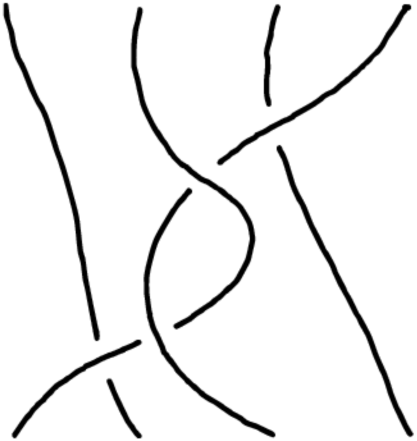
FROM BRAIDS TO QUANTIZATION



our aim:

- quantization of Poisson-Lie groups (or Poisson-Hopf algebras)
- quantization of moduli spaces of flat connections
- how it fits together

The toolbox: braids, Drinfeld associators

Braid groups

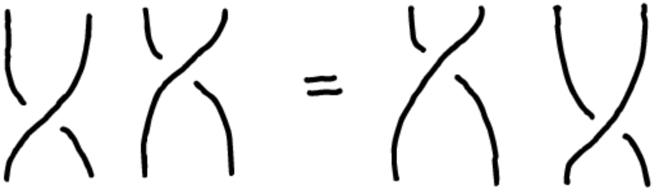
B_n = braids with n strands $= \pi_1((\mathbb{C}^n \setminus \text{diags})/S_n)$

group product = vertical composition

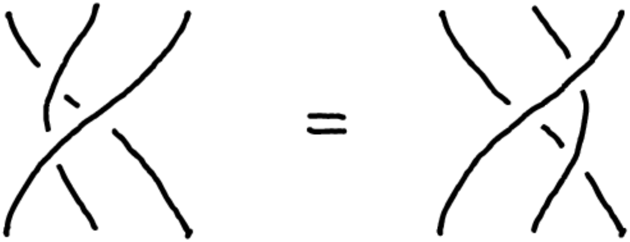
generators: $s_i =$

relations:

$s_i s_j = s_j s_i$
if $|i-j| \geq 2$



$s_i \, s_{i+1} \, s_i =$
$= s_{i+1} \, s_i \, s_{i+1}$



Monoidal categories

= categories with an "associative" product

$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}, \quad 1_{\mathcal{C}} \in \mathcal{C}$

associativity: natural iso $(X \otimes Y) \otimes Z \xrightarrow[]{\overset{\gamma_{XYZ}}{\sim}} X \otimes (Y \otimes Z)$

satisfying the pentagon:

$\Downarrow$

any rebracketing path gives the same result

$$\begin{tikzcd}
 & ((X\,Y)\,Z)\,W \arrow[ld] \arrow[rr] & & (X\,Y)(Z\,W) \arrow[rd] & \\
(X\,(Y\,Z))\,W \arrow[rd] & & & & X\,(Y\,(Z\,W)) \\
 & & X((Y\,Z)\,W) \arrow[rru] & &
\end{tikzcd}$$

Examples: $(\mathrm{Vect}_k, \otimes, K)$, $(\mathrm{Set}, \sqcup, \emptyset)$, $(\mathrm{Set}, \times, \{*\})$

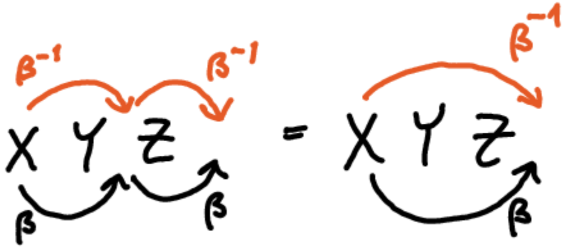
Braided monoidal categories

BMC = a monoidal categ. with a natural isomorphism

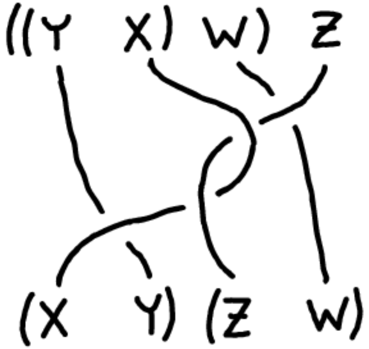
$\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$

$\beta_{X,Y} =$

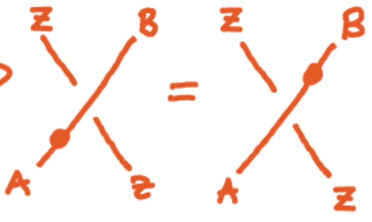
such that
(hexagon relations)



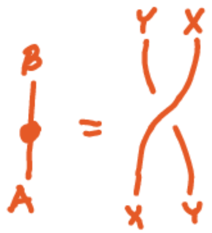
⇒ a natural iso for (parenthesised) braids :



proof: naturality ⇒

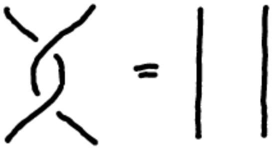


use for



symmetric monoidal category (SMC):

a BMC s.t.



drawing: $\beta_{X,Y} =: \sigma_{X,Y} =$

⇒ permutations instead of braids

Examples of SMCs: $(\mathrm{Vect}, \otimes)$ $(\mathrm{Set}, \times)$ $(\mathrm{Set}, \sqcup)$ ...

An example of a true BMC: a free one (parenthesized braids)

objects: $(\bullet\ (\bullet\ \bullet))\ (\bullet\ \bullet)$ $\qquad 1 =$ empty

morphisms:

$\boxed{A} \otimes \boxed{B} = \boxed{A \mid B}$

Monoidal functors

$F : \mathcal{C} \to \mathcal{D}$ is <u>monoidal</u> if : a natural iso

$$F(X) \otimes F(Y) \xrightarrow[(*)]{\sim} F(X \otimes Y)$$ is given

s.t.

$$\begin{tikzcd}
(F(X)\,F(Y))\,F(Z) \arrow[r] \arrow[d, "\gamma"'] & F(XY)\,F(Z) \arrow[r] & F((XY)\,Z) \arrow[d, "F(\gamma)"] \\
F(X)\,(F(Y)\,F(Z)) \arrow[r] & F(X)\,F(YZ) \arrow[r] & F(X\,(YZ))
\end{tikzcd}$$

and a compatible iso $F(1_{\mathcal{C}}) \xrightarrow[(*)]{\sim} 1_{\mathcal{D}}$

F is <u>lax monoidal</u> : (*) don't have to be isomorphisms

F is <u>braided monoidal</u> if

$$\begin{tikzcd}
F(X)\,F(Y) \arrow[r, "\beta"] \arrow[d] & F(Y)\,F(X) \arrow[d] \\
F(XY) \arrow[r, "F(\beta)"] & F(YX)
\end{tikzcd}$$

Infinitesimal braids, or chord diagrams

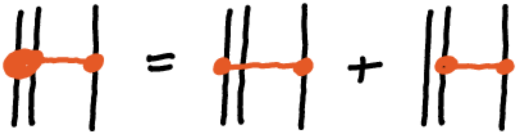
An *infinitesimally braided category* = a linear SMC $\mathcal{C}$ with natural transf. $t_{X,Y} : X \otimes Y \to X \otimes Y$ s.t.

$\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \varepsilon\, t_{X,Y})$ $\quad \varepsilon^2 = 0$

is a braiding in $\mathcal{C}$ and $t_{X,Y} = t_{Y,X}$

Drawing: $t_{X,Y} =$

what it means:



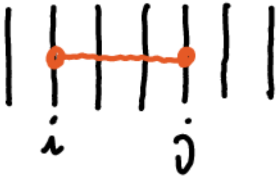
Example: $\mathfrak{g}$ a Lie algebra, $t \in (S^2\mathfrak{g})^{\mathfrak{g}} \subset \mathfrak{g} \otimes \mathfrak{g}$

$\mathcal{C} = U\mathfrak{g}\text{-mod}$, $\quad t_{X,Y} = \rho_X \otimes \rho_Y (t) \in \mathrm{End}(X \otimes Y)$

The algebra of infinitesimal pure braids:

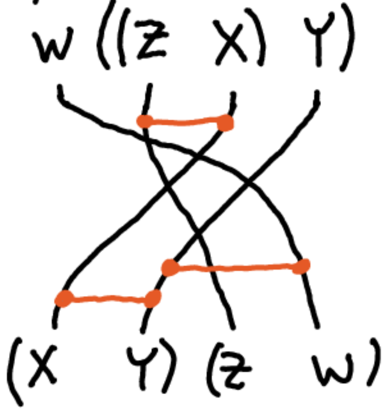
$\mathcal{A}_n$ generated by t_{ij}'s $(1 \leq i,j \leq n,\ i \neq j,\ t_{ji} = t_{ij})$

relations: $[t_{ij}, t_{k\ell}] = 0$ if i, j, k, ℓ all different $\qquad [t_{ij}, t_{ik} + t_{jk}] = 0$

$t_{ij} =$ 

$\mathcal{A}_n$ is a cocommutative Hopf algebra (t_{ij}'s primitive)

Operations in a iBMC:



= parenthesised permutations × $\mathcal{A}_n$

Drinfeld associators

Recall: $\mathcal{C}$ is an iBMC $\Leftrightarrow$ $\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \varepsilon t_{X,Y})$ is a braiding $(\varepsilon^2 = 0)$

The problem: extend the 1st order deformation to a true deformation (i.e.: remove $\varepsilon^2=0$)

Theorem (Drinfeld): $\exists\ \Phi \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ such that $\beta^{new}_{X,Y} := \sigma_{X,Y} \circ \exp\left(\frac{\hbar}{2} t_{X,Y}\right)$ and $\gamma^{new}_{X,Y,Z} := \gamma^{old}_{X,Y,Z} \circ \Phi(\hbar t_{X,Y}, \hbar t_{Y,Z})$ make every $\mathcal{C}$ to a BMC.

$(\gamma_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z))$

Φ is called a Drinfeld associator

additional requirement: $\Delta \Phi = \Phi \otimes \Phi$ (for x, y primitive)

Where associators come from:

KZ-connection $A_n \in \Omega^1(\mathbb{C}^n \setminus \text{diags}) \otimes \mathcal{A}_n$

$$A_n = \sum_{k<\ell} t_{k\ell} \frac{d(z_k - z_\ell)}{z_k - z_\ell}$$

is flat

$$\mathrm{hol}_{\circlearrowleft} A_2 = \exp(2\pi i\, t_{12})$$

$$\boxed{\Phi_{KZ}(t_{12}, t_{23}) := \lim_{\varepsilon \to 0_+} \varepsilon^{-t_{23}} \cdot \mathrm{hol}\, A_3 \cdot \varepsilon^{t_{12}}}$$

then $\Phi(x,y) := \Phi_{KZ}\left(\frac{x}{2\pi i}, \frac{y}{2\pi i}\right) \in \mathbb{C}\langle\langle x, y \rangle\rangle$

is an associator

Other sources of associators:

Kontsevich's deformation quantization

$$\Rightarrow \Phi_{AT} \in \mathbb{R}\langle\langle x,y\rangle\rangle$$

$\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}})$ generic $\left(\sigma^{p-1}(e^{2\pi i/p^2}) \neq e^{2\pi i/p^2},\ p \neq 2\right)$

$$\Rightarrow \Phi_{\sigma} \in \mathbb{Q}_p\langle\langle x,y\rangle\rangle$$

Drinfeld: free & transitive actions

$$\begin{tikzcd} GT \arrow[d, equal] & \circlearrowright \mathrm{Assoc} \circlearrowleft & GRT \arrow[d, equal] \\ \mathrm{Aut}(\text{parenthesised braids}) & & \mathrm{Aut}(\text{parenthesised } i\text{-braids}) \end{tikzcd}$$

$\Rightarrow \Phi \in \mathbb{Q}\langle\langle x,y\rangle\rangle$ exists

Brown: an "explicit" $\Phi_B \in \mathbb{Q}\langle\langle x,y\rangle\rangle$

QUANTIZATION OF POISSON-HOPF ALGEBRAS

Or : Deformation quantization of Poisson-Lie groups

What we shall do:

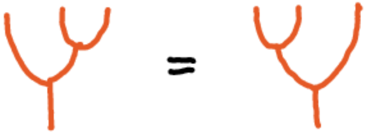
- express Hopf algebras in terms of braids
- express Poisson-Hopf algebras in terms of infinitesimal braids
- Drinfeld associators $\rightsquigarrow$ quantization of Poisson-Hopf algebras to Hopf algebras

Hopf algebras

A Hopf algebra $\mathcal{H}$ in a BMC $\mathcal{C}$ is:

a monoid (unital algebra): $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$, $\eta : 1_{\mathcal{C}} \to \mathcal{H}$

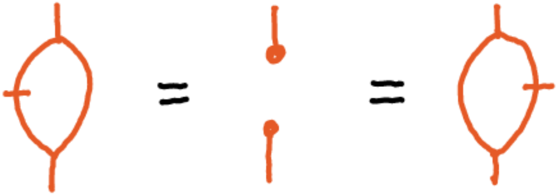
a comonoid (counital coalgebra): $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$, $\varepsilon : \mathcal{H} \to 1_{\mathcal{C}}$



compatibility:

antipode:

$S : \mathcal{H} \to \mathcal{H}$



we shall demand S invertible

Poisson-Hopf algebras

A Poisson-Hopf algebra $\mathcal{H}$ in a linear SMC $\mathcal{C}$ is: a commutative Hopf algebra (

) with a Poisson bracket $\{,\}$ s.t. $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a Poisson algebra morphism

Poisson bracket on $\mathcal{H} \otimes \mathcal{H}$:

Geometrically: $\mathcal{C} = \text{Vect}$, $\mathcal{H} = C^\infty(G)$

G a Lie group, with $\{,\}$ – Poisson-Lie group

e.g. the $ax+b$ group, $\{a,b\} = ab$

The problem

Given a Poisson-Hopf algebra $(\mathcal{H}, m_0, \Delta_0, \{,\}, \eta, \varepsilon)$ construct $m_\hbar = m_0 + \hbar m_1 + \hbar^2 m_2 + \dots$, $\Delta_\hbar = \Delta_0 + \hbar^2 \Delta_2 + \dots$, $S_\hbar = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots$ s.t. $m_1 - m_1^{op} = \{,\}$, so that $(\mathcal{H}, m_\hbar, \Delta_\hbar, S_\hbar, \eta, \varepsilon)$ is a Hopf algebra

m_i's, Δ_i's, S_i's should be given by universal expressions in terms of $m_0, \Delta_0, S_0, \{,\}, \eta, \varepsilon$. The SMC $\mathcal{C}$ needs to be $\mathbb{Q}$-linear (formulas have denominators)

Etingof - Kazhdan '95: solution for $\mathcal{H} = (U\mathfrak{g})^*$

Our method: joint work with Ján Pulmann

The nerve of a group G

is the functor $F : \mathrm{FinSet}^{op} \to \mathrm{Set}$

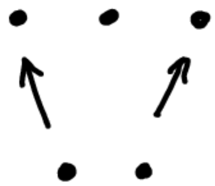
$$F(X) := \{ g : X \times X \to G \mid g(a,b)\, g(b,c) = g(a,c),\ g(a,a) = 1 \}$$

$$F(X) \cong G^{|X|-1} : \quad \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "g(1,2)"] & \underset{2}{\bullet} \arrow[r, "g(2,3)"] & \underset{3}{\bullet} \arrow[r, "g(3,4)"] & \underset{4}{\bullet} \end{tikzcd}$$

For a general F: $\quad F(X) \to F(\bullet\bullet)^{|X|-1} \quad (*)$

Proposition: F is the nerve of a group iff $(*)$ is a bijection $\forall X$. In that case $G = F(\bullet\bullet)$, the product is

$$F(\bullet\bullet) \times F(\bullet\bullet) \cong F(\bullet\bullet\bullet) \xrightarrow{F(\quad)} F(\bullet\bullet)$$

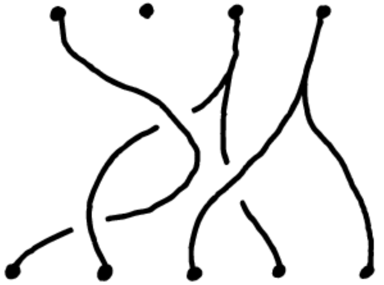


Category BrSet

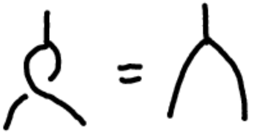
Motivation: nerves of Hopf algebras

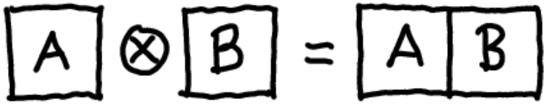
- FinSet needs to be replaced with BrSet

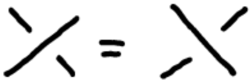
BrSet is the BMC with morphisms



the BMC freely generated by a commutative monoid



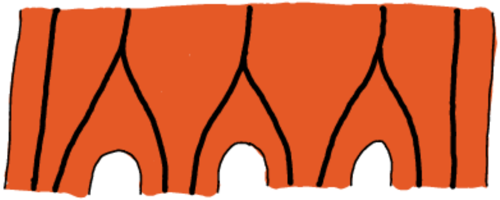


(Imposing  gives FinSet)

The nerve of a Hopf algebra

Theorem: Hopf algebras in a BMC $\mathcal{C}$ are equivalent to braided lax monoidal functors $F : \mathrm{Br\,Set} \to \mathcal{C}$

s.t.



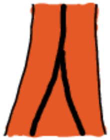
$: F(\cdot\cdot)^{n} \xrightarrow[\varphi_n]{} F(\cdot^{n+1})$ $(n=4)$

is an iso $(\forall n)$ and
$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r] \arrow[rr, bend right, "\Psi"'] & F(\emptyset) \arrow[r] & F(\cdot) \end{tikzcd}$$
are isos.

Getting $\mathcal{H}$ from F: $\mathcal{H} = F(\bullet\bullet)$

$\eta =$

$\varepsilon = \psi^{-1} \circ$



$\Delta = \varphi_2^{-1} \circ$

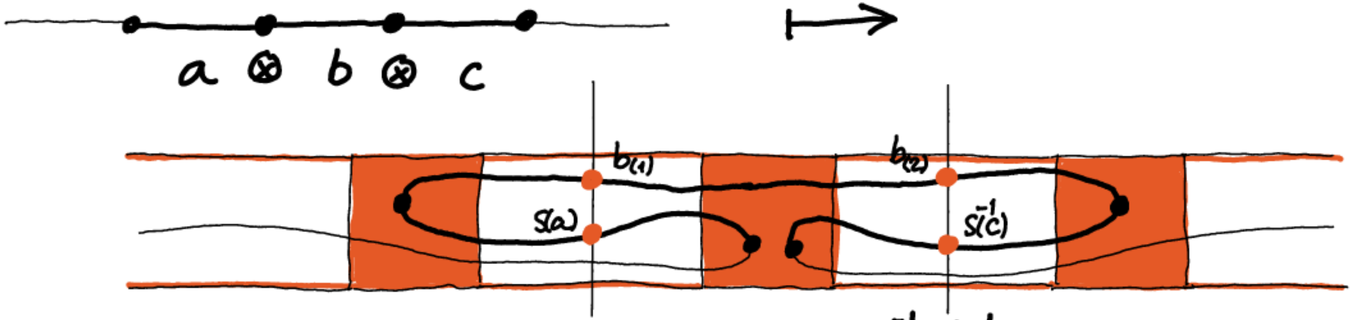
$m =$

$S =$

Constructing the nerve of a Hopf algebra $\mathcal{H} \in \mathcal{C}$

objects: $F(\bullet^n) = \mathcal{H}^{n-1}$, $F(\emptyset) = 1_{\mathcal{C}}$

morphisms: $F\Big($  $\Big) : \mathcal{H}^3 \longrightarrow \mathcal{H}^2$



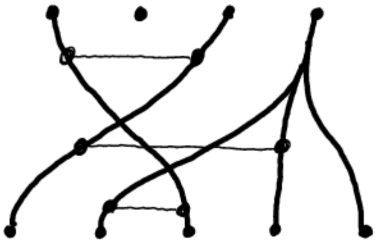
$a \otimes b \otimes c \longmapsto S(a)\,b_{(1)} \otimes S^{-1}(c)\,b_{(2)}$

F monoidal: $F(\bullet^m)F(\bullet^n) \to F(\bullet^{m+n})$, $\mathcal{H}^{m-1}\mathcal{H}^{n-1} \to \mathcal{H}^{m+n-1}$

= insertion of 1

The nerve of a Poisson-Hopf algebra

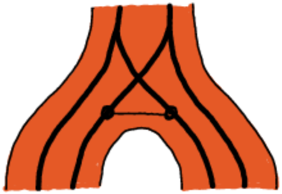
CSet : the i-braided version of FinSet / BrSet



 $= 0$

(analogous to  $=$  in BrSet)

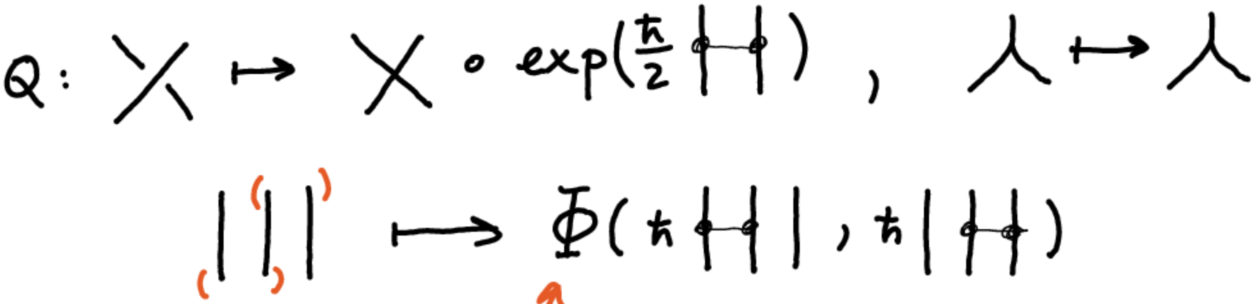
Theorem: Poisson-Hopf algebras in $\mathcal{C}$ are equivalent to i-braided lax monoidal functors $F: \mathrm{CSet} \to \mathcal{C}$ s.t. φ_n's and $1_{\mathcal{C}} \to F(\emptyset) \to F(\cdot)$ are isos.

$\{,\} =$ 

Quantization of Poisson-Hopf algebras

$$\begin{tikzcd}
\text{(Pa) BrSet} \arrow[r, "Q"] \arrow[rr, bend right, "\text{Hopf}"'] & \text{CSet} \arrow[r, "\text{Poisson-Hopf}"] & \mathcal{C}
\end{tikzcd}$$

the parenthesized version of BrSet



a Drinfeld associator

monoidal struct.: $Q(X)\,Q(Y) \longrightarrow Q(XY)$ is the identity

- Φ group-like $\Rightarrow$ quantization compatible with tensor products of Hopf algebras

- Hopf algebras in BMCs:

  recall: if $\mathcal{C}$ is an iBMC, Φ will turn it to a BMC $\mathcal{C}^{\Phi}_{\hbar}$

  then: if $\mathcal{H}$ is a Poisson-Hopf algebra in $\mathcal{C}$, Q quantizes it to a Hopf algebra in $\mathcal{C}^{\Phi}_{\hbar}$

- nerves can be defined also for groupoids
  - Q quantizes semi-commutative Hopf algebroids

MODULI SPACES OF FLAT CONNECTIONS ON SURFACES

plan:

- (quasi-) Poisson structures on moduli spaces for surfaces with a decorated boundary

- their deformation quantization

Atiyah-Bott symplectic structure

ingredients: connected Lie group G

invariant $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$

closed oriented surface Σ

moduli space $\mathcal{M}(\Sigma ; G) = \mathrm{Hom}(\pi_1(\Sigma), G) / G$

$= \{$isom. classes of flat G-bundles over $\Sigma\}$

has a natural symplectic form ω_{AB}

$\mathcal{M}(\Sigma ; G) = \Omega^1(\Sigma, \mathfrak{g}) /\!/$ gauge group

cst sympl. form $\omega(\alpha, \beta) = \int_\Sigma \langle \alpha, \beta \rangle$

$$T_{[P,A]} \mathcal{M}(\Sigma ; G) = H^1(\Sigma ; \mathrm{ad}_P)$$

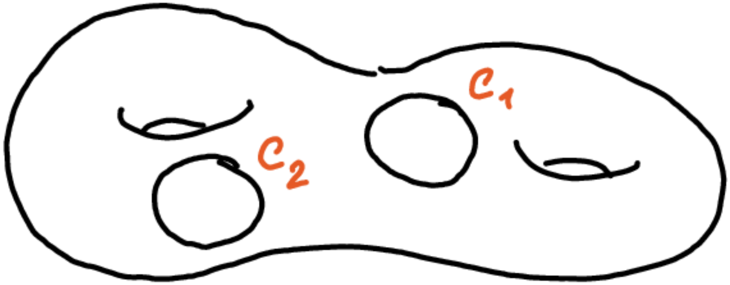
$$\omega_{AB}([\alpha], [\beta]) = \int_\Sigma \langle \alpha \overset{\wedge}{,} \beta \rangle$$ intersection pairing

Surfaces with boundary

Σ cpt., oriented, with a boundary

$\Rightarrow \mathcal{M}(\Sigma; G)$ is Poisson

symplectic leaves = conjugacy classes of bdry holonomies



$$T^*_{P,A}\mathcal{M}(\Sigma; G) = H_1(\Sigma; \mathrm{ad}^*_P)$$

Poisson bivector = intersection pairing

Quasi-Poisson structures

a $\mathfrak{g}$-qPoisson manifold = a $\mathfrak{g}$-manifold M with a $\mathfrak{g}$-invariant bivector field π s.t.

$$\frac{1}{2}[\pi,\pi] = \varphi_M \qquad \text{where } \varphi \in \Lambda^3\mathfrak{g} \text{ is } \frac{1}{4}[\,,\,]$$

[Alekseev - Kosmann-Schwarzbach - Meinrenken]

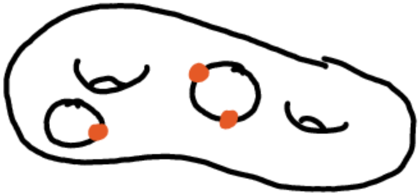
Motivation/quantization: $U\mathfrak{g}$-mod is i-braided, if $*$ is an associative product on $C^\infty(M) \in U\mathfrak{g}\text{-mod}^{\bar{\Phi}}_{\hbar}$
$f*g = fg + \hbar B_1(f,g) + \dots$, then

$$\{f,g\} := B_1(f,g) - B_1(g,f)$$

is $\mathfrak{g}$-qPoisson $\quad \left(\pi(df,dg) = \{f,g\} \right)$

[Enriquez - Etingof]

Surfaces with marked points



$V \subset \partial\Sigma$ a finite set

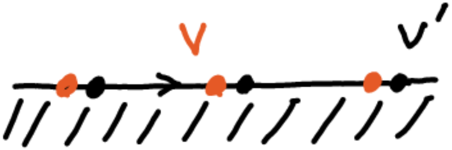
$\mathcal{M}(\Sigma, V; G)$ = flat G-bundles $P \to \Sigma$, trivialized over V

$G^V \circlearrowright \mathcal{M}(\Sigma, V; G)$ by changing the trivialization

$\mathcal{M}(\Sigma, V; G)$ is $\mathfrak{g}^V$-qPoisson, π is the skew-sym. part of

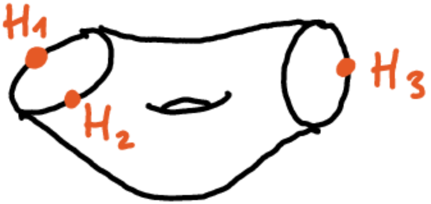
$$H_1(\Sigma, V; \mathrm{ad}^*) \times \underset{\displaystyle H_1(\Sigma, V; \mathrm{ad}^*)}{\underset{\cong}{H_1(\Sigma, V'; \mathrm{ad}^*)}} \to \mathbb{R}$$

$V' = V$ moved a bit along $\partial\Sigma$



Reduction and surfaces with a boundary

If M is G-qPoisson and $H \subset G$ is coisotropic ($\mathfrak{h}^\perp \subset \mathfrak{h}$)
then M/H is Poisson

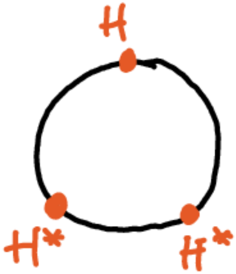


- reduction of the structure group to H_i's over the marked points gives a Poisson moduli space

symplectic leaves (supposing $\mathfrak{h}_i$'s Lagrangian : $\mathfrak{h}_i^\perp = \mathfrak{h}_i$):
fix boundary holonomies $\in H_i \backslash G / H_j$

example : $\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{g}$ Lagrangian
$\mathfrak{h} \cap \mathfrak{h}^* = 0$ (a Manin triple)

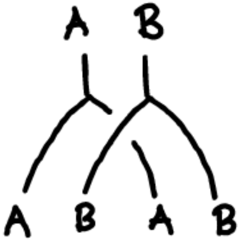
[with David Li-Bland]



$\cong H$ Poisson-Lie group

Fusion

Motivation: if A, B are monoids in a BMC $\mathcal{C}$ then so is $A \otimes B$:



If M_1, M_2 are $\mathfrak{g}$-qPoisson then $M_1 \times M_2$ with

$$\pi = \pi_1 + \pi_2 - \frac{1}{2} (e_i)_{M_1} \wedge (e^i)_{M_2}$$

is also $\mathfrak{g}$-qPoisson (the fusion of M_1 and M_2)

- $\mathfrak{g}$-qPoiss. manifolds form a monoidal category

More generally: if M is $\mathfrak{g} \oplus \mathfrak{g}$-qPoisson then it is $\mathfrak{g}$-qPoisson with the diagonal action and with

$$\pi^{new} = \pi^{old} - \frac{1}{2} \rho_1(e_i) \wedge \rho_2(e^i)$$

Building blocks and gluing

the building block: a disk with 2 marked points

$\pi = 0$

corner connected sum = fusion

In this way we can build any surface with marked points and thus compute the qPoisson structure

Quantization: repeat this procedure for associative products

Quantization of $\mathfrak{g}$-qPoisson manifolds

to associative products $*$ on $C^\infty(M)$ in the BMC $\mathcal{C} = U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}$

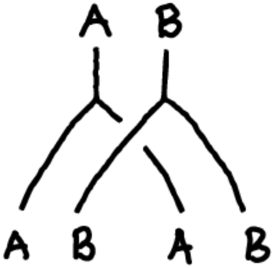
$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\Phi"] \arrow[d] & & A\otimes(A\otimes A) \arrow[d] \\
A\otimes A \arrow[r] & A & A\otimes A \arrow[l]
\end{tikzcd}$$

an easy method [Li-Bland, Š]:

- "commutative" $\mathfrak{g}$-qPoisson manifolds: $\mathfrak{g} \circlearrowright M$ has coisotropic stabilizers (e.g. $M = G/\overset{\substack{\text{coisotropic}\\ \swarrow}}{H}$), $\pi = 0$

  $\to$ no deformation needed, $C^\infty(M)$ is associative in $\mathcal{C}$ (and commutative)

- fusion: if $A = C^\infty(M)$, $B = C^\infty(N)$ are already quantized then $A \otimes B = C^\infty(M \times N)$ is



also: if a $\mathfrak{g}\oplus\mathfrak{g}$-qPoisson M is quantized, we get a quantization of the fused $\mathfrak{g}$-qPoisson M

Quantization of moduli spaces, at last

$+ \cdots \; \mathfrak{g} = (\mathfrak{g}, \langle , \rangle)$, $- \cdots \; \bar{\mathfrak{g}} = (\mathfrak{g}, -\langle , \rangle)$

$\mathcal{C} = \mathcal{U}(\mathfrak{g}^{V_+} \oplus \bar{\mathfrak{g}}^{V_-})\text{-mod}^{\Phi}_{\hbar}$

blocks: 

$= G = (G \times \bar{G}) / G_{diag}$ is commutative

− no quantization needed

fusion: 

, 

− apply the quantum fusion

i.e. $*_{new} = *_{old} \circ$ 

Poisson manifolds which we quantize

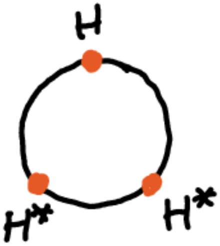
recall: if M is G-qPoisson and $K \subset G$ is coisotropic
then M/K is Poisson

if $C^\infty(M)$ is quantized to an algebra in $U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}$
then $C^\infty(M)^K = C^\infty(M/K)$ is a quantization of M/K

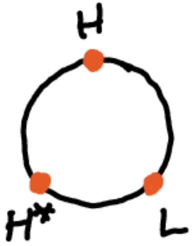
For moduli spaces: M = a moduli space,

$K \subset G^{V_+} \times \bar{G}^{V_-}$

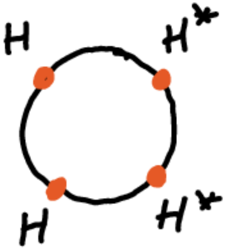
e.g.:



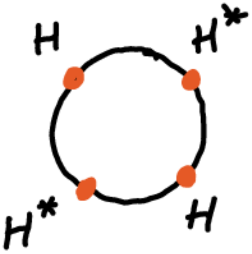
Poisson-Lie group H



Poisson homog. space of H



Drinfeld double



Heisenberg double

Functoriality and other problems

- is our quantization of moduli spaces independent of the choices of gluing?

  [Pulmann]: yes, though we need to modify fusion

- qPoisson structure on moduli spaces is functorial w.r.t. embedding of surfaces – is quantization functorial?

- [Ben-Zvi-Brochier-Jordan] our method should be more conceptual when formulated in terms of factorization homology