundle over M,
- and ∇ an *invariant* linear connection in E.

We compare ∇ with the **fundamental derivative** D given by the canonical flat Cartan connection on G/H,

$$\nabla_{\Pi(\xi)}\,\Phi = \mathrm{D}_\xi\,\Phi + C_\xi\,\Phi, \qquad \forall \xi \in \mathcal{A}(M),\ \Phi \in \Gamma(E), \tag{C}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras corresponding to G and H respectively,

- $\mathcal{A}(M) = G \times_H \mathfrak{g}$ is the adjoint tractor bundle over G/H,
- $\Pi\colon \mathcal{A}(M) \longrightarrow \mathcal{T}(M)$ is the canonical projection,
- $\mathrm{D}\colon \mathcal{A}(M) \otimes \Gamma(E) \longrightarrow E$ is the fundamental derivative,
- and $C\colon \mathcal{A}(M) \otimes E \longrightarrow E$ is a tensorial mapping.

Homogeneous spaces

Suppose that $M = G/H$ is a simply connected homogeneous space,

- $E \longrightarrow M$ a finite-dimensional *homogeneous* vector bundle over M,
- and ∇ an *invariant* linear connection in E.

We compare ∇ with the **fundamental derivative** D given by the canonical flat Cartan connection on G/H,

$$\nabla_{\Pi(\xi)}\,\Phi = \mathrm{D}_\xi\,\Phi + C_\xi\,\Phi, \qquad \forall \xi \in \mathcal{A}(M),\ \Phi \in \Gamma(E), \tag{C}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras corresponding to G and H respectively,

- $\mathcal{A}(M) = G \times_H \mathfrak{g}$ is the adjoint tractor bundle over G/H,
- $\Pi\colon \mathcal{A}(M) \longrightarrow \mathcal{T}(M)$ is the canonical projection,
- $\mathrm{D}\colon \mathcal{A}(M) \otimes \Gamma(E) \longrightarrow E$ is the fundamental derivative,
- and $C\colon \mathcal{A}(M) \otimes E \longrightarrow E$ is a tensorial mapping.

Invariance of ∇ implies that C must be G*-equivariant* and hence, by abuse of notation, we can identify it with

- an H-equivariant linear mapping $C\colon \mathfrak{g} \otimes E_o \longrightarrow E_o$,

where E_o is the fiber of E at the origin $o \in M$.

Homogeneous spaces

Using the Ricci identity we can write the integrability conditions in form

$$[C_\xi, C_\nu]\,\phi = C_{\{\xi,\nu\}}\,\phi, \tag{CI$_0$}$$

$$[C_\xi, C_\nu]\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi = C_{\{\xi,\nu\}}\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi, \tag{CI$_k$}$$

$$\forall \xi, \nu, \zeta_1, \ldots, \zeta_k \in \mathfrak{g},$$

where [,] denotes the commutator in $\mathcal{L}(E_o, E_o)$ and {,} the Lie bracket in $\mathfrak{g}$.

Homogeneous spaces

Using the Ricci identity we can write the integrability conditions in form

$$[C_\xi, C_\nu]\,\phi = C_{\{\xi,\nu\}}\,\phi, \tag{CI$_0$}$$

$$[C_\xi, C_\nu]\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi = C_{\{\xi,\nu\}}\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi, \tag{CI$_k$}$$

$$\forall \xi, \nu, \zeta_1, \ldots, \zeta_k \in \mathfrak{g},$$

where [,] denotes the commutator in $\mathcal{L}(E_o, E_o)$ and {,} the Lie bracket in $\mathfrak{g}$.

- $S_o \subseteq E_o$ is the maximal subspace on which the linear mapping C restricts to a **representation of $\mathfrak{g}$**.

The subbundle $S \subseteq E$ becomes a (flat) **tractor bundle** over G/H via $\exp C$ and the parallel sections are explicitly given by the formula

$$\Phi(go) = L_g((\exp C)(g^{-1})\,\phi), \qquad \forall g \in G,\ \phi \in S_o, \tag{F}$$

where L_g denotes the left action of G on E.

Homogeneous spaces

Using the Ricci identity we can write the integrability conditions in form

$$[C_\xi, C_\nu]\,\phi = C_{\{\xi,\nu\}}\,\phi, \tag{CI$_0$}$$

$$[C_\xi, C_\nu]\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi = C_{\{\xi,\nu\}}\,C_{\zeta_k}\cdots C_{\zeta_1}\,\phi, \tag{CI$_k$}$$

$$\forall \xi, \nu, \zeta_1, \ldots, \zeta_k \in \mathfrak{g},$$

where [,] denotes the commutator in $\mathcal{L}(E_o, E_o)$ and {,} the Lie bracket in $\mathfrak{g}$.

- $S_o \subseteq E_o$ is the maximal subspace on which the linear mapping C restricts to a **representation of $\mathfrak{g}$**.

The subbundle $S \subseteq E$ becomes a (flat) **tractor bundle** over G/H via $\exp C$ and the parallel sections are explicitly given by the formula

$$\Phi(go) = L_g((\exp C)(g^{-1})\,\phi), \qquad \forall g \in G,\ \phi \in S_o, \tag{F}$$

where L_g denotes the left action of G on E.
The regularity of S_x^l is satisfied thanks to homogeneity, so the sequence $S_o^0 \supseteq \cdots \supseteq S_o^l \supseteq \cdots \supseteq S_o$ definitely stabilizes once it does not decrease.
Finding S_o is therefore *completely algorithmic*. We have implemented this algorithm in CAS for Kiling forms, spinors and spinor-valued forms.

Cone construction

The **ε-metric cone** over a pseudo-Riemannian manifold (M, g) is

- the warped product $\overline{M} = M \times \mathbb{R}_+$ with metric $\overline{g} = r^2 g + \varepsilon \, \mathrm{d}r^2$,

where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$.

Cone construction

The ***ε*-metric cone** over a pseudo-Riemannian manifold (M, g) is

- the warped product $\overline{M} = M \times \mathbb{R}_+$ with metric $\overline{g} = r^2 g + \varepsilon\, \mathrm{d}r^2$,

where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$.

We associate to a vector field X on M a vector field $\overline{X}$ on $\overline{M}$ by

$$\overline{X} = \frac{1}{r}\, p_1^*(X), \tag{MC1}$$

where p_1^* denotes pull-back along the canonical projection $p_1 \colon \overline{M} \longrightarrow M$.

Cone construction

The **ε-metric cone** over a pseudo-Riemannian manifold (M, g) is

- the warped product $\overline{M} = M \times \mathbb{R}_+$ with metric $\overline{g} = r^2 g + \varepsilon\, \mathrm{d}r^2$,

where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$.

We associate to a vector field X on M a vector field $\overline{X}$ on $\overline{M}$ by

$$\overline{X} = \tfrac{1}{r}\, p_1^*(X), \tag{MC1}$$

where p_1^* denotes pull-back along the canonical projection $p_1 \colon \overline{M} \longrightarrow M$.

The corresponding Levi-Civita connections on M and $\overline{M}$ are related by

$$\begin{aligned} \overline{\nabla^g}_{\overline{X}}\, \overline{Y} &= \tfrac{1}{r}\big(\overline{\nabla^g_X Y} - \varepsilon g(X, Y)\, \partial_r\big), & \overline{\nabla^g}_{\partial_r}\, \overline{X} &= 0, \\ \overline{\nabla^g}_{\overline{X}}\, \partial_r &= \tfrac{1}{r}\, \overline{X}, & \overline{\nabla^g}_{\partial_r}\, \partial_r &= 0. \end{aligned} \tag{MC2}$$

Cone construction

The ***ε*-metric cone** over a pseudo-Riemannian manifold (M, g) is

- the warped product $\overline{M} = M \times \mathbb{R}_+$ with metric $\overline{g} = r^2 g + \varepsilon\, \mathrm{d}r^2$,

where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$.

We associate to a vector field X on M a vector field $\overline{X}$ on $\overline{M}$ by

$$\overline{X} = \tfrac{1}{r}\, p_1^*(X), \tag{MC1}$$

where p_1^* denotes pull-back along the canonical projection $p_1 \colon \overline{M} \longrightarrow M$.

The corresponding Levi-Civita connections on M and $\overline{M}$ are related by

$$\begin{aligned} \overline{\nabla^g_{\overline{X}}}\, \overline{Y} &= \tfrac{1}{r}\big(\overline{\nabla^g_X Y} - \varepsilon g(X, Y)\, \partial_r\big), & \overline{\nabla^g_{\partial_r}}\, \overline{X} &= 0, \\ \overline{\nabla^g_{\overline{X}}}\, \partial_r &= \tfrac{1}{r}\, \overline{X}, & \overline{\nabla^g_{\partial_r}}\, \partial_r &= 0. \end{aligned} \tag{MC2}$$

The so called cone constructions establish a correspondence between

- solutions of natural systems of PDEs like the Killing equations on M
- and parallel sections of suitable vector bundles over $\overline{M}$.

In fact we can view the natural bundles over $\overline{M}$ as **tractor bundles** over M.

Killing forms

From now on let (M, g) be a pseudo-Riemannian manifold,
∇^g the Levi-Civita connection and $\mathcal{R}^g$ its curvature.

Definition

A p-form α on M is a **Killing form** if there exists a $(p+1)$-form β, such that

$$\nabla^g_X \alpha = X \lrcorner \beta. \tag{KF1}$$

Killing forms

From now on let (M, g) be a pseudo-Riemannian manifold,
∇^g the Levi-Civita connection and $\mathcal{R}^g$ its curvature.

Definition

A p-form α on M is a **Killing form** if there exists a $(p+1)$-form β, such that

$$\nabla^g_X \alpha = X \lrcorner \beta. \tag{KF1}$$

Prolongation

$$\nabla^g_X \beta = \frac{1}{p} \mathcal{R}^g_X \wedge \alpha, \tag{KF2}$$

$$\mathcal{R}^g_X \wedge \alpha = \sum_{i=1}^{n} e^i \wedge (\mathcal{R}^g_{X, e_i} \alpha).$$

Killing forms

From now on let (M, g) be a pseudo-Riemannian manifold,
∇^g the Levi-Civita connection and $\mathcal{R}^g$ its curvature.

Definition

A p-form α on M is a **Killing form** if there exists a $(p+1)$-form β, such that

$$\nabla^g_X \alpha = X \lrcorner \beta. \tag{KF1}$$

Prolongation

$$\nabla^g_X \beta = \frac{1}{p} \mathcal{R}^g_X \wedge \alpha, \tag{KF2}$$

$$\mathcal{R}^g_X \wedge \alpha = \sum_{i=1}^{n} e^i \wedge (\mathcal{R}^g_{X, e_i} \alpha).$$

1st curvature condition

$$\mathcal{R}^g_{X,Y} \alpha + \frac{1}{p}\Big(X \lrcorner (\mathcal{R}^g_Y \wedge \alpha) - Y \lrcorner (\mathcal{R}^g_X \wedge \alpha)\Big) = 0. \tag{KF3}$$

- The condition is *void* for 1-forms.

Killing forms

Definition

A Killing p-form α on M is **special** if additionally holds

$$\nabla^g_X \beta = -cX^* \wedge \alpha, \tag{KF4}$$

for some constant $c \in \mathbb{R}$ and where X^* is the metric dual of X.

The equation (KF4) considered alone (with exchanged roles of α and β) is just a Hodge star dualization of (KF1) called **$*$-Killing equation**.

Killing forms

Definition

A Killing p-form α on M is **special** if additionally holds

$$\nabla^g_X \beta = -cX^* \wedge \alpha, \tag{KF4}$$

for some constant $c \in \mathbb{R}$ and where X^* is the metric dual of X.

The equation (KF4) considered alone (with exchanged roles of α and β) is just a Hodge star dualization of (KF1) called **$*$-Killing equation**.

Cone construction

We associate to a p-form α on M a p-form $\overline{\alpha}$ on $\overline{M}$ by

$$\overline{\alpha} = r^p p_1^*(\alpha). \tag{MC3}$$

Proposition

Let α be a p-form and β a $(p+1)$-form on M. The $(p+1)$-form Θ on $\overline{M}$,

$$\Theta = \mathrm{d}r \wedge \overline{\alpha} + \overline{\beta}, \tag{KF5}$$

*is **parallel** if and only if α together with β is **special Killing** with $c = \varepsilon$.*

(Semmelmann 2003)

Killing spinors

Further suppose that (M, g) is a Spin manifold and denote by ∇^g and $\mathcal{R}^g$ also the corresponding spin connection and its curvature respectively.

Definition

A spinor field Ψ on M is a **Killing spinor** if

$$\nabla^g_X \Psi = aX \cdot \Psi, \tag{KS1}$$

for some constant $a \in \mathbb{C}$ called the **Killing number** and where '$\cdot$' denotes the Clifford multiplication.

- The prolongation is trivial since (KS1) is already closed.

Killing spinors

Further suppose that (M, g) is a Spin manifold and denote by ∇^g and $\mathcal{R}^g$ also the corresponding spin connection and its curvature respectively.

Definition

A spinor field Ψ on M is a **Killing spinor** if

$$\nabla^g_X \Psi = aX \cdot \Psi, \tag{KS1}$$

for some constant $a \in \mathbb{C}$ called the **Killing number** and where '$\cdot$' denotes the Clifford multiplication.

- The prolongation is trivial since (KS1) is already closed.

1^{st} curvature condition

$$\mathcal{R}^g_{X,Y} \Psi + a^2(X \cdot Y - Y \cdot X) \cdot \Psi, = 0. \tag{KS2}$$

- Implies that (M, g) must be *Einstein*. (Friedrich 1980)

Killing spinors

Cone construction

The cone $\overline{M}$ is clearly homotopy equivalent to M, hence any spin structure on M determines a unique spin structure on $\overline{M}$. We denote

- by Σ and $\overline{\Sigma}$ the associated spinor bundles on M and $\overline{M}$ respectively.

Killing spinors

Cone construction

The cone $\overline{M}$ is clearly homotopy equivalent to M, hence any spin structure on M determines a unique spin structure on $\overline{M}$. We denote

- by Σ and $\overline{\Sigma}$ the associated spinor bundles on M and $\overline{M}$ respectively.

The pullback bundle $p_1^*(\Sigma)$ is naturally a subbundle of $\overline{\Sigma}$ and we associate to a spinor field Ψ on M spinor fields $\overline{\Psi}_\pm$ on $\overline{M}$ by

$$\overline{\Psi}_\pm = (1 \mp \sqrt{\varepsilon}\partial_r) \cdot p_1^*(\Psi). \tag{MC4}$$

The two choices of sign of the square root yield inequivalent though analogous results and we consider both of them.

Killing spinors

Cone construction

The cone $\overline{M}$ is clearly homotopy equivalent to M, hence any spin structure on M determines a unique spin structure on $\overline{M}$. We denote

- by Σ and $\overline{\Sigma}$ the associated spinor bundles on M and $\overline{M}$ respectively.

The pullback bundle $p_1^*(\Sigma)$ is naturally a subbundle of $\overline{\Sigma}$ and we associate to a spinor field Ψ on M spinor fields $\overline{\Psi}_\pm$ on $\overline{M}$ by

$$\overline{\Psi}_\pm = (1 \mp \sqrt{\varepsilon}\partial_r) \cdot p_1^*(\Psi). \tag{MC4}$$

The two choices of sign of the square root yield inequivalent though analogous results and we consider both of them.

Proposition

*Let Ψ be a spinor field on M. The associated spinor field $\overline{\Psi}_\pm$ on $\overline{M}$ is **parallel** if and only if Ψ is **Killing** with $a = \pm\frac{1}{2}\sqrt{\varepsilon}$.* *(Bär 1993; Bohle 2003)*

Killing spinor-valued forms

Definition

A spinor-valued p-form Φ on M is a **Killing spinor-valued form** if there exists a spinor-valued $(p+1)$-form Ξ, such that

$$\nabla^g_X \Phi = aX \cdot \Phi + X \lrcorner \Xi, \tag{KSF1}$$

for some constant $a \in \mathbb{C}$ called the **Killing number**.

Killing spinor-valued forms

Definition

A spinor-valued p-form Φ on M is a **Killing spinor-valued form** if there exists a spinor-valued $(p+1)$-form Ξ, such that

$$\nabla^g_X \Phi = aX \cdot \Phi + X \lrcorner \Xi, \tag{KSF1}$$

for some constant $a \in \mathbb{C}$ called the **Killing number**.

Prolongation

We absorb the Clifford multiplication term into covariant derivative ∇^a,

$$\nabla^a_X \Psi = \nabla^g_X \Psi - aX \cdot \Psi, \qquad \forall \Psi \in \Gamma(\Sigma), \tag{KSF2}$$

and extend it to spinor-valued forms by the Levi-Civita connection.

Killing spinor-valued forms

Definition

A spinor-valued p-form Φ on M is a **Killing spinor-valued form** if there exists a spinor-valued $(p+1)$-form Ξ, such that

$$\nabla^g_X \Phi = aX \cdot \Phi + X \lrcorner \Xi, \tag{KSF1}$$

for some constant $a \in \mathbb{C}$ called the **Killing number**.

Prolongation

We absorb the Clifford multiplication term into covariant derivative ∇^a,

$$\nabla^a_X \Psi = \nabla^g_X \Psi - aX \cdot \Psi, \qquad \forall \Psi \in \Gamma(\Sigma), \tag{KSF2}$$

and extend it to spinor-valued forms by the Levi-Civita connection.

$$\nabla^g_X \Xi = aX \cdot \Xi + \frac{1}{p}\left(\mathcal{R}^a_X \wedge \Phi - \frac{1}{2(p+1)} X \lrcorner (\mathcal{R}^a \wedge \Phi)\right), \tag{KSF3}$$

$$\mathcal{R}^a_X \wedge \Phi = \sum_{i=1}^n e^i \wedge (\mathcal{R}^a_{X,e_i} \Phi), \qquad \mathcal{R}^a \wedge \Phi = \sum_{i=1}^n e^i \wedge (\mathcal{R}^a_{e_i} \wedge \Phi),$$

where $\mathcal{R}^a$ denotes the curvature of ∇^a.

Killing spinor-valued forms

1st curvature condition

$$\mathcal{R}^{a}_{X,Y}\,\Phi + \tfrac{1}{p}\Big(X \lrcorner (\mathcal{R}^{a}_{Y} \wedge \Phi) - Y \lrcorner (\mathcal{R}^{a}_{X} \wedge \Phi) - - \tfrac{1}{p+1}\, X \lrcorner Y \lrcorner (\mathcal{R}^{a} \wedge \Phi)\Big) = 0. \tag{KSF4}$$

- The condition is again *void* for spinor-valued 1-forms.

Killing spinor-valued forms

1st curvature condition

$$\mathcal{R}^{a}_{X,Y}\,\Phi + \tfrac{1}{p}\Big(X \lrcorner (\mathcal{R}^{a}_{Y} \wedge \Phi) - Y \lrcorner (\mathcal{R}^{a}_{X} \wedge \Phi) - \\ - \tfrac{1}{p+1}\, X \lrcorner Y \lrcorner (\mathcal{R}^{a} \wedge \Phi)\Big) = 0. \tag{KSF4}$$

- The condition is again *void* for spinor-valued 1-forms.

Definition

A Killing spinor-valued p-form Φ on M is **special** if additionally holds

$$\nabla^{g}_{X}\,\Xi = aX \cdot \Xi - cX^{*} \wedge \Phi, \tag{KSF5}$$

for some constant $c \in \mathbb{R}$.

As in the case of ordinary forms, the equation (KSF5) considered alone (with exchanged roles of Φ and Ξ) is just a Hodge star dualization of (KSF1) called **$*$-Killing equation**.

Killing spinor-valued forms

Cone construction

We associate to a spinor-valued p-form Φ on M spinor-valued p-forms $\overline{\Phi}_\pm$ on the ε-metric cone $\overline{M}$ by

$$\overline{\Phi}_\pm = r^p(1 \mp \sqrt{\varepsilon}\partial_r) \cdot p_1^*(\Phi). \tag{MC5}$$

Proposition

Let Φ be a spinor-valued p-form and Ξ a spinor-valued $(p+1)$-form on M. The spinor-valued $(p+1)$-form $\Theta_\pm$ on $\overline{M}$,

$$\Theta_\pm = \mathrm{d}r \wedge \overline{\Phi}_\pm + \overline{\Xi}_\pm, \tag{KSF6}$$

is ***parallel*** *if and only if Φ together with Ξ is* ***special Killing*** *with $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ and $c = \varepsilon$.*

(Somberg, Zima 2016)

Constant curvature spaces

Let $\overline{M} = \mathbb{R}^{n+1} \setminus \{0\}$ with the standard inner product $\overline{g}$ of signature $(\overline{p}, \overline{q})$, and (M, g) be the pseudo-Riemannian submanifold

$$M = \left\{x \in \overline{M} \;\middle|\; \overline{g}(x, x) = \varepsilon\right\}, \tag{CC1}$$

with the inherited metric g, where $\varepsilon = \pm 1$.

Constant curvature spaces

Let $\overline{M} = \mathbb{R}^{n+1} \setminus \{0\}$ with the standard inner product $\overline{g}$ of signature $(\overline{p}, \overline{q})$, and (M, g) be the pseudo-Riemannian submanifold

$$M = \left\{x \in \overline{M} \;\middle|\; \overline{g}\,(x, x) = \varepsilon\right\}, \tag{CC1}$$

with the inherited metric g, where $\varepsilon = \pm 1$.

- M has **constant curvature** ε.
- M can be viewed as the **homogeneous space** $\mathrm{O}(\overline{p}, \overline{q}) \,/\, \mathrm{O}(p, q)$.
- $\overline{M}$ is the ***ε*-metric cone** over M.
- The pullbacks of natural bundles over $\overline{M}$ along the embedding $i \colon M \hookrightarrow \overline{M}$ are trivial and can be viewed as **tractor bundles** over M.

Constant curvature spaces

Let $\overline{M} = \mathbb{R}^{n+1} \setminus \{0\}$ with the standard inner product $\overline{g}$ of signature $(\overline{p}, \overline{q})$, and (M, g) be the pseudo-Riemannian submanifold

$$M = \left\{x \in \overline{M} \;\middle|\; \overline{g}\,(x, x) = \varepsilon\right\}, \tag{CC1}$$

with the inherited metric g, where $\varepsilon = \pm 1$.

- M has **constant curvature** ε.
- M can be viewed as the **homogeneous space** $\mathrm{O}(\overline{p}, \overline{q}) \,/\, \mathrm{O}(p, q)$.
- $\overline{M}$ is the **ε-metric cone** over M.
- The pullbacks of natural bundles over $\overline{M}$ along the embedding $i \colon M \hookrightarrow \overline{M}$ are trivial and can be viewed as **tractor bundles** over M.

The Levi-Civita connection on $\overline{M}$ is just the usual *partial derivative* ∂.

- Restriction of ∂ to M is the canonical (flat) **tractor connection**.

The Levi-Civita connection on M and its curvature are given by

$$\nabla^g_X = \partial_X + \varepsilon(x \wedge X), \qquad \mathcal{R}^g_{X,Y} = \varepsilon(X \wedge Y), \tag{CC2}$$

where $x \wedge X$ and $X \wedge Y$ are respective elementary matrices in $\mathfrak{so}(\overline{p}, \overline{q})$.

Constant curvature spaces

Killing forms

$$\left.\begin{aligned}\alpha(x) &= x \lrcorner \Theta \\ \beta(x) &= \pi(\Theta)\end{aligned}\right\} \quad\Leftrightarrow\quad \Theta = \varepsilon x^* \wedge \alpha(x) + \beta(x) \tag{CC3}$$

Killing spinors

$$\Psi(x) = \tfrac{1}{2}\,\varphi_\pm \cdot \Theta \quad\Leftrightarrow\quad \Theta = \varphi_\mp \cdot \Psi(x) \tag{CC4}$$

Killing spinor-valued forms

$$\left.\begin{aligned}\Phi(x) &= \tfrac{1}{2}\,\varphi_\pm \cdot (x \lrcorner \Theta) \\ \Xi(x) &= \tfrac{1}{2}\,\varphi_\pm \cdot \pi(\Theta)\end{aligned}\right\} \quad\Leftrightarrow\quad \Theta = \varphi_\mp \cdot \big(\varepsilon x^* \wedge \Phi(x) + \Xi(x)\big) \tag{CC5}$$

where we denote

$$\pi(\Theta) = \Theta - \varepsilon x^* \wedge (x \lrcorner \Theta), \qquad \varphi_\pm = 1 \pm \sqrt{\varepsilon}x.$$

Constant curvature spaces

Killing forms

$$\left.\begin{aligned} \alpha(x) &= x \lrcorner \Theta \\ \beta(x) &= \pi(\Theta) \end{aligned}\right\} \quad \Leftrightarrow \quad \Theta = \varepsilon x^* \wedge \alpha(x) + \beta(x) \tag{CC3}$$

Killing spinors

$$\Psi(x) = \tfrac{1}{2}\, \varphi_{\pm} \cdot \Theta \quad \Leftrightarrow \quad \Theta = \varphi_{\mp} \cdot \Psi(x) \tag{CC4}$$

Killing spinor-valued forms

$$\left.\begin{aligned} \Phi(x) &= \tfrac{1}{2}\, \varphi_{\pm} \cdot (x \lrcorner \Theta) \\ \Xi(x) &= \tfrac{1}{2}\, \varphi_{\pm} \cdot \pi(\Theta) \end{aligned}\right\} \quad \Leftrightarrow \quad \Theta = \varphi_{\mp} \cdot \big(\varepsilon x^* \wedge \Phi(x) + \Xi(x)\big) \tag{CC5}$$

where we denote

$$\pi(\Theta) = \Theta - \varepsilon x^* \wedge (x \lrcorner \Theta), \qquad \varphi_{\pm} = 1 \pm \sqrt{\varepsilon} x.$$

- All the solutions are *special* and the *Killing number* is $a = \pm\frac{1}{2}\sqrt{\varepsilon}$.
- The *1st curvature conditions* rule out other solutions except for spinor-valued forms in degree $p = 1$.

New solutions

Algebraic decomposition

We can decompose the space of spinor-valued forms using the technique of *Howe dual pairs*. The degree raising and lowering algebraic operators,

$$\gamma \cdot \wedge \Phi = \sum_{i=1}^{n} e^i \wedge (e_i \cdot \Phi), \qquad \gamma^{*}\!\cdot \lrcorner\, \Phi = \sum_{i,j=1}^{n} g^{ij} e_i \lrcorner\, (e_j \cdot \Phi), \tag{G1}$$

are Spin-equivariant and generate a Lie algebra isomorphic to $\mathfrak{sl}(2)$.

New solutions

Algebraic decomposition

We can decompose the space of spinor-valued forms using the technique of *Howe dual pairs*. The degree raising and lowering algebraic operators,

$$\gamma\cdot \wedge \Phi = \sum_{i=1}^{n} e^i \wedge (e_i \cdot \Phi), \qquad \gamma^*_{\cdot} \lrcorner \Phi = \sum_{i,j=1}^{n} g^{ij} e_i \lrcorner (e_j \cdot \Phi), \tag{G1}$$

are Spin-equivariant and generate a Lie algebra isomorphic to $\mathfrak{sl}(2)$.

- A spinor-valued p-form Φ is called **primitive** if $\gamma^*_{\cdot} \lrcorner \Phi = 0$.
- Projection on the primitive component in degree $p = 1$ is given by

$$\pi_{\mathrm{Tw}}(\Phi) = \Phi + \frac{1}{n}\gamma\cdot \wedge (\gamma^*_{\cdot} \lrcorner \Phi). \tag{G2}$$

New solutions

Algebraic decomposition

We can decompose the space of spinor-valued forms using the technique of *Howe dual pairs*. The degree raising and lowering algebraic operators,

$$\gamma\cdot \wedge \Phi = \sum_{i=1}^{n} e^i \wedge (e_i \cdot \Phi), \qquad \gamma^*\!\cdot \lrcorner\, \Phi = \sum_{i,j=1}^{n} g^{ij} e_i \lrcorner\, (e_j \cdot \Phi), \tag{G1}$$

are Spin-equivariant and generate a Lie algebra isomorphic to $\mathfrak{sl}(2)$.

- A spinor-valued p-form Φ is called **primitive** if $\gamma^*\!\cdot \lrcorner\, \Phi = 0$.
- Projection on the primitive component in degree $p = 1$ is given by

$$\pi_{\mathrm{Tw}}(\Phi) = \Phi + \frac{1}{n}\, \gamma\cdot \wedge (\gamma^*\!\cdot \lrcorner\, \Phi). \tag{G2}$$

Higher curvature conditions

The higher curvature conditions for Killing spinor-valued forms in degree $p = 1$ allow only one other possibility on M besides $a = \pm\frac{1}{2}\sqrt{\varepsilon}$,

$$a = \pm\frac{3}{2}\sqrt{\varepsilon}, \qquad \Xi = \mp\sqrt{\varepsilon}\,(\gamma\cdot \wedge \Phi). \tag{CC6}$$

New solutions

The new solutions in degree $p = 1$ hence satisfy a stronger equation,

$$\nabla^g_X \Phi = \pm\sqrt{\varepsilon}\left(\tfrac{3}{2}\, X \cdot \Phi - X \lrcorner (\gamma\cdot \wedge \Phi)\right). \tag{KSF7}$$

Cone construction

They correspond to constant **primitive spinor-valued 1-forms** on $\overline{M}$,

$$\begin{aligned} \Phi(x) = \tfrac{1}{2}\left(\varphi_\pm \cdot \pi(\Theta) \pm \sqrt{\varepsilon}\,\gamma\cdot \wedge (\varphi_\pm \cdot (x \lrcorner \Theta))\right) \qquad &\Leftrightarrow \\ \Leftrightarrow \qquad \Theta = \overline{\pi_{\mathrm{Tw}}}\left(\varphi_\mp \cdot \Phi(x)\right). \end{aligned} \tag{CC7}$$

New solutions

The new solutions in degree $p = 1$ hence satisfy a stronger equation,

$$\nabla^g_X \Phi = \pm\sqrt{\varepsilon}\left(\tfrac{3}{2}\, X \cdot \Phi - X \lrcorner (\gamma\cdot \wedge \Phi)\right). \tag{KSF7}$$

Cone construction

They correspond to constant **primitive spinor-valued 1-forms** on $\overline{M}$,

$$\begin{aligned}\Phi(x) = \tfrac{1}{2}\left(\varphi_\pm \cdot \pi(\Theta) \pm \sqrt{\varepsilon}\,\gamma\cdot \wedge (\varphi_\pm \cdot (x \lrcorner \Theta))\right) \qquad &\Leftrightarrow \\ \Leftrightarrow \qquad \Theta = \overline{\pi_{\mathrm{Tw}}}\left(\varphi_\mp \cdot \Phi(x)\right).& \end{aligned} \tag{CC7}$$

- Do not attain the maximal dimension from the prolongation (KSF3).
- Not special in the sense of (KSF5).
- *Not spanned by tensor products* $\alpha \otimes \Psi$ of K. spinors and forms.
- $\Phi(x)$ cannot take values solely in the primitive component.

New solutions

The new solutions in degree $p = 1$ hence satisfy a stronger equation,

$$\nabla^g_X \Phi = \pm\sqrt{\varepsilon}\left(\tfrac{3}{2} X \cdot \Phi - X \lrcorner (\gamma \cdot \wedge \Phi)\right). \tag{KSF7}$$

Cone construction

They correspond to constant **primitive spinor-valued 1-forms** on $\overline{M}$,

$$\begin{aligned} \Phi(x) = \tfrac{1}{2}\left(\varphi_\pm \cdot \pi(\Theta) \pm \sqrt{\varepsilon}\, \gamma \cdot \wedge (\varphi_\pm \cdot (x \lrcorner \Theta))\right) \qquad \Leftrightarrow \\ \Leftrightarrow \qquad \Theta = \overline{\pi_{\mathrm{Tw}}}\left(\varphi_\mp \cdot \Phi(x)\right). \end{aligned} \tag{CC7}$$

- Do not attain the maximal dimension from the prolongation (KSF3).
- Not special in the sense of (KSF5).
- *Not spanned by tensor products* $\alpha \otimes \Psi$ of K. spinors and forms.
- $\Phi(x)$ cannot take values solely in the primitive component.

Comparing (CC7) with (CC5) we get that the new solutions Φ are just a transformation of **special Killing forms** (Φ', Ξ') **in order** $p = 0$,

$$\Phi = \Xi' \pm \sqrt{\varepsilon}\, \gamma \cdot \wedge \Phi'. \tag{KSF8}$$

THANK YOU FOR YOUR ATTENTION!