CHARLES UNIVERSITY PRAGUE

faculty of mathematics and physics

Jan Šťovíček

Noncommutative algebraic geometry based on quantum flag manifolds

Part I.

(joint with Réamonn Ó Buachalla and Adam-Christiaan van Roosmalen)

39th Winter School Geometry and Physics, Srní, January 14th, 2019

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Affine varieties

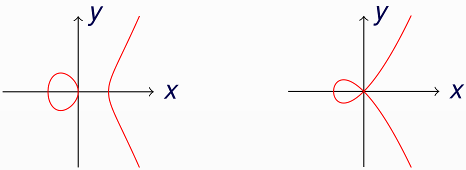
- Let $\mathbb{C}$ be the field of complex numbers and $n \geq 1$.
- A complex affine variety $V \subseteq \mathbb{C}^n$ is just the solution set of a system of polynomial equations, i.e.

$$V = \{P \in \mathbb{C}^n \mid f_i(P) = 0 \text{ for each } i \in I\},$$

where

$$f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n] \text{ for each } i \in I.$$

- The real part of $V \subseteq \mathbb{C}^2$ may look like this



$y^2 - x(x-1)(x+1)$ (smooth)

$y^2 - x^2(x+1)$ (singular)

Maps between varieties

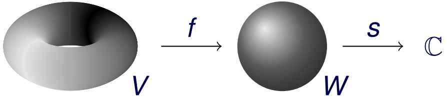
- A map $f\colon V \to W$ of affine varieties ($V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^\ell$) if polynomial if there exist $f_1, f_2, \ldots, f_\ell \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ such that

$$f(P) = \big(f_1(P), f_2(P), \ldots, f_\ell(P)\big) \text{ for each } P \in V.$$

- If $V \subseteq \mathbb{C}^n$ is an affine variety, the coordinate ring $\mathbb{C}[V]$ of V is the set of all polynomial maps $f\colon V \to \mathbb{C}$.
- $\mathbb{C}[V]$ is a $\mathbb{C}$-algebra with pointwise operations and as such $\mathbb{C}[V] \cong \mathbb{C}[x_1, x_2, \ldots, x_n]/\{f \text{ such that } f|_V \equiv 0\}$ ($\mathbb{C}[V]$ is a finitely generated $\mathbb{C}$-algebra).

Maps control affine varieties

- To each polynomial map $f\colon V \to W$ we may naturally assign a homomorphism of $\mathbb{C}$-algebras $f^*\colon \mathbb{C}[W] \to \mathbb{C}[V]$ given by $f^*(s) = s \circ f$:



- Fact: This assignment induces a bijection between
  1. polynomial maps $V \to W$ and
  2. $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[W] \to \mathbb{C}[V]$.
- A reformulation: There is a full embedding of categories

$$\text{Varieties}_{\mathbb{C}} \longrightarrow (\text{Alg}_{\mathbb{C}})^{\text{op}}.$$

- Hilbert's Nullstellensatz tells us what the image is: These are precisely finitely generated $\mathbb{C}$-algebras R which are reduced: $(\forall\, s \in R)(\forall n \geq 1)(s^n = 0 \implies s = 0)$.
- Analogy with Gelfand-Naimark, $X \leftrightarrow C(X)$ (X compact Hausdorff topological space, $C(X)$ the C^*-algebra of continuous maps $X \to \mathbb{C}$).

The dictionary between algebra and geometry

Affine geometry	Algebra
points of V	maps of $\mathbb{C}$-algebras $\mathbb{C}[V] \to \mathbb{C}$
Cartesian product $V \times W$	tensor product $\mathbb{C}[V] \otimes \mathbb{C}[W]$
affine algebraic groups (such as SL_n)	commutative Hopf algebras
$(\mu\colon G \times G \to G, 1_G \in G)$	$(\mathbb{C}[G] \xrightarrow{\Delta} \mathbb{C}[G] \otimes \mathbb{C}[G], \mathbb{C}[G] \xrightarrow{\varepsilon} \mathbb{C})$

Theorem (Serre, 1955)

For a complex affine variety V, there is a bijective correspondence between

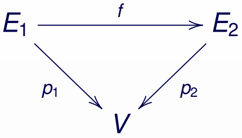
1. *algebraic vector bundles $p\colon E \to V$ and*
2. *certain finitely generated projective $\mathbb{C}[V]$-modules (i.e. direct summands of free $\mathbb{C}[V]$-modules $\mathbb{C}[V]^n$, $n \geq 1$).*

The bijection assigns to a vector bundle p its $\mathbb{C}[V]$-module of sections

$$P = \{s\colon V \to E \text{ polynomial map} \mid p \circ s = 1_V\}.$$

(Quasi-)coherent sheaves

- Problem: Vector bundles do not form an abelian category. More concretely, the image of a map of vector bundles



may not be a vector bundle (the ranks of f may differ between fibers).
- Morally, the category of coherent sheaves $\operatorname{coh} V$ is the smallest abelian category containing $\operatorname{Vect} V$. Dictionary:

Affine geometry	Algebra
vector bundles over V	fin. gen. proj. $\mathbb{C}[V]$-modules
coherent sheaves on V	all fin. gen. $\mathbb{C}[V]$-modules
quasi-coherent sheaves on V	all $\mathbb{C}[V]$-modules

- Algebraic principle: If we want to understand properties of a ring R, it is a good idea to study the category of R-modules.

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Projective varieties

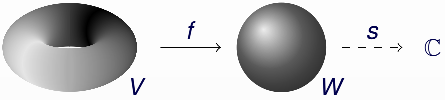
- We can define similarly projective algebraic varieties. Projective space:
$$\mathbb{P}^n_{\mathbb{C}} = \{(a_0 : a_1 : \cdots : a_n) \mid (\exists i)(a_i \neq 0)\}.$$
- A complex projective variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ is the solution set of a system of homogeneous polynomial equations,
$$V = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n_{\mathbb{C}} \mid f_i(a_0, a_1, \ldots, a_n) = 0 \text{ for each } i \in I\}.$$
Here: A polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ is homogeneous if all non-zero terms have the same total degree.
- Similarly, we can take the ideal
$$I(V) = (f \text{ homogeneous} | f_V \equiv 0) \subseteq \mathbb{C}[x_0, x_1, \ldots, x_n]$$
and the homogeneous coordinate ring $S(V) := \mathbb{C}[x_0, x_1, \ldots, x_n]/I(V)$.
- Warning: The elements $f \in S(V)$ typically do not define functions $S(V) \to \mathbb{C}$. Conceptual problem: No holomorphic non-constant maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}$ by Liouville's theorem!

Regular functions

- Observation: If V is a projective variety and $f, g \in \mathbb{C}[x_0, x_1, \dots, x_n]$ homogeneous of the same degree, then
$$(a_0 : a_1 : \cdots : a_n) \longmapsto \frac{f(a_0, a_1, \dots, a_n)}{g(a_0, a_1, \dots, a_n)} \qquad (*)$$
defines a partial function $V \dashrightarrow \mathbb{C}$.
- Zariski topology on V: the closed sets are the algebraic subsets of V.
- A function $f \colon U \to \mathbb{C}$, $U \subseteq V$ Zariski open, is regular if it is Zariski locally of the form $(*)$.
- What structure should a projective variety actually carry?
- A ringed space is a pair $(V, \mathcal{O}_V)$ such that V is a topological space and $\mathcal{O}_V$ is a sheaf of rings:
  1. for each $U \subseteq V$ we have a ring $\mathcal{O}_V(U)$,
  2. for each $U' \subseteq U \subseteq V$ we have a homomorphism
  $$\mathrm{res}^U_{U'} \colon \mathcal{O}_V(U) \to \mathcal{O}_V(U'),$$
  3. subject to certain axioms.

  (For complex varieties, we have a sheaf of $\mathbb{C}$-algebras!)

- A homomorphism of projective varieties is if a map $f\colon V \to W$ which is Zariski locally computed by ratios of homogeneous polynomials.
- Formally: f is a homomorphisms of varieties if
  1. f is Zariski continuous, and
  2. For each $s \in \mathcal{O}_W(U)$, we have $s \circ f \in \mathcal{O}_V(f^{-1}(U))$.



Related example

If M is a smooth real manifold, M has a structure of ringed space with

$$\mathcal{O}_M(U) = \{s\colon M \to \mathbb{R} \mid s \text{ smooth}\}.$$

A map $f\colon M \to N$ of smooth manifolds is smooth if and only if it satisfies (1) and (2) above.

Vector bundles and Serre's theorem

- If V is a projective variety and $p\colon E \to V$ is an algebraic vector bundle, E might have no non-zero global sections.
- We should consider sections over open subsets $U \subseteq V$:

$$\mathscr{V}(U) = \{s\colon U \to E \mid f \circ s = 1_U\}.$$

- Each $\mathscr{V}(U)$ is an $\mathcal{O}_V(U)$-module, and restrictions $\mathrm{res}^U_{U'}\colon \mathscr{V}(U) \to \mathscr{V}(U')$ are compatible with the module structure.
- Serre, 1955: There is a bijection between
  1. algebraic vector bundles $p\colon E \to V$ and
  2. certain sheaves of $\mathcal{O}_V$-modules such that Zariski locally, $\mathscr{V}(U)$ is a finitely generated projective $\mathcal{O}_V(U)$-module.
- The category of vector bundles can be extended to an abelian category:

$$\mathrm{Vect}\, V \subseteq \mathrm{coh}\, V \subseteq \mathrm{Qcoh}\, V.$$

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Example: Grassmannians

- The set $\mathrm{Gr}_{n,r}$ of r-dimensional vector subspaces of $\mathbb{C}^n$ naturally forms a subset of a projective space via the embedding

$$\iota\colon \mathrm{Gr}_{n,r} \longrightarrow \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1},$$
$$V = \langle v_1, v_2, \dots, v_r \rangle \longmapsto \langle v_1 \wedge v_2 \wedge \cdots \wedge v_r \rangle.$$

  (We fix a basis of $\Lambda^r \mathbb{C}^n$ and assign to V its Plücker coordinates.)

- The image of ι is well-known to be a the zero set of quadratic homogeneous polynomials, e.g.

$$\mathrm{Gr}_{4,2} = \{(a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34}) \in \mathbb{P}^5_{\mathbb{C}} \mid a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\}.$$

Grassmannians as flag varieties

- Representation-theoretic point of view: $\Lambda^r \mathbb{C}^n$ is naturally a representation of $\mathfrak{sl}_n$; it is the r$^{\text{th}}$ fundamental representation $V(\varpi_r)$.
- The image of $\iota\colon \mathrm{Gr}_{n,r} \to \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1}$ is identified with the orbit $SL_n \cdot v$ of a highest weight vector $v \in V(\varpi_r)$ and the homogeneous coordinate ring is explicitly given as

$$S(\mathrm{Gr}_{n,r}) \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^*.$$

- This generalizes to all flag manifolds F: They are complex projective varieties given by quadratic homogeneous polynomials with the coordinate ring of the form

$$S(F) \cong \bigoplus_{k=0}^{\infty} V(k\lambda)^*.$$

  where λ is the sum of the fundamental weights for F.

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Part II.

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39^{th} Winter School Geometry and Physics, Srní, January 16^{th}, 2019

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Homogeneous coordinate rings

- Let $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ and

$$S(V) = \mathbb{C}[x_0, x_1, \ldots, x_n]/(f \text{ homogeneous}, f|_V \equiv 0)$$

be its homogeneous coordinate ring. Then

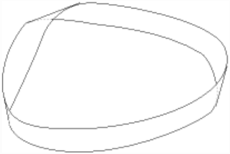
$$S(V) = \bigoplus_{n=0}^{\infty} S(V)_n$$

is naturally $\mathbb{Z}$-graded.

- Question: We know that the elements of $S(V)$ are not functions on V. What are they?
- The homogeneous parts $S(V)_n$, $n \geq 0$ are global sections of certain line bundles $\mathscr{L}_n$.
- So every projective variety is the set of zeros of sections in line bundles.

The tautological bundle

- There is an important line bundle over $\mathbb{P}^n_{\mathbb{C}}$, the tautological bundle $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$.
- It is dual to $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-1) \subseteq \mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{C}}}$, whose the fiber over $(a_0 : a_1 : \cdots : a_n)$ is the line $\langle a_0, a_1, \ldots, a_n \rangle \subseteq \mathbb{C}^{n+1}$.

$\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$: 

- If $\iota\colon V \subseteq \mathbb{P}^n_{\mathbb{C}}$, consider the restricted line bundle $\mathscr{L} := \iota^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$. This is an example of what is called ample (algebraic geometry) or positive (in the context of Kähler manifolds) line bundle.
- Fact: $S(V) \cong \bigoplus_{n=0}^{\infty} \Gamma(V, \mathscr{L}^{\otimes n})$. The homogeneous coordinate ring is the direct sum of global sections of tensor powers of $\mathscr{L}$.

- The twist functor: If $\mathscr{F} \in \operatorname{Qcoh} V$ and $n \in \mathbb{Z}$, put

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_V} \mathscr{L}^{\otimes n}$$

(that is, $\mathscr{F}(n)(U) := \mathscr{F}(U) \otimes_{\mathcal{O}_V(U)} \mathscr{L}(U)^{\otimes n}$ on U in an open basis of V).

- The graded module associated to a sheaf: If $\mathscr{F} \in \operatorname{Qcoh} V$, we put

$$\Gamma_*(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathscr{F}(n)) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}\big((\mathscr{L}^*)^{\otimes n}, \mathscr{F}\big).$$

Example: $\Gamma_*(\mathcal{O}_V) \cong S(V)$.

Theorem (Serre, 1955)

1. *The functor* $\Gamma_* \colon \operatorname{Qcoh} V \to \operatorname{Mod}^{\mathbb{Z}} S(V)$ *is fully faithful.*
2. Γ_* *has an exact left adjoint* $Q \colon \operatorname{Mod}^{\mathbb{Z}} S(V) \to \operatorname{Qcoh} V$ *which satisfies a universal property:* $\operatorname{Qcoh} V = \operatorname{Mod}^{\mathbb{Z}} S(V)/\operatorname{Mod}^{\mathbb{Z}}_0 S(V)$ *(Serre quotient).*
3. *Similarly,* $\operatorname{coh} V = \operatorname{mod}^{\mathbb{Z}} S(V)/\operatorname{mod}^{\mathbb{Z}}_0 S(V)$.

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- We have $SL_n/P \cong \mathrm{Gr}_{n,r}$, where

$$P = \left(\begin{array}{c|c} P_r & Q \\ \hline 0 & P_{n-r} \end{array} \right),$$

  where $P_r \in M_r(\mathbb{C})$ and $P_{n-r} \in M_{n-r}(\mathbb{C})$. The bijection sends the coset UP, $U = (u_{ij})_{i,j=1}^n \in SL_n$ to the linear hull of the first r columns of U.
- If we view $\mathrm{Gr}_{n,r} \subseteq \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1}$ via the Plücker embedding, the quotient map $SL_n \twoheadrightarrow \mathrm{Gr}_{n,r}$ sends $U = (u_{ij})_{i,j=1}^n$ to a point with homogeneous coordinates $\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r}$, one for each sequence $i_1 < i_2 < \cdots < i_r$.
- In terms of coordinate rings, this shows that $S(\mathrm{Gr}_{n,r})$ coincides with the subring

$$\mathbb{C}\left[\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r \right] \subseteq \mathbb{C}[SL_n].$$

Quantum Grassmannians

- In terms of coordinate rings, this shows that $S(\mathrm{Gr}_{n,r})$ coincides with the subring

$$\mathbb{C}\left[\sum_{\sigma}(-1)^{\mathrm{sgn}(\sigma)}u_{\sigma(i_1),1}u_{\sigma(i_2),2}\cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r\right] \subseteq \mathbb{C}[SL_n].$$

- We have $\mathbb{C}[SL_n] = U(\mathfrak{sl}_n)^\circ$ ($(-)^\circ$ is the Hopf dual).
- Quantum deformation: We can deform $\mathbb{C}[SL_n]$ to $U_q(\mathfrak{sl}_n)^\circ$ and define $S_q[\mathrm{Gr_{n,r}}]$ as the subring

$$\mathbb{C}\left[\sum_{\sigma}(-q)^{\ell(\sigma)}u_{\sigma(i_1),1}u_{\sigma(i_2),2}\cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r\right] \subseteq U_q(\mathfrak{sl}_n)^\circ.$$

- Representation-theoretic perspective: Again $S_q[\mathrm{Gr_{n,r}}] \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^*$, where $V(k\varpi_r)$ is the corresponding representation of $U_q(\mathfrak{sl}_n)$.

Quantized homogeneous coordinate rings of flags

- One can do the same for all flags (Soibelman 1992, Taft and Towber 1991, Lakshmibai and Reshetikin 1992, Braveman 1994, ...).
- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the corresponding complex simply connected algebraic group and P a parabolic subgroup. Then the flag $F = G/P$ is a projective variety and

$$\bigoplus_{k=0}^{\infty} V(k\lambda)^* \cong S_q(F) \subseteq U_q(\mathfrak{g})^\circ,$$

  where λ is the sum of fundamental weights for F and $V(k\lambda)$ are the corresponding finite dimensional representations of $U_q(\mathfrak{g})$.
- One can define a quantization for the category of coherent sheaves: $\mathrm{coh}_q F := \mathrm{mod}^{\mathbb{Z}} S_q(F)/\mathrm{mod}_0^{\mathbb{Z}} S_q(F)$.
- This is an abelian category and we can, for instance, define and study the analogue of the sheaf cohomology as well as other algebraic properties.

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Compact Lie versus algebraic groups

- Aim: Relate the quantized algebraic and differential geometry.
- We have $SU_n \subseteq SL_n$, where
  1. SL_n is a complex affine algebraic group,
  2. SU_n is a real compact Lie group but it is also a real algebraic group!
- Rings of functions in place:
  1. For SL_n we have the complex coordinate ring $\mathbb{C}[SL_n]$,
  2. For SU_n we have the hierarchy

     $$C(SU_n) \supseteq C^\infty(SU_n) \supseteq \mathcal{O}(SU_n)$$

     where $\mathcal{O}(SU_n)$ is the ring of polynomial functions $s\colon SU_n \to \mathbb{C}$ of real algebraic varieties.
  3. The magic here: $\mathbb{C}[SL_n] \cong \mathcal{O}(SU_n)$ (via the restriction of $s\colon SL_n \to \mathbb{C}$ to SU_n).

A pocket dictionary: algebraic to differential geometry

- A "cultural" problem:
  1. In differential geometry, a compact complex manifold is a real manifold with an extra structure (flat connection $\overline{\partial}\colon C^\infty(V) \to \Omega^{(0,1)}$).
  2. In algebraic geometry, one usually encounters only polynomial or rational (so holomorphic) functions.
- To relate the two, we need a meeting point of (1) and (2).
- We have $\mathrm{Gr}_{n,r} \cong SL_n/P \cong SU_n/L$, where

$$P = \left(\begin{array}{c|c} P_r & Q \\ \hline 0 & P_{n-r} \end{array}\right) \quad and \quad L = P \cap SU_n = \left(\begin{array}{c|c} L_r & 0 \\ \hline 0 & L_{n-r} \end{array}\right).$$

- Now:
  1. The expression $\mathrm{Gr}_{n,r} \cong SL_n/P$ allows to view the Grassmannian as a projective complex algebraic variety.
  2. The expression $\mathrm{Gr}_{n,r} \cong SU_n/L$ allows to view the Grassmannian as a affine real algebraic variety.
- The meeting point: Try to view a complex algebraic variety V as a real algebraic variety with a "complex structure" (a flat connection $\overline{\partial}\colon \mathcal{O}(V) \to \Omega^{(0,1)}$).

Dolbeault dg algebra

- If V is a complex manifold, we have the Dolbeault complex:

$$0 \longrightarrow C^\infty(V) \xrightarrow{\overline{\partial}} \Omega^{(0,1)} \xrightarrow{\overline{\partial}} \Omega^{(0,2)} \xrightarrow{\overline{\partial}} \cdots$$

- We can wedge forms ($\wedge\colon \Omega^{(0,i)} \otimes \Omega^{(0,j)} \longrightarrow \Omega^{(0,i+j)}$). Then $\Omega^{(0,\bullet)} = \bigoplus_i \Omega^{(0,i)}$ is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$.
- Moreover, we have the graded Leibniz rule: $\overline{\partial}(\omega_i \wedge \omega_j) = \overline{\partial}(\omega_i) \wedge \omega_j + (-1)^i \omega_i \wedge \overline{\partial}(\omega_j)$ for each $\omega_i \in \Omega^{(0,i)}$ and $\omega_j \in \Omega^{(0,j)}$. In other words, $(\Omega^{(0,\bullet)}(V), \wedge, \overline{\partial})$ is a differential graded (dg) algebra.
- If $V = \mathrm{Gr}_{n,r} = SU_n/L$, then

$$\mathcal{O}(\mathrm{Gr}_{n,r}) \subseteq C^\infty(\mathrm{Gr}_{n,r}) \subseteq C(\mathrm{Gr}_{n,r})$$

  and $\mathcal{O}(\mathrm{Gr}_{n,r})$ is dense with respect to $|| - ||_\infty$.
- Now, the Dolbeault dg algebra for $\mathrm{Gr}_{n,r}$ does restrict to real algebraic sections:

$$0 \longrightarrow \mathcal{O}(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \cdots$$

The differential calculus of Heckenberger and Kolb

- The Dolbeault dg algebra for $\mathrm{Gr}_{n,r}$ does restrict to real algebraic sections:

$$0 \longrightarrow \mathcal{O}(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{\mathrm{alg}} \xrightarrow{\overline{\partial}} \cdots$$

- and can be quantized:

$$0 \longrightarrow \mathcal{O}_q(\mathrm{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{q} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{q} \xrightarrow{\overline{\partial}} \cdots$$

  ($\big(\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,r}), \wedge, \overline{\partial}\big)$ is a dg algebra again).

- If we impose some more natural conditions on $\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,r})$, it is unique (Heckenberger and Kolb, 2006)!
- In fact, Heckenberger and Kolb quantized the Dolbeault dg algebra for all compact Hermitian symmetric flags.

The Koszul-Malgrange theorem

Theorem (Koszul and Malgrange, 1958)

Let V be a compact complex manifold. Then there is a bijective correspondence between

1. *holomorphic vector bundles $p\colon E \to V$ and*
2. *smooth complex vector bundles equipped with a flat connection $\nabla_E\colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(V)} \Omega^{(0,1)}$, where*

$$\Gamma^\infty(E) = \{s\colon V \to E \text{ smooth map} \mid p \circ s = 1_V\}.$$

The holomorphic sections of E are precisely ∇_E.

- By a version of the Serre-Swan theorem, $\Gamma^\infty(E)$ is a finitely generated projective $C^\infty(V)$-module.
- Define quantized algebraic vector bundles over $\mathrm{Gr}_{n,r}$ as flat connections $\nabla\colon P \to P \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}$, where P is a finitely generated projective $\mathcal{O}_q(\mathrm{Gr}_{n,r})$-module.

The first match (quantized alg. vs. diff. geometry)

- Recall: On $\mathrm{Gr}_{n,r} = SU_{n+1}/L$, we have only one reasonable quantized Dolbeault dg algebra $(\Omega_q^{(0,\bullet)}, \wedge, \overline{\partial})$.
- Since $\mathrm{Gr}_{n,r}$ is homogeneous, one can use representation theory of $\mathfrak{l}$ to construct quantum deformations $\mathscr{L}_{n,q}$ of tensor powers $\mathscr{L}^{\otimes n}$ of the tautological bundle $\mathscr{L}$.
- That is, there are finitely generated projective $L_{n,q}$ are finitely generated projective $\mathcal{O}_q(\mathrm{Gr}_{n,r})$-modules and certain flat connections, unique by Ó Buachalla and Mrozinski,

$$\nabla_{\mathscr{L}_{n,q}} \colon L_{n,q} \longrightarrow L_{n,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}.$$

Theorem (Ó Buachalla and Mrozinski, 2017)

For each $n \geq 0$, We have $S_q(\mathrm{Gr}_{n,r})_n \cong \ker \nabla_{\mathscr{L}_{n,q}}$.
So the holomorphic sections of line bundles based on the Heckenberger-Kolb calculus and the Koszul-Malgrange theorem agree with the older "naive" construction of the quantized coordinate ring.

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Part III.

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Coherent sheaves after Pali

- Recall: If V is a compact complex manifold, then a holomorphic vector bundle $p\colon E \to V$ can be equivalently given via a flat connection
$$\nabla_E \colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(V)} \Omega^{(0,1)}.$$
(Koszul and Malgrange, 1958).
- There is a generalization for coherent sheaves over V:

Theorem (Pali, 2006)

Given a compact complex manifold V and the sheaf $\mathcal{O}_V^\infty$ of smooth complex-valued functions, there is a bijective correspondence between

1. *analytic coherent sheaves on V and*
2. *flat connections $\nabla_{\mathscr{G}} \colon \mathscr{G} \to \mathscr{G} \otimes_{\mathcal{O}_V^\infty} \Omega^{(0,1)}$, where the sheaf $\mathcal{O}_V^\infty$-modules locally admits a resolution*

$$0 \to (\mathcal{O}_V^\infty|_U)^{n_k} \to \cdots \to (\mathcal{O}_V^\infty|_U)^{n_1} \to (\mathcal{O}_V^\infty|_U)^{n_0} \to \mathscr{G}|_U \to 0.$$

Theorem (Pali, 2006)

Given a compact complex manifold V and the sheaf $\mathcal{O}_V^\infty$ of smooth complex-valued functions, there is a bijective correspondence between

1. *analytic coherent sheaves on V and*
2. *flat connections $\nabla_{\mathscr{G}} : \mathscr{G} \to \mathscr{G} \otimes_{\mathcal{O}_V^\infty} \Omega^{(0,1)}$, where the sheaf $\mathcal{O}_V^\infty$-modules locally admits a resolution*

$$0 \to (\mathcal{O}_V^\infty|_U)^{n_k} \to \cdots \to (\mathcal{O}_V^\infty|_U)^{n_1} \to (\mathcal{O}_V^\infty|_U)^{n_0} \to \mathscr{G}|_U \to 0.$$

- If V embeds into a projective space, then V is a smooth complex projective algebraic variety (Chow, 1949).
- In that case, the categories of analytic and algebraic coherent sheaves are equivalent (Serre's GAGA Theorem, 1956).
- Pali actually proves that $\operatorname{coh} V$ is equivalent to the category of the flat connections as above.

The category of Beggs and Smith

- This motivated Beggs and Smith (2012) to define an abelian category category $\mathrm{Hol}(A)$ for a non-commutative complex structure $(\Omega^\bullet(A), \mathrm{d} = \partial + \overline{\partial})$ (e.g. $A = \mathcal{O}_q(\mathrm{Gr}_{n,r})$ as before):
- The objects are flat connections $\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}$ and the morphisms are given by $f \colon M \to N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\nabla_M} & M \otimes_A \Omega^{(0,1)} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes\Omega^{(0,1)}} \\ N & \xrightarrow[\nabla_N]{} & N \otimes_A \Omega^{(0,1)}. \end{array}$$

- First approximation to the differential description of the category of coherent sheaves: require that M have a finite projective resolution over A

$$0 \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0.$$

- Denote this full subcategory of $\mathrm{Hol}(A)$ by $\mathrm{hol}(A)$.

A catch

- Unlike in Pali's case, the algebraic category $\mathrm{hol}(\mathcal{O}(V))$ is too big to model $\mathrm{coh}\, V$ even for projective spaces $V = \mathbb{P}^n_{\mathbb{C}}$.
- For a connection $\nabla_M \colon M \to M \otimes_{\mathcal{O}(V)} \Omega^{(0,1)}$, $m \in M$ and $s \in \mathcal{O}(V)$, we have

$$\nabla_M(ms) = \nabla_M(m)s + m\bar{\partial}(s).$$

  If $v \colon M \to M \otimes_{\mathcal{O}(V)} \Omega^{(0,1)}$ is a homomorphism of $\mathcal{O}(V)$-modules, then $\nabla'_M = \nabla_M + v$ is again a connection.
- In this way, we can construct flat connections with infinite dimensional space of holomorphic global sections

$$\Gamma(M, \nabla'_M) := \ker \nabla'_M.$$

- This never happens for a coherent sheaf over a projective variety!

Differential coherent sheaves

- Classically, if V is a projective variety, then each $\mathscr{F} \in \operatorname{coh} V$ has a presentation of the form

$$(\mathscr{L}^{\otimes t_1})^{n_1} \to (\mathscr{L}^{\otimes t_0})^{n_0} \to \mathscr{F} \to 0$$

($\mathscr{L} \in \operatorname{coh} V$ an ample line bundle).

- For $\mathcal{O}_q(\mathrm{Gr}_{n,r})$, we have a unique quantization for line bundles

$$\nabla_{\mathscr{L}_{n,q}} \colon L_{n,q} \longrightarrow L_{n,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}.$$

(Ó Buachalla and Mrozinski, 2017).

- So we can define the category $\operatorname{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}$ of differential coherent sheaves as the subcategory of $\mathrm{Hol}(\mathcal{O}_q(\mathrm{Gr}_{n,r}))$ consisting of the connections $\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}$ admitting a presentation

$$\begin{array}{ccccccc}
L_{t_1,q}^{n_1} & \longrightarrow & L_{t_0}^{n_0} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \\
L_{t_1,q}^{n_1} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & L_{t_0,q}^{n_0} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & 0
\end{array}$$

Comparison: algebraic and differential sheaves

- For $\mathcal{O}_q(\mathrm{Gr}_{n,r})$, we have
  1. Algebraic coherent sheaves $\mathrm{coh}_q\mathrm{Gr}_{n,r} = \mathrm{mod}^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})/\mathrm{mod}_0^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})$, and
  2. Differential coherent sheaves $\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r} = \{\nabla_M\colon M \to M \otimes_A \Omega^{(0,1)}\}$.
- The aim is to show that the categories are equivalent.
- For this we need that certain cohomologies vanish in $\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}$. More precisely, we focus on cohomologies of the dg $\Omega^{(0,\bullet)}$-module

$$0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots$$

  which we obtain by Leibniz rule because ∇_M is flat (Dolbeault cohomology).

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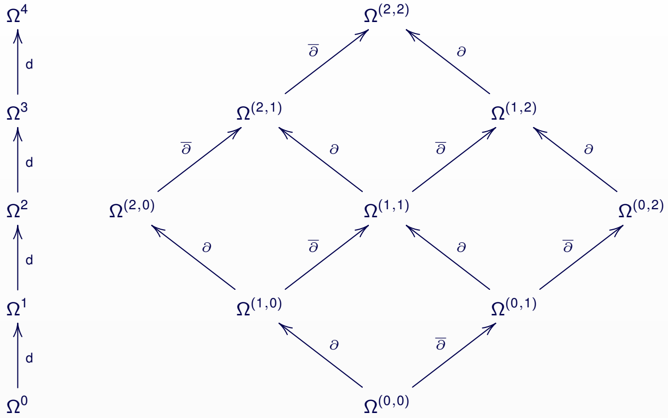
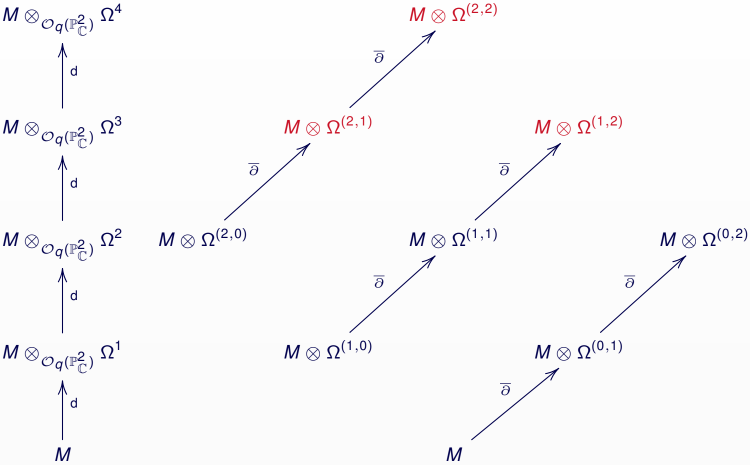
- The complex structure on (quantized or not) $\mathbb{P}^2_{\mathbb{C}}$:



Complex structure and ‘holomorphic’ connections

- If $(\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}) \in \mathrm{Hol}(A)$, we tensor over the dg $\Omega^{(0,\bullet)}$-module with the diamond. Example for $A = \mathcal{O}_q(\mathbb{P}^2_{\mathbb{C}})$:



- Kodaira vanishing (under extra assumptions!)

Kodaira vanishing

Theorem (Ó Buachalla, Š., van Roosmalen)

Suppose we have a (non-commutative) Kähler differential calculus (such as the one for $\mathcal{O}_q(\mathrm{Gr}_{n,r})$) and let (M, ∇_M) be a positive Hermitian vector bundle. Then $H^{(a,b)}(M) = 0$ for all $a + b > d$, where d is the dimension of the calculus.

- The non-commutative Kähler structure is defined via a closed real central form $\kappa \in \Omega^{(1,1)}$ such that $L = \kappa \wedge -$ induces isomorphisms $L^{n-k}\colon \Omega^k \to \Omega^{2n-k}$ for each k.

Theorem (Krutov, Ó Buachalla, Strung)

The line bundles $\nabla_{\mathscr{L}_{t,q}}\colon L_{t,q} \to L_{t,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$ over $\mathcal{O}_q(\mathrm{Gr}_{n,r})$ are positive (= ample) for $t > 0$.

- For quantum Grassmannians, we have the following version of the Bott-Borel-Weil theorem:

Theorem (Ó Buachalla, Š., van Roosmalen)

For $t \geq 0$ and the line bundle $\nabla_{\mathscr{L}_{t,q}} : L_{t,q} \to L_{t,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$, we have

$$H^0(L_{t,q}) = V(t\varpi_r) \quad \text{and} \quad H^i(L_{t,q}) = 0 \text{ for all } i > 0.$$

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Theorem (Artin and Zhang 1994, Polishchuk 2005)

Suppose that $\mathcal{A}$ is an abelian category. Suppose further that we have fixed object $\mathcal{O}_\mathcal{A}$ (an abstract structure sheaf) and an autoequivalence $(1)\colon \mathcal{A} \to \mathcal{A}$ (an abstract twist functor), such that:

1. *$\mathcal{O}_\mathcal{A}$ is noetherian and $\mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A})$ is a noetherian $\mathrm{End}_\mathcal{A}(\mathcal{O}_\mathcal{A})$-module for each $M \in \mathcal{A}$.*
2. *For each $M \in \mathcal{A}$, there are integers $t_1, t_2, \dots, t_m$ and an epimorphism $\bigoplus_{i=1}^m \mathcal{O}_\mathcal{A}(-t_i) \twoheadrightarrow M$.*
3. *For each epimorphism $M \twoheadrightarrow N$ in $\mathcal{A}$, there is an integer n_0 such that for every $n \geq n_0$, the map*
$$\mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, M(n)) \to \mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, N(n))$$
is surjective.

Then $\mathcal{A} \simeq \mathrm{mod}^\mathbb{Z} S(\mathcal{A})/\mathrm{mod}_0^\mathbb{Z} S(\mathcal{A})$ for $S(\mathcal{A}) = \bigoplus_{n=0}^\infty \mathrm{Hom}(\mathcal{O}_\mathcal{A}, \mathcal{O}_\mathcal{A}(n))$ (an abstract homogeneous coordinate ring).

- Now we just put everything together.
- For $\mathcal{A} = \mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}$, the abstract structure sheaf will be $\mathcal{O}_{\mathcal{A}} = (\overline{\partial}\colon \mathcal{O}_q(\mathrm{Gr}_{n,r}) \to \Omega^{(0,1)})$ and we construct a twist functor such that $\mathcal{O}_{\mathcal{A}}(1) = (\nabla_{\mathscr{L}_{1,q}}\colon L_{1,q} \to L_{1,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)})$.
- Now we apply to Bott-Borel-Weil theorem for quantized Grassmannians to obtain

Theorem (Ó Buachalla, Š., van Roosmalen)

The categories $\mathrm{coh}_q\mathrm{Gr}_{n,r} = \mathrm{mod}^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})/\mathrm{mod}_0^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})$ *and* $\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r} = \{\nabla_M\colon M \to M \otimes_A \Omega^{(0,1)}\}$ *are equivalent via*

$$\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r} \xrightarrow{\Gamma_*} \mathrm{coh}_q\mathrm{Gr}_{n,r},$$

$$(\nabla_M\colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}) \longmapsto \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{coh}_q^{\overline{\partial}}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M(n)).$$

Dolbeault vs. sheaf cohomology

- The Bott-Borel-Weil theorem implies more.
- For each $\nabla_M \colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$, we can apply two cohomology theories:
  1. The Dolbeault cohomology—as before, from the complex
  $$0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots$$
  2. The intrinsic cohomology in the abelian category $\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}$:
  $$\mathrm{Ext}^n_{\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M)$$
  (abstract sheaf cohomology).

Finiteness for cohomology

Theorem (Ó Buachalla, Š., van Roosmalen)

For each coherent sheaf $\nabla_M \colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$ over a quantum Grassmannian and for each $n \geq 0$, the two cohomologies are isomorphic:

1. $H^n(0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots)$ *and*
2. $\mathrm{Ext}^n_{\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M)$.

Corollary

The Dolbeault cohomology of a coherent sheaf is finite dimensional over $\mathbb{C}$.

Thank you for your attention!