Killing equations on Riemannian spaces of constant curvature

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Introduction

Prolongation procedure

Integrability conditions

Homogeneous spaces

Cone construction

Killing equations

Constant curvature spaces
Introduction

**Killing equations** are certain natural systems of partial differential equations defined on a (pseudo-)Riemannian or Spin manifold.

**Different types**

- Killing vectors are infinitesimal generators of isometries.
- Symmetric Killing tensors and Killing forms are direct generalizations.
- Killing spinors arise on Spin manifolds of constant scalar curvature.
- Killing spinor-valued forms are a combination of K. spinors and forms.
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**Properties and applications**

- Overdetermined systems of PDEs
- Intrinsic relationship to the *curvature*
- Related to some special additional geometric structures: Sasakian, nearly Kähler, nearly parallel $G_2$-manifolds
- Integrals of motion for the geodesic equation
- In physics: general relativity, super-gravity, super-symmetry, regarded as “hidden symmetries”
Prolongation procedure

Prolongation procedure transforms a wide class of overdetermined systems of PDEs into the equation for a parallel section. It works roughly in the following steps:

1. Introduce new indeterminates so that the first derivative of all indeterminates is completely determined by algebraic terms.
2. Absorb the algebraic terms into a covariant derivative.

Typically it produces tractor bundles with a modified tractor connection.
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Elementary approach

In simple cases the prolongation can be deduced ad hoc by repeated differentiations and projections on suitable components.

- The curvature appears from skew-symmetrizing a second derivative.
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**General construction**

There is a systematic construction based on the Lie algebra cohomology.

- Main assumption is that the highest weight component of the derivative (the “Twistor operator”) is prescribed to vanish.
- The basic construction can be further refined in order to gain desired invariance properties. (Branson, Čap, Eastwood and Gover 2006)
Integrability conditions

Let $M$ be a connected smooth manifold,

- $E \to M$ a finite-dimensional smooth vector bundle over $M$,
- and $\nabla$ an \textit{arbitrary} linear connection in $E$.

We consider the equation for a \textbf{parallel section} $\Phi \in \Gamma(E)$,

$$\nabla_X \Phi = 0, \quad \forall X \in \mathcal{T}(M).$$  \hfill (E)
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$$\nabla_X \Phi = 0, \quad \forall X \in \mathfrak{T}(M). \quad (E)$$

The *integrability conditions* of $(E)$ say simply that $\Phi$ has to be annihilated by the curvature $R$ of $\nabla$ and its derivatives:

$$R_{X,Y} \Phi = 0, \quad (I_0)$$

$$((\nabla_{Z_k}(\ldots(\nabla_{Z_1}(R_{X,Y}))\ldots)) \Phi = 0, \quad \forall X, Y, Z_1, \ldots, Z_k \in \mathfrak{X}(M). \quad (I_k)$$
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The conditions are tensorial in $\Phi$ so we can evaluate them just pointwise and define subspaces consisting of admissible values in the fibers $E_x$,

$$S_x = \{ \phi \text{ such that (I}_0\text{) and (I}_k\text{) hold at } x \} \subseteq E_x. \quad \text{(S)}$$

Note that $\dim S_x$ is not generally constant but only upper semi-continuous.
Existence results

Under a simple regularity assumption on $S_x$ the integrability conditions are in fact sufficient for the existence of at least a local solution.

**Theorem**

*If $\dim S_x$ is constant on some neighborhood of $x_0 \in M$ then for each $\phi \in S_{x_0}$ exists a unique local solution $\Phi$ of $(E)$ such that $\Phi(x_0) = \phi$.*

In detail, the hypothesis ensures that $S_x$ form a smooth vector subbundle $S \subseteq E$ and $\nabla$ consequently restricts to a *flat connection* in $S$. 
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**Proposition**

If $(M, E, \nabla)$ are real-analytic then $\dim S_x$ is constant on $M$. 
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**Proposition**

If $(M, E, \nabla)$ are real-analytic then $\dim S_x$ is constant on $M$.

In general, there is no upper bound on the order $k$ of differentiation in $(I_k)$ which needs to be considered, even in the real-analytic case. Let us define

$$S_x^l = \{ \phi \text{ such that } (I_0) \text{ and } (I_k) \text{ hold at } x \text{ for } k \leq l \} \subseteq E_x. \quad (S^l)$$

With assumption on $S_x^l$ we can determine $S_x$ in a finite number of steps.

**Proposition**

If $\dim S_x^l$ is constant on some neighborhood of $x_0 \in M$ and $S_{x_0}^l = S_{x_0}^{l-1}$ then $S_x = S_x^l = S_{x}^{l-1}$ on a (possibly smaller) neighborhood of $x_0$. 
Homogeneous spaces

Suppose that $M = G/H$ is a simply connected homogeneous space,

- $E \to M$ a finite-dimensional *homogeneous* vector bundle over $M$,
- and $\nabla$ an *invariant* linear connection in $E$. 

Invariance of $\nabla$ implies that $C$ must be $G$-equivariant and hence, by abuse of notation, we can identify it with an $H$-equivariant linear mapping $C : g \otimes E_o \to E_o$, where $E_o$ is the fiber of $E$ at the origin $o \in M$. 


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We compare $\nabla$ with the **fundamental derivative** $D$ given by the canonical flat Cartan connection on $G/H$,

$$
\nabla_{\Pi(\xi)} \Phi = D\xi \Phi + C\xi \Phi, \quad \forall \xi \in \mathcal{A}(M), \Phi \in \Gamma(E),
$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras corresponding to $G$ and $H$ respectively,

- $\mathcal{A}(M) = G \times_H \mathfrak{g}$ is the adjoint tractor bundle over $G/H$,
- $\Pi: \mathcal{A}(M) \to \mathcal{T}(M)$ is the canonical projection,
- $D: \mathcal{A}(M) \otimes \Gamma(E) \to E$ is the fundamental derivative,
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Homogeneous spaces

Using the Ricci identity we can write the integrability conditions in form

\[
\begin{align*}
[C_\xi, C_\nu] \phi &= C_{\{\xi, \nu\}} \phi, \\
[C_\xi, C_\nu] C_{\zeta_k} \cdots C_{\zeta_1} \phi &= C_{\{\xi, \nu\}} C_{\zeta_k} \cdots C_{\zeta_1} \phi,
\end{align*}
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where \([,]\) denotes the commutator in \(\mathcal{L}(E_o, E_o)\) and \({,}\) the Lie bracket in \(\mathfrak{g}\).

\(\forall \xi, \nu, \zeta_1, \ldots, \zeta_k \in \mathfrak{g},\)

\((\text{CI}_0)\)

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\( S_o \subseteq E_o \) is the maximal subspace on which the linear mapping \( C \) restricts to a representation of \( \mathfrak{g} \).

The subbundle \( S \subseteq E \) becomes a (flat) tractor bundle over \( G/H \) via \( \exp C \) and the parallel sections are explicitly given by the formula

\[
\Phi(g \phi) = L_g((\exp C)(g^{-1}) \phi), \quad \forall g \in G, \phi \in S_o, \quad (F)
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where \( L_g \) denotes the left action of \( G \) on \( E \).
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The regularity of \(S_x^l\) is satisfied thanks to homogeneity, so the sequence \(S^0_o \supseteq \cdots \supseteq S^l_o \supseteq \cdots \supseteq S_o\) definitely stabilizes once it does not decrease.

Finding \(S_o\) is therefore completely algorithmic. We have implemented this algorithm in CAS for Killing forms, spinors and spinor-valued forms.
Cone construction

The \emph{\(\varepsilon\)-metric cone} over a pseudo-Riemannian manifold \((M, g)\) is

1. the warped product \(\overline{M} = M \times \mathbb{R}_+\) with metric \(\overline{g} = r^2 g + \varepsilon \, dr^2\),

where \(r\) is the coordinate function on \(\mathbb{R}_+\) and \(\varepsilon = \pm 1\).
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We associate to a vector field $X$ on $M$ a vector field $\overline{X}$ on $\overline{M}$ by

$$\overline{X} = \frac{1}{r} p_1^*(X), \quad (MC1)$$

where $p_1^*$ denotes pull-back along the canonical projection $p_1 : \overline{M} \to M$. 

The so-called cone constructions establish a correspondence between

- solutions of natural systems of PDEs like the Killing equations on $M$
- parallel sections of suitable vector bundles over $M$.

In fact we can view the natural bundles over $M$ as tractor bundles over $M$. 
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$$\overline{\nabla}^g_X \overline{Y} = \frac{1}{r} \left( \nabla^g_X Y - \varepsilon g(X, Y) \, \partial_r \right), \quad \overline{\nabla}^g_{\partial_r} \overline{X} = 0,$$

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In fact we can view the natural bundles over \( \overline{M} \) as tractor bundles over \( M \).
Killing forms

From now on let \((M, g)\) be a pseudo-Riemannian manifold, \(\nabla^g\) the Levi-Civita connection and \(\mathcal{R}^g\) its curvature.

**Definition**

A \(p\)-form \(\alpha\) on \(M\) is a **Killing form** if there exists a \((p + 1)\)-form \(\beta\), such that

\[
\nabla^g_X \alpha = X \lrcorner \beta.
\]

(KF1)
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**Prolongation**

\[
\nabla^g_X \beta = \frac{1}{p} \mathcal{R}^g_X \wedge \alpha,
\]

(KF2)

\[
\mathcal{R}^g_X \wedge \alpha = \sum_{i=1}^{n} e^i \wedge (\mathcal{R}^g_{X, e_i} \alpha).
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(KF3)
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**1st curvature condition**

\[
\mathcal{R}^g_{X,Y} \alpha + \frac{1}{p} \left( X \lrcorner (\mathcal{R}^g_Y \wedge \alpha) - Y \lrcorner (\mathcal{R}^g_X \wedge \alpha) \right) = 0.
\]

(KF3)

- The condition is *void* for 1-forms.
Killing forms

**Definition**
A Killing $p$-form $\alpha$ on $M$ is **special** if additionally holds

$$
\nabla^g_X \beta = -cX^* \wedge \alpha,
$$

(KF4)

for some constant $c \in \mathbb{R}$ and where $X^*$ is the metric dual of $X$.

The equation (KF4) considered alone (with exchanged roles of $\alpha$ and $\beta$) is just a Hodge star dualization of (KF1) called **$*$-Killing equation**.
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A Killing $p$-form $\alpha$ on $M$ is **special** if additionally holds

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for some constant $c \in \mathbb{R}$ and where $X^\ast$ is the metric dual of $X$.

The equation (KF4) considered alone (with exchanged roles of $\alpha$ and $\beta$) is just a Hodge star dualization of (KF1) called **$\ast$-Killing equation**.

**Cone construction**

We associate to a $p$-form $\alpha$ on $M$ a $p$-form $\bar{\alpha}$ on $\overline{M}$ by

$$\bar{\alpha} = r^p p_1^*(\alpha).$$

(MC3)

**Proposition**

Let $\alpha$ be a $p$-form and $\beta$ a $(p + 1)$-form on $M$. The $(p + 1)$-form $\Theta$ on $\overline{M}$,

$$\Theta = dr \wedge \bar{\alpha} + \bar{\beta},$$

(KF5)

is **parallel** if and only if $\alpha$ together with $\beta$ is **special Killing** with $c = \varepsilon$.

(Semmelmann 2003)
Killing spinors

Further suppose that \((M, g)\) is a Spin manifold and denote by \(\nabla^g\) and \(\mathcal{R}^g\) also the corresponding spin connection and its curvature respectively.

**Definition**

A spinor field \(\Psi\) on \(M\) is a **Killing spinor** if

\[
\nabla^g_X \Psi = aX \cdot \Psi,
\]

for some constant \(a \in \mathbb{C}\) called the **Killing number** and where ‘\(\cdot\)’ denotes the Clifford multiplication.

- The prolongation is trivial since (KS1) is already closed.
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**1st curvature condition**

\[
\mathcal{R}^g_{X, Y} \Psi + a^2 (X \cdot Y - Y \cdot X) \cdot \Psi, = 0. \tag{KS2}
\]

- Implies that \((M, g)\) must be **Einstein**. (Friedrich 1980)
Killing spinors

Cone construction
The cone $\bar{M}$ is clearly homotopy equivalent to $M$, hence any spin structure on $M$ determines a unique spin structure on $\bar{M}$. We denote

- by $\Sigma$ and $\bar{\Sigma}$ the associated spinor bundles on $M$ and $\bar{M}$ respectively.

Proposition
Let $\Psi$ be a spinor field on $M$. The associated spinor field $\Psi^{\pm}$ on $\bar{M}$ is parallel if and only if $\Psi$ is Killing with $a = \pm \frac{1}{2} \sqrt{\varepsilon} \partial_r$.

(Bär 1993; Bohle 2003)
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- by $\Sigma$ and $\overline{\Sigma}$ the associated spinor bundles on $M$ and $\overline{M}$ respectively.

The pullback bundle $p_1^*(\Sigma)$ is naturally a subbundle of $\overline{\Sigma}$ and we associate to a spinor field $\Psi$ on $M$ spinor fields $\overline{\Psi}_\pm$ on $\overline{M}$ by

$$\overline{\Psi}_\pm = (1 \mp \sqrt{\varepsilon} \partial_r) \cdot p_1^*(\Psi).$$

(MC4)

The two choices of sign of the square root yield inequivalent though analogous results and we consider both of them.
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(Bär 1993; Bohle 2003)
Killing spinor-valued forms

Definition
A spinor-valued $p$-form $\Phi$ on $M$ is a Killing spinor-valued form if there exists a spinor-valued $(p + 1)$-form $\Xi$, such that

$$\nabla^g_X \Phi = aX \cdot \Phi + X \llap\lrcorner} \Xi,$$  \hspace{1cm} (KSF1)

for some constant $a \in \mathbb{C}$ called the Killing number.
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Prolongation
We absorb the Clifford multiplication term into covariant derivative \( \nabla^a \),

\[
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(KSF2)

\( \forall \Psi \in \Gamma(\Sigma) \),

and extend it to spinor-valued forms by the Levi-Civita connection.
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$$\nabla^g_X \Xi = aX \cdot \Xi + \frac{1}{p} \left( \mathcal{R}^a_X \wedge \Phi - \frac{1}{2(p+1)} X \lrcorner (\mathcal{R}^a \wedge \Phi) \right),$$

where $\mathcal{R}^a$ denotes the curvature of $\nabla^a$. 
Killing spinor-valued forms

1\textsuperscript{st} curvature condition

\[ \mathcal{R}^a_{X, Y} \Phi + \frac{1}{p} \left( X \lrcorner (\mathcal{R}^a_Y \wedge \Phi) - Y \lrcorner (\mathcal{R}^a_X \wedge \Phi) - \right. \]
\[ \left. - \frac{1}{p+1} X \lrcorner Y \lrcorner (\mathcal{R}^a \wedge \Phi) \right) = 0. \]

(KSF4)

- The condition is again \textit{void} for spinor-valued 1-forms.
Killing spinor-valued forms

1st curvature condition

\[ R^a_{X,Y} \Phi + \frac{1}{p} \left( X \downarrow (R^a_Y \wedge \Phi) - Y \downarrow (R^a_X \wedge \Phi) - \right. \]

\[ \left. - \frac{1}{p+1} X \downarrow Y \downarrow (R^a \wedge \Phi) \right) = 0. \]  
(KSF4)

- The condition is again void for spinor-valued 1-forms.

Definition

A Killing spinor-valued \( p \)-form \( \Phi \) on \( M \) is special if additionally holds

\[ \nabla^g_X \Xi = aX \cdot \Xi - cX^* \wedge \Phi, \]  
(KSF5)

for some constant \( c \in \mathbb{R} \).

As in the case of ordinary forms, the equation (KSF5) considered alone (with exchanged roles of \( \Phi \) and \( \Xi \)) is just a Hodge star dualization of (KSF1) called \( * \)-Killing equation.
Killing spinor-valued forms

**Cone construction**

We associate to a spinor-valued $p$-form $\Phi$ on $M$ spinor-valued $p$-forms $\overline{\Phi}_\pm$ on the $\varepsilon$-metric cone $\overline{M}$ by

$$
\overline{\Phi}_\pm = r^p (1 \mp \sqrt{\varepsilon} \partial_r) \cdot p^*_1(\Phi).
$$

**(MC5)**

**Proposition**

Let $\Phi$ be a spinor-valued $p$-form and $\Xi$ a spinor-valued $(p + 1)$-form on $M$. The spinor-valued $(p + 1)$-form $\Theta_\pm$ on $\overline{M}$,

$$
\Theta_\pm = dr \wedge \overline{\Phi}_\pm + \overline{\Xi}_\pm,
$$

**(KSF6)**

is **parallel** if and only if $\Phi$ together with $\Xi$ is **special Killing** with

$a = \pm \frac{1}{2} \sqrt{\varepsilon}$ and $c = \varepsilon$.

**(Somberg, Zima 2016)**
Constant curvature spaces

Let $\overline{M} = \mathbb{R}^{n+1} \setminus \{0\}$ with the standard inner product $\overline{g}$ of signature $(\overline{p}, \overline{q})$, and $(M, g)$ be the pseudo-Riemannian submanifold

$$M = \{ x \in \overline{M} \mid \overline{g}(x, x) = \varepsilon \}, \quad \text{(CC1)}$$

with the inherited metric $g$, where $\varepsilon = \pm 1$. 
Constant curvature spaces

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\[ M = \{ x \in \overline{M} \mid \bar{g}(x, x) = \varepsilon \}, \tag{CC1} \]

with the inherited metric $g$, where $\varepsilon = \pm 1$.

- $M$ has constant curvature $\varepsilon$.
- $M$ can be viewed as the homogeneous space $O(\bar{p}, \bar{q}) / O(p, q)$.
- $\overline{M}$ is the $\varepsilon$-metric cone over $M$.
- The pullbacks of natural bundles over $\overline{M}$ along the embedding $i : M \hookrightarrow \overline{M}$ are trivial and can be viewed as tractor bundles over $M$. 
**Constant curvature spaces**

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- The pullbacks of natural bundles over $\overline{M}$ along the embedding $i: M \hookrightarrow \overline{M}$ are trivial and can be viewed as **tractor bundles** over $M$.

The Levi-Civita connection on $\overline{M}$ is just the usual **partial derivative** $\partial$.

- Restriction of $\partial$ to $M$ is the canonical (flat) **tractor connection**.

The Levi-Civita connection on $M$ and its curvature are given by

$$\nabla^g_X = \partial_X + \varepsilon (x \wedge X), \quad R^g_{X,Y} = \varepsilon (X \wedge Y), \quad \text{(CC2)}$$

where $x \wedge X$ and $X \wedge Y$ are respective elementary matrices in $\mathfrak{so}(\overline{p}, \overline{q})$. 
Constant curvature spaces

Killing forms

\[
\begin{align*}
\alpha(x) &= x \cdot \Theta \\
\beta(x) &= \pi(\Theta)
\end{align*}
\]  \quad \Leftrightarrow \quad \Theta = \varepsilon x^* \wedge \alpha(x) + \beta(x) \quad \text{(CC3)}

Killing spinors

\[
\Psi(x) = \frac{1}{2} \varphi_{\pm} \cdot \Theta \quad \Leftrightarrow \quad \Theta = \varphi_{\mp} \cdot \Psi(x) \quad \text{(CC4)}
\]

Killing spinor-valued forms

\[
\begin{align*}
\Phi(x) &= \frac{1}{2} \varphi_{\pm} \cdot (x \cdot \Theta) \\
\Xi(x) &= \frac{1}{2} \varphi_{\pm} \cdot \pi(\Theta)
\end{align*}
\]  \quad \Leftrightarrow \quad \Theta = \varphi_{\mp} \cdot (\varepsilon x^* \wedge \Phi(x) + \Xi(x)) \quad \text{(CC5)}

where we denote

\[
\pi(\Theta) = \Theta - \varepsilon x^* \wedge (x \cdot \Theta), \quad \varphi_{\pm} = 1 \pm \sqrt{\varepsilon} x.
\]
## Constant curvature spaces

### Killing forms

\[
\begin{align*}
\alpha(x) &= x \perp \Theta \\
\beta(x) &= \pi(\Theta)
\end{align*}
\]

\(\iff\)

\[
\Theta = \varepsilon x^* \wedge \alpha(x) + \beta(x) \quad (CC3)
\]

### Killing spinors

\[
\Psi(x) = \frac{1}{2} \varphi_{\pm} \cdot \Theta 
\]

\(\iff\)

\[
\Theta = \varphi_{\mp} \cdot \Psi(x) \quad (CC4)
\]

### Killing spinor-valued forms

\[
\begin{align*}
\Phi(x) &= \frac{1}{2} \varphi_{\pm} \cdot (x \perp \Theta) \\
\Xi(x) &= \frac{1}{2} \varphi_{\pm} \cdot \pi(\Theta)
\end{align*}
\]

\(\iff\)

\[
\Theta = \varphi_{\mp} \cdot \left( \varepsilon x^* \wedge \Phi(x) + \Xi(x) \right) \quad (CC5)
\]

where we denote

\[
\pi(\Theta) = \Theta - \varepsilon x^* \wedge (x \perp \Theta), \quad \varphi_{\pm} = 1 \pm \sqrt{\varepsilon} x.
\]

- All the solutions are *special* and the *Killing number* is \(a = \pm \frac{1}{2} \sqrt{\varepsilon}\).
- The 1st *curvature conditions* rule out other solutions except for spinor-valued forms in degree \(p = 1\).
New solutions

Algebraic decomposition

We can decompose the space of spinor-valued forms using the technique of *Howe dual pairs*. The degree raising and lowering algebraic operators,

\[
\gamma \cdot \wedge \Phi = \sum_{i=1}^{n} e^i \wedge (e_i \cdot \Phi), \quad \gamma^\ast \downarrow \Phi = \sum_{i,j=1}^{n} g^{ij} e_i \downarrow (e_j \cdot \Phi),
\]

are Spin-equivariant and generate a Lie algebra isomorphic to $\mathfrak{sl}(2)$. 

▶ A spinor-valued $p$-form $\Phi$ is called primitive if $\gamma^\ast \downarrow \Phi = 0$.

▶ Projection on the primitive component in degree $p = 1$ is given by

\[
\pi_T \chi^p(\Phi) = \Phi + \frac{1}{n} \gamma \cdot \wedge (\gamma^\ast \downarrow \Phi).
\]
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- A spinor-valued \(p\)-form \(\Phi\) is called \textbf{primitive} if \(\gamma^* \cdot \Phi = 0\).
- Projection on the primitive component in degree \(p = 1\) is given by

\[
\pi_{Tw}(\Phi) = \Phi + \frac{1}{n} \gamma \cdot \wedge (\gamma^* \cdot \Phi).
\]
New solutions

Algebraic decomposition
We can decompose the space of spinor-valued forms using the technique of \textit{Howe dual pairs}. The degree raising and lowering algebraic operators,

\[ \gamma \cdot \wedge \Phi = \sum_{i=1}^{n} e^i \wedge (e_i \cdot \Phi), \quad \gamma^\ast \cdot \downarrow \Phi = \sum_{i,j=1}^{n} g^{ij} e_i \downarrow (e_j \cdot \Phi), \quad (G1) \]

are Spin-equivariant and generate a Lie algebra isomorphic to $sl(2)$.

- A spinor-valued $p$-form $\Phi$ is called \textbf{primitive} if $\gamma^\ast \cdot \downarrow \Phi = 0$.
- Projection on the primitive component in degree $p = 1$ is given by

\[ \pi_{Tw}(\Phi) = \Phi + \frac{1}{n} \gamma \cdot \wedge (\gamma^\ast \cdot \downarrow \Phi). \quad (G2) \]

Higher curvature conditions
The higher curvature conditions for Killing spinor-valued forms in degree $p = 1$ allow only one other possibility on $M$ besides $a = \pm \frac{1}{2} \sqrt{\epsilon}$,

\[ a = \pm \frac{3}{2} \sqrt{\epsilon}, \quad \Xi = \mp \sqrt{\epsilon} (\gamma \cdot \wedge \Phi). \quad (CC6) \]
New solutions

The new solutions in degree $p = 1$ hence satisfy a stronger equation,

$$\nabla^g_X \Phi = \pm \sqrt{\varepsilon} \left( \frac{3}{2} X \cdot \Phi - X \perp (y \cdot \wedge \Phi) \right). \quad \text{(KSF7)}$$

Cone construction

They correspond to constant primitive spinor-valued 1-forms on $\overline{M}$,

$$\Phi(x) = \frac{1}{2} \left( \varphi_{\pm} \cdot \pi(\Theta) \pm \sqrt{\varepsilon} y \cdot \wedge (\varphi_{\pm} \cdot (x \perp \Theta)) \right) \iff \Theta = \pi_{\text{Tw}}(\varphi_{\mp} \cdot \Phi(x)). \quad \text{(CC7)}$$
New solutions

The new solutions in degree \( p = 1 \) hence satisfy a stronger equation,

\[
\nabla_X^g \Phi = \pm \sqrt{\epsilon} \left( \frac{3}{2} X \cdot \Phi - X \cdot (\gamma \cdot \wedge \Phi) \right).
\]

(KSF7)

Cone construction

They correspond to constant primitive spinor-valued 1-forms on \( \overline{M} \),

\[
\Phi(x) = \frac{1}{2} \left( \varphi_{\pm} \cdot \pi(\Theta) \pm \sqrt{\epsilon} \gamma \cdot \wedge (\varphi_{\pm} \cdot (x \cdot \Theta)) \right) \quad \iff \quad \Theta = \overline{\pi_{TW}}(\varphi_{\mp} \cdot \Phi(x)).
\]

(CC7)

- Do not attain the maximal dimension from the prolongation (KSF3).
- Not special in the sense of (KSF5).
- Not spanned by tensor products \( \alpha \otimes \Psi \) of K. spinors and forms.
- \( \Phi(x) \) cannot take values solely in the primitive component.
New solutions

The new solutions in degree $p = 1$ hence satisfy a stronger equation,

$$
\nabla_X^g \Phi = \pm \sqrt{\epsilon} \left( \frac{3}{2} X \cdot \Phi - X \wedge (y \cdot \wedge \Phi) \right).
$$  \hspace{1cm} \text{(KSF7)}

Cone construction

They correspond to constant \textbf{primitive spinor-valued 1-forms} on $\overline{M}$,

$$
\Phi(x) = \frac{1}{2} \left( \varphi_{\pm} \cdot \pi(\Theta) \pm \sqrt{\epsilon} y \cdot \wedge (\varphi_{\pm} \cdot (x \wedge \Theta)) \right) \quad \Leftrightarrow \\
\Leftrightarrow \quad \Theta = \overline{\pi_{TW}}(\varphi_{\mp} \cdot \Phi(x)).
$$  \hspace{1cm} \text{(CC7)}

- Do not attain the maximal dimension from the prolongation (KSF3).
- Not special in the sense of (KSF5).
- \textit{Not spanned by tensor products $\alpha \otimes \Psi$ of K. spinors and forms.}
- $\Phi(x)$ cannot take values solely in the primitive component.

Comparing (CC7) with (CC5) we get that the new solutions $\Phi$ are just a transformation of \textbf{special Killing forms $(\Phi', \Xi')$ in order $p = 0$},

$$
\Phi = \Xi' \pm \sqrt{\epsilon} y \cdot \wedge \Phi'.
$$  \hspace{1cm} \text{(KSF8)}
THANK YOU FOR YOUR ATTENTION!