# Quantum Flag Manifolds: From Quantum Groups to Noncommutative Geometry

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*39th Winter School Geometry and Physics 2019 - Srní*

# 1: $C^*$-algebras and Noncommutative Topology

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  - $C(X)$ is complete with respect to $\|\cdot\|_\infty$

- The norm is *sub-multiplicative*, which is to say

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$$* : C(X) \to C(X), \qquad f \mapsto f^*$$

where

$$f^*(x) := \overline{f(x)}, \qquad \text{for all } x \in X.$$

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- Moreover, we have the very important identity

$$\|f^* f\|_\infty = \|f\|_\infty^2.$$

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A $*$-*algebra* $(A, *)$ is a complex algebra $A$ endowed with a conjugate linear **anti-multiplicative** involution $* : A \to A$.

### Example

A basic noncommutative example of a $*$-algebra is the matrices $M_n(\mathbb{C})$ endowed with the conjugate transpose, or more generally, the bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ endowed with the adjoint operation.

### Definition

A $C^*$*-algebra* is a unital Banach algebra $(\mathcal{A}, \|\cdot\|)$, together with a $*$-algebra structure on $\mathcal{A}$, such that

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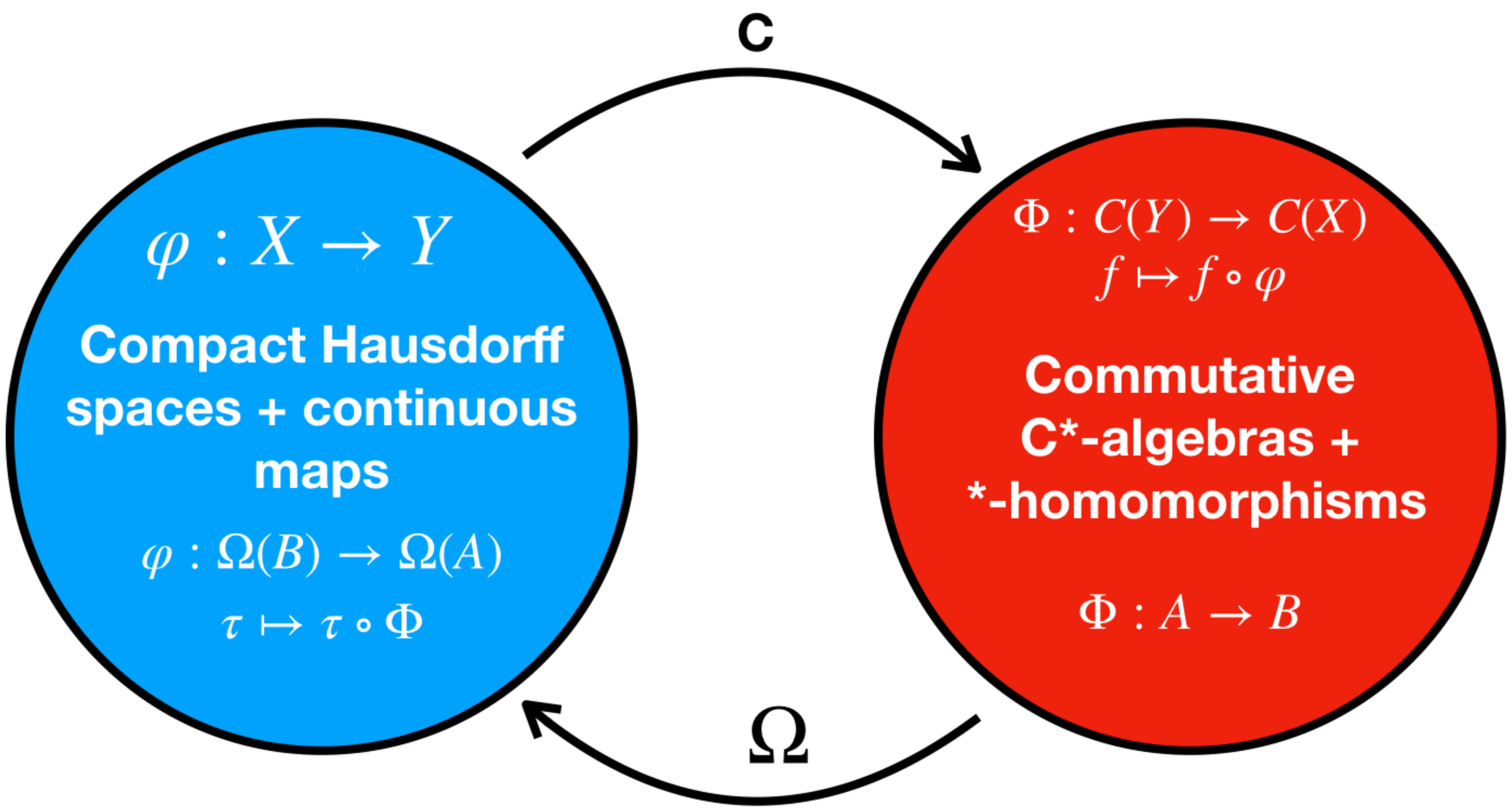
## Theorem (Gelfand–Naimark '43)

*Every commutative $C^*$-algebra is isomorphic to $C(X)$, for some compact Hausdorff space $X$.*

In fact, we get a duality of categories:




$\Omega(A) := \{\tau : A \to \mathbb{C} \mid \tau$ **a *-homomorphism**$\}$,
**equipped with weak-* topology**

**Dictionary** between topological structures and $C^*$-algebraic structures:

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| | | |
|---|---|---|
| compact space X | $\longleftrightarrow$ | comm. $C^*$-algebra $C(X)$ |
| locally compact space $X$ | $\longleftrightarrow$ | non-unital comm. $C^*$-algebra $C_0(X)$ |
| homeomorphism | $\longleftrightarrow$ | $*$-isomorphism |
| image of a function | $\longleftrightarrow$ | spectrum of an element |
| positive function | $\longleftrightarrow$ | positive element |
| regular Borel measure | $\longleftrightarrow$ | bounded linear functionals on $C(X)$ |
| one-point compactification of a space | $\longleftrightarrow$ | unitisation of $C(X)$ |
| open subset of $X$ | $\longleftrightarrow$ | ideal of $C(X)$ |
| $X$ is connected | $\longleftrightarrow$ | $C(X)$ is projectionless |
| $X$ is metrisable | $\longleftrightarrow$ | $C(X)$ is separable |
| vector bundle over $X$ | $\longleftrightarrow$ | finite projective module over $C(X)$ |
| measure space | $\longleftrightarrow$ | comm. von Neumann algebra |

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The algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ endowed with the adjoint operation and the *operator norm*

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**Example**

Any norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

**Theorem (Gelfand–Naimark–Segal '43)**

*Every $C^*$-algebra $\mathcal{A}$ admits a faithful $*$-representation*

$$\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

*for some Hilbert space $\mathcal{H}$.*

- For a commutative $C^*$-algebra $C(X)$ such a representation is given by the elements of $C(X)$ acting by multiplication on the square integrable functions $L^2(X, \mu)$, where $\mu$ is a Borel measure.

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  1. Connes' celebrated classification of injective von Neumann factors used noncommutative analogues of measure and ergodic theory.
  2. Elliott's classification program for simple separable nuclear $C^*$-algebras uses noncommutative topological $K$-theory and Winter's noncommutative topological covering dimension.

# 2: Compact Quantum Groups

- **Question:** What is a compact topological group in Gelfand–Naimark terms?

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- **Question:** What is a compact topological group in Gelfand–Naimark terms?
- **On the topological side:** It is an object $G$ in the category of compact Hausdorff spaces, together with a 4-tuple of morphisms $(m, \bullet^{-1}, \eta, \varepsilon)$

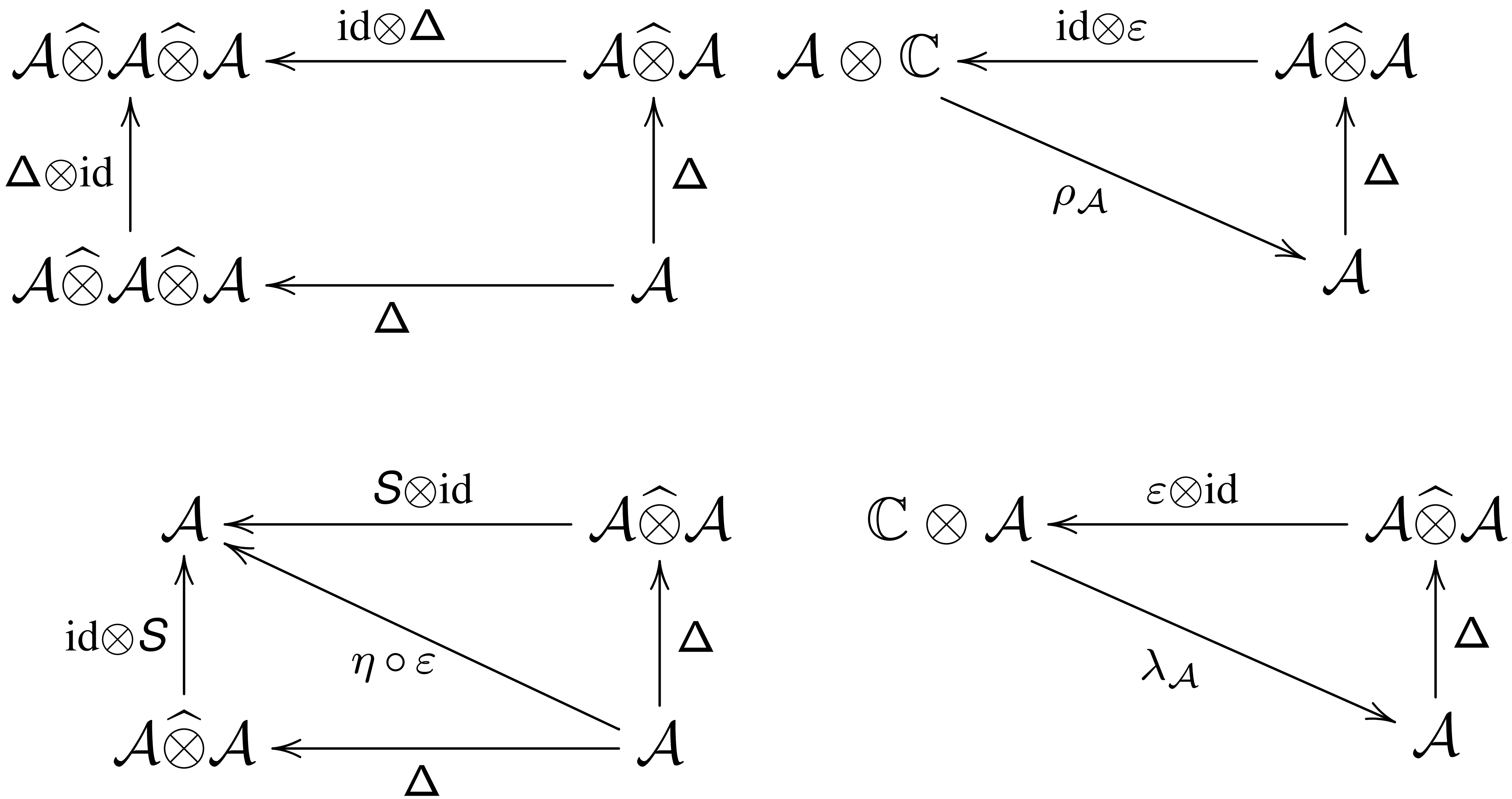
$$m : G \times G \to G, \qquad \bullet^{-1} : G \to G,$$
$$\iota : \{\bullet\} \to G \qquad e : G \to \{\bullet\}.$$

such that the following diagrams commute:

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mathrm{id} \times m} & G \times G \\
{\scriptstyle m \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle m} \\
G \times G & \xrightarrow[m]{} & G
\end{array}
\qquad
\begin{array}{ccc}
G \times \{\bullet\} & \xrightarrow{\mathrm{id} \times \eta} & G \times G \\
& \searrow^{\rho_G} & \downarrow {\scriptstyle m} \\
& & G
\end{array}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{\bullet^{-1} \times \mathrm{id}} & G \\
{\scriptstyle \mathrm{id} \times \bullet^{-1}} \downarrow & \searrow^{\iota \circ e} & \downarrow {\scriptstyle m} \\
G \times G & \xrightarrow[m]{} & G
\end{array}
\qquad
\begin{array}{ccc}
\{\bullet\} \times G & \xrightarrow{\varepsilon \circ \eta} & G \times G \\
& \searrow^{\lambda_G} & \downarrow {\scriptstyle m} \\
& & G
\end{array}
$$

Thus the dual structure in the category of commutative $C^*$-algebras is an object $\mathcal{A}$ and a 4-tuple $(\Delta, S, \varepsilon, \eta)$, where $\mathcal{A} = C(G)$ is a $C^*$-algebra with the following commutative diagrams:



- An important observation is the following:
  - Denote by $\mathcal{O}(G)$ the algebra of *representable functions* on $G$, that is, the functions generated by the coordinate functions of all the finite-dimensional representations $\rho : G \to M_k(\mathbb{C})$.
  - Note that $\mathcal{O}(G) \subseteq C(G)$.
  - It holds that $\Delta(\mathcal{O}(G)) \subseteq \mathcal{O}(G) \otimes \mathcal{O}(G)$, and $S(\mathcal{O}(G)) \subseteq \mathcal{O}(G)$.
- This gives us the definition of a Hopf algebra.

## Definition

A *Hopf algebra* is a 4-tuple $(C, \Delta, S, \varepsilon)$, where $C$ is a vector space, and

$$\Delta : C \to C \otimes C; \qquad S : C \to C, \qquad \varepsilon : C \to \mathbb{C},$$

are linear maps (called the *coproduct*, *antipode*, and *counit* respectively), satisfying the following axioms:

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
2. $(S \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes S) \circ \Delta = \varepsilon$,
3. $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$.

# 3: Drinfel'd–Jimbo Quantum Groups

- In Leningrad in the 1980 physicists working on the quantum inverse scattering method discovered

$$U_q(\mathfrak{sl}_2).$$

- The work of Vladimir Drinfeld and Michio Jimbo would generalise this to

$$U_q(\mathfrak{g}), \qquad \text{for } \mathfrak{g} \text{ any complex semisimple Lie algebra.}$$

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## Definition

We denote by $U_q(\mathfrak{sl}_2)$ the free noncommutative algebra generated by $E, F, K$, and $K^{-1}$, subject to the relations

$$KE = q^2 EK, \qquad KF = q^{-2} FK,$$

$$[E, F] - \frac{K - K^{-1}}{q - q^{-1}}.$$

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- **Warning:** When $q = 1$ the relations of $U_q(\mathfrak{sl}_2)$ are not well-defined! However, there exists an alternative (slightly more complicated) presentation of the algebra which is well-defined for $q = 1$, and forms a double cover of $U(\mathfrak{sl}_2)$.

## Definition

A Hopf algebra structure on $U_q(\mathfrak{sl})_2$ is defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$
$$S(E) = -EK^{-1}, \quad S(K) = K^{-1}, \quad S(F) = -FK,$$
$$\varepsilon(E) = \varepsilon(F) = 0.$$

- It was shown by Drinfeld and Jimbo that, for every semisimple complex Lie algebra $\mathfrak{g}$:
    - There exists a $q$-deformed universal enveloping algebra $U_q(\mathfrak{g})$.
    - For $q = 1$, $U_1(\mathfrak{g})$ forms a rank($\mathfrak{g}$)-fold cover of $U(\mathfrak{g})$.
    - $U_q(\mathfrak{g})$ comes endowed with a Hopf algebra structure, deforming that Hopf algebra structure of $U(\mathfrak{g})$.
- Moreover, for $q \in \mathbf{R}$, it admits a $*$-algebra structure, whose fixed points identify the compact real form of $\mathfrak{g}$.

- Some facts about the finite dimensional modules of $U_q(\mathfrak{g})$, when $q \in \mathbf{R} \backslash \{-1\}$:
  - The category of modules is semisimple and the irreducible modules are classified by the dominant weights of $\mathfrak{g}$.

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    mapping irreducibles to irreducibles.
  - Dimensions and characters remain unchanged.
  - The category ${}_{U_q(\mathfrak{g})}\mathrm{Mod}$ has a monoidal structure, defined for $V$ and $W$ irreducibles, and $v \in V$ and $w \in W$, according to

$$X \triangleright v \otimes w = \sum_i (X_i \triangleright v) \otimes (X_i' \triangleright w),$$

    where $\Delta(X) = \sum_i X_i \otimes X_i'$.

- Moreover, $Q(V \otimes W) \simeq Q(V) \otimes Q(W)$.
- This is *not* an equivalence of monoidal categories!!!

- The category admits the structure of a braided monoidal category, which is not symmetric when $q \neq 1$.

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- While the $q$-deformation of $U_q(\mathfrak{g})$ is not unique, work of Kazhdan–Wenzl, Wenzl–Tuba, and Liu (more or less) shows that, the monoidal category ${}_{U_q(\mathfrak{g})}\mathrm{Mod}$ is the unique monoidal deformation of ${}_{U(\mathfrak{g})}\mathrm{Mod}$.

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- A very interesting feature of a braided monoidal category is that it allows us to define a braided notion of dimension. This very important in applications to knots, and will arise later in our treatment of Dirac operator spectra.

# 5: Quantum Coordinate Algebras and Woronowicz

- For a Hopf algebra $H$, consider the linear dual $H^*$.
- We can dualise comultiplication of $H$ to a multiplication on $H^*$, such that for $f, g \in H^*$,

$$f * g(h) := \sum_i f(h_i) g(h_i'), \quad \text{where } h \in H,\ \Delta(h) = \sum_i h_i \otimes h_i'.$$

- With respect to this multiplication, the counit $\varepsilon_H$ is the unit $1_{H^*}$.
- The unit $1_H$ dualises to a counit $\varepsilon_{H^*}$

$$\varepsilon_{H^*} : H^* \to \mathbb{C}, \qquad f \mapsto f(1_H).$$

- We can dualise multiplication to a map

$$\Delta : H^* \to (H \otimes H)^*, \qquad \Delta(f)(h, g) := f(hg).$$

- When $H$ is infinite dimensional $(H \otimes H)^* \neq H^* \otimes H^*$, so $\Delta$ is not a comultiplication. However, there does exist a smallest $H^\circ \subseteq H^*$ such that

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- It can be shown that

$$H^\circ \simeq \bigoplus_{\alpha \in \widehat{H}} V_\alpha \otimes V_\alpha^\vee,$$

where $\widehat{H}$ denotes the finite dimensional representations of $H$, $V_\alpha$ the left module, $V_\alpha^\vee$ the dual right module.

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where $\widehat{H}$ denotes the finite dimensional representations of $H$, $V_\alpha$ the left module, $V_\alpha^\vee$ the dual right module. So the Hopf dual can also be viewed as a type of 'Peter–Weyl dual'.

### Definition

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**Theorem**

*The $*$-structure of $U_q(\mathfrak{g})$ dualises to a $*$-algebra structure on $\mathcal{O}_q(G)$. Moreover, $\mathcal{O}_q(G)$ admits a unique completion to a $C^*$-algebra $C_q(G)$.*

- **Problem:** The maps $S$ and $\varepsilon$ are not in general bounded operators, and hence, do not admit an extension to the completion of $\mathcal{O}_q(G)$.

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- Thus our guess for the definition of a **compact quantum group** was too naive.
- This is where we need to look to Woronowicz for help . . .

## Definition (Woronowicz '87)

A *compact quantum group* is a pair $(\mathcal{A}, \Delta)$, where $\mathcal{A}$ is a $C^*$-algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\text{min}} \mathcal{A}$ is a $*$-homomorphism such that

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$
2. $(1 \otimes_{\text{min}} \mathcal{A})\Delta(\mathcal{A})$ and $(\mathcal{A} \otimes_{\text{min}} 1)\Delta(\mathcal{A})$ are dense in $\mathcal{A} \otimes_{\text{min}} \mathcal{A}$.

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## Theorem

*For every Drinfeld–Jimbo quantised enveloping algebra $U_q(\mathfrak{g})$, the pair $(C_q(G), \Delta)$ is a compact quantum group.*

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## Theorem

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## Theorem

*Every compact quantum group contains a dense Hopf algebra.*


Naimark
Gelfand
Woronowicz
$SU_q(2)$
Jimbo
Drinfeld

**Question**

What about a noncommutative generalisation of compact differentiable manifolds, or even Lie groups?

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**Question**

Can we construct motivating examples from Drinfeld–Jimbo quantum groups?


Connes
Jimbo
Drinfeld

# Quantum Flag Manifolds: From Quantum Groups to Noncommutative Geometry II

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- The Gelfand–Naimark Theorem:

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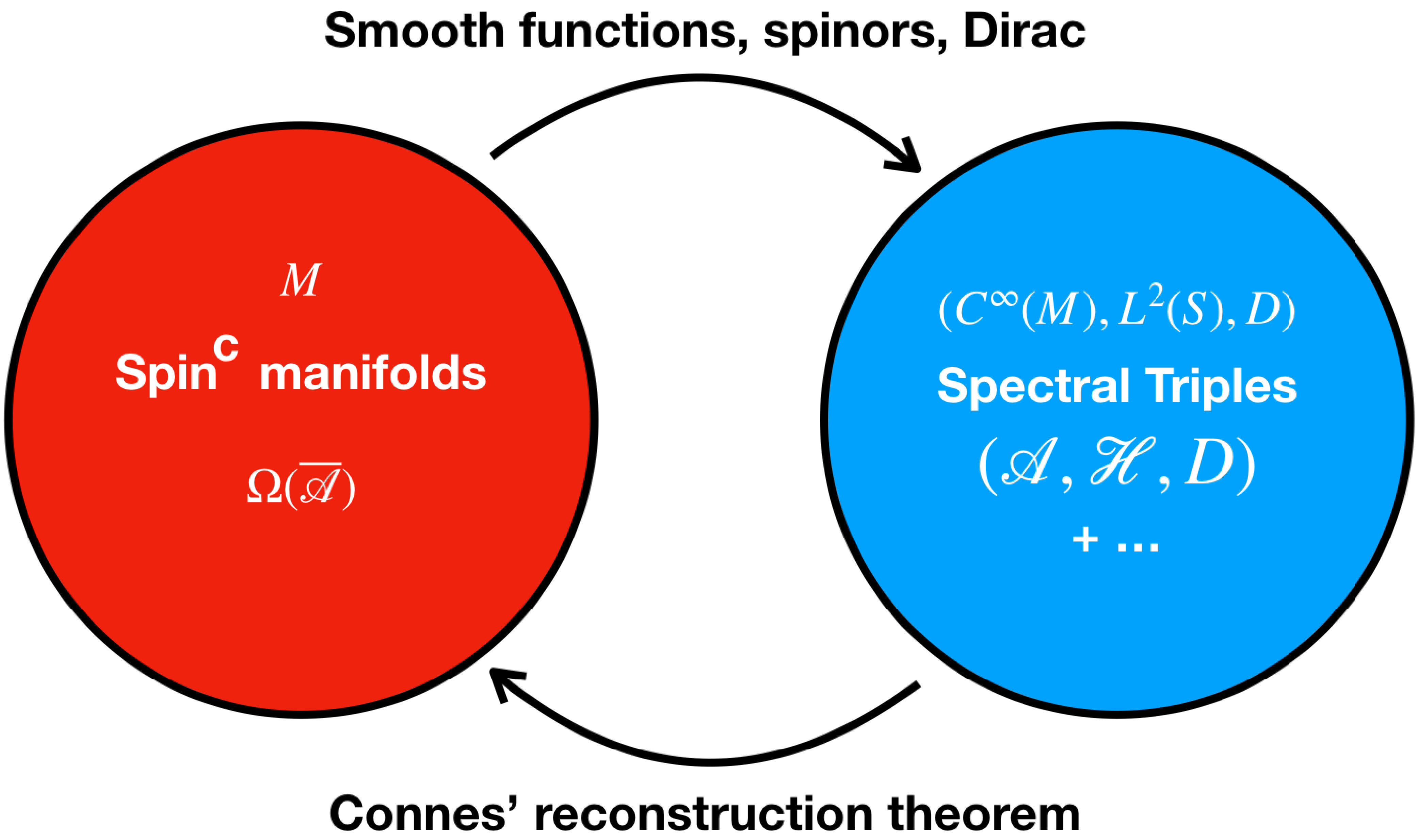
- Woronowicz's Theorem:

  Compact topological Groups $\longleftrightarrow$ Commutative Compact Quantum Groups

- **Question** Can we express differential structures on a compact Hausdorff space in terms of some $C^*$-algebraic differential structure on $C(X)$?


Smooth functions, spinors, Dirac
$M$
Spin$^c$ manifolds
$\Omega(\overline{\mathscr{A}})$
$(C^\infty(M), L^2(S), D)$
Spectral Triples
$(\mathscr{A}, \mathscr{H}, D)$
+ ...
Connes' reconstruction theorem

- But what is a spectral triple?

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## Definition

A **spectral triple** is a triple $(A, \mathcal{H}, D)$, where

- $A$ is a dense $*$-subalgebra of a $C^*$-algebra,
- $\mathcal{H}$ is a Hilbert space with a faithful $*$-representation $\rho : A \to \mathcal{B}(\mathcal{H})$
- $D$ is a densely defined unbounded self-adjoint operator $D : \mathrm{dom}(D) \to \mathcal{H}$, such that

$$[D, a] \in \mathcal{B}(\mathcal{H}), \text{ for all } a \in A, \quad \text{and} \quad (1 - D^2)^{-1} \in \mathcal{K}(\mathcal{H}).$$

- $\mathcal{K}(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$, i.e. the norm closure of the finite rank operators

### Example

For a compact Riemannian spin manifold $M$, we have a spectral triple

$$\big(C^\infty(M), L^2(\mathbf{S}), D\big),$$

where $L^2(\mathbf{S})$ is the space of square integrable sections of the spinor bundle of $M$, and $D$ is the Dirac operator.

## Example

For a compact Riemannian spin manifold $M$, we have a spectral triple

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## Example

For a compact Hermitian manifold $M$, we have a spectral triple

$$\big(C^\infty(M), L^2(\Omega^{(0,\bullet)}), D_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^\dagger\big),$$

where $\mathrm{d} = \partial + \overline{\partial}$, and $\overline{\partial}^\dagger$ is the adjoint of $\overline{\partial}$.

## Example

Noncommutative spectral triples arise in the study of foliated manifolds.

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One of the motivating examples of a noncommutative spectral triple is constructed over the noncommutative torus: For $\theta \in \mathbb{R}$, the noncommutative torus $A_\theta$ is the $C^*$-subalgebra of $\mathcal{B}(L^2(S^1))$, the algebra of bounded linear operators of square-integrable functions on the unit circle, generated by the unitary elements $U$ and $V$, where

$$U(f)(z) = zf(z) \qquad \text{and} \qquad V(f)(z) = f(e^{2\pi i\theta}z).$$

This implies the noncommutative relation $VU = e^{2\pi i\theta}UV$.

- What about examples from quantum groups?


Connes
Jimbo
Drinfeld

- Despite a large number of important contributions over the last thirty years, there is still no consensus on how to construct a spectral triple for $\mathcal{O}_q(SU_2)$, probably the most basic example of a quantum group!

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- In contrast the Podleś sphere $\mathcal{O}_q(S^2)$ admits a canonical spectral triple which directly $q$-deforms the classical Dolbeault–Dirac operator of the 2-sphere. Moreover, it is the most widely and consistently accepted example of a spectral triple in the Drinfeld–Jimbo setting.

# Drinfeld–Jimbo Quantised Enveloping Algebras $U_q(\mathfrak{g})$

Let $(a_{ij})_{ij}$ denote the Cartan matrix of $\mathfrak{g}$.

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The *quantised enveloping algebra* $U_q(\mathfrak{g})$ is generated by

$$E_i, F_i, K_i, K_i^{-1}, \qquad i = 1, \dots, r;$$

subject to the relations

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \qquad K_i F_j = q_i^{-a_{ij}} F_j K_i, \qquad K_i K_j = K_j K_i,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i};$$

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along with the *quantum Serre relations*.

# Hopf algebra structure on $U_q(\mathfrak{g})$

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$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$
$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \qquad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

A $^*$-structure for $U_q(\mathfrak{g})$, called the *compact real form* , is given by

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# Quantum Flag Manifolds

- For $S$ a subset of simple roots, we have the *quantum Levi subalgebra*

$$U_q(\mathfrak{l}_S) := \langle K_i, E_j, F_j \,|\, i = 1, \ldots, r; j \in S \rangle$$

## Definition

For $S$ a subset of simple roots of $\mathfrak{g}$, the corresponding *quantum flag manifold* is the invariant subspace

$$\begin{aligned} \mathcal{O}_q(G/L_S) :=& \mathcal{O}_q(G)^{U_q(\mathfrak{l}_S)} \\ =& \big\{ g \in \mathcal{O}_q(G) | g \triangleleft X = \varepsilon(X) g, \forall X \in U_q(\mathfrak{l}_S) \big\}. \end{aligned}$$

- Let's look at the simplest example $\mathcal{O}_q(S^2)$, where $U_q(\mathfrak{l}_{\varpi_1}) = \langle K, K^{-1} \rangle$

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- By direct calculation, it can be shown that $\mathcal{O}_q(S^2)$ is generated by the three elements $b_+$, $b_0$, and $b_-$ subject to the relations

$$
\begin{aligned}
b_{\pm} b_3 &= q^{\pm 2} b_3 b_{\pm} + (1 - q^{\pm 2}) b_{\pm}, \\
q^2 b_- b_+ &= q^{-2} b_+ b_- + (q - q^{-1})(b_3 - 1), \\
b_3^2 &= b_3 + q b_- b_+.
\end{aligned}
$$

- For $q = 1$, with respect to the variables

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b_{\pm} = \pm(x \pm iy), \qquad b_3 = z + \frac{1}{2},
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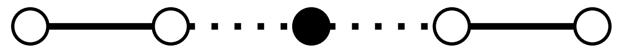
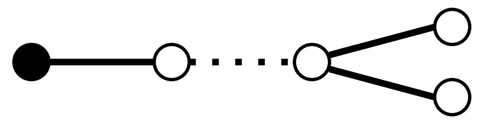
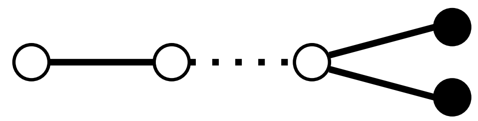
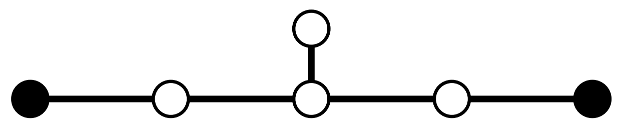
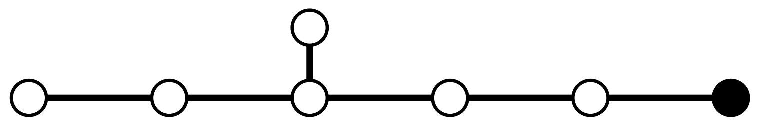
the relations reduce to

$$x^2 + y^2 + z^2 = 1.$$

## Compact Quantum Hermitian Symmetric Spaces

| | | | |
|---|---|---|---|
| $A_n$ |  | $\mathcal{O}_q(\mathrm{Gr}_{n,r})$ | quantum Grassmanian |
| $B_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n+1})$ | odd quantum quadric |
| $C_n$ |  | $\mathcal{O}_q(\mathrm{L}_n)$ | symmetric q.-Lagrangian Grassmannian |
| $D_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n})$ | even quantum quadric |
| $D_n$ |  | $\mathcal{O}_q(\mathrm{S}_n)$ | quantum spinor variety |
| $E_6$ |  | $\mathcal{O}_q(\mathbb{OP}^2)$ | quantum Cayley plane |
| $E_7$ |  | $\mathcal{O}_q(\mathrm{F})$ | quantum Freudenthal variety |

# Noncommutative Differential Calculi

## Definition

A pair $(\Omega^\bullet, d)$ is called a **differential graded algebra** if $\Omega^\bullet = \bigoplus_{k \in \mathbf{N}_0} \Omega^k$ is an $\mathbf{N}_0$-graded algebra, and $d$ is a degree 1 map such that $d^2 = 0$, and

$$\mathrm{d}(\omega \wedge \nu) = \mathrm{d}(\omega) \wedge \nu + (-1)^k \omega \wedge \mathrm{d}(\nu), \qquad (\omega \in \Omega^k, \nu \in \Omega^\bullet).$$

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$$\mathrm{d}(\omega \wedge \nu) = \mathrm{d}(\omega) \wedge \nu + (-1)^k \omega \wedge \mathrm{d}(\nu), \qquad (\omega \in \Omega^k, \nu \in \Omega^\bullet).$$

A **differential calculus** over an algebra $A$ is a differential algebra $(\Omega(A), \mathrm{d})$ such that

$$\Omega^k = \mathrm{span}_{\mathbb{C}}\{a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_k \,|\, a_0, \ldots, a_k \in A\}.$$

## Definition

A **differential $*$-calculus** is a differential calculus endowed with a conjugate linear, involutive, graded anti-algebra map which commutes with the differential.

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### Definition

We say that a differential calculus $\Omega^\bullet_q(G/L_S)$ over $\mathcal{O}_q(G/L_S)$ is **covariant** if the action $U_q(\mathfrak{g}) \times \mathcal{O}_q(G/L_S)$ extends to a (necessarily unique) algebra map $U_q(\mathfrak{g}) \times \Omega^\bullet_q(G/L_S)$ such that

$$X \triangleright (\mathrm{d}m) := \mathrm{d}(X \triangleright m), \qquad \text{for all } m \in M.$$

**Theorem (Heckenberger, Kolb '06)**

*For each compact quantum Hermitian symmetric flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique equivariant differential calculus $\Omega_q^\bullet(G/L)$ of classical dimension.*

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- In a little more detail: The first-order part of $\Omega_q(G/L)$ is the direct sum of two covariant irreducible finite-dimensional first-order differential calculi, and these were classified by Heckenberger and Kolb as the only such first-order calculi.

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- The total differential calculi were then constructed as the universal extensions of these first-order calculi.

# Noncommutative Complex Structures

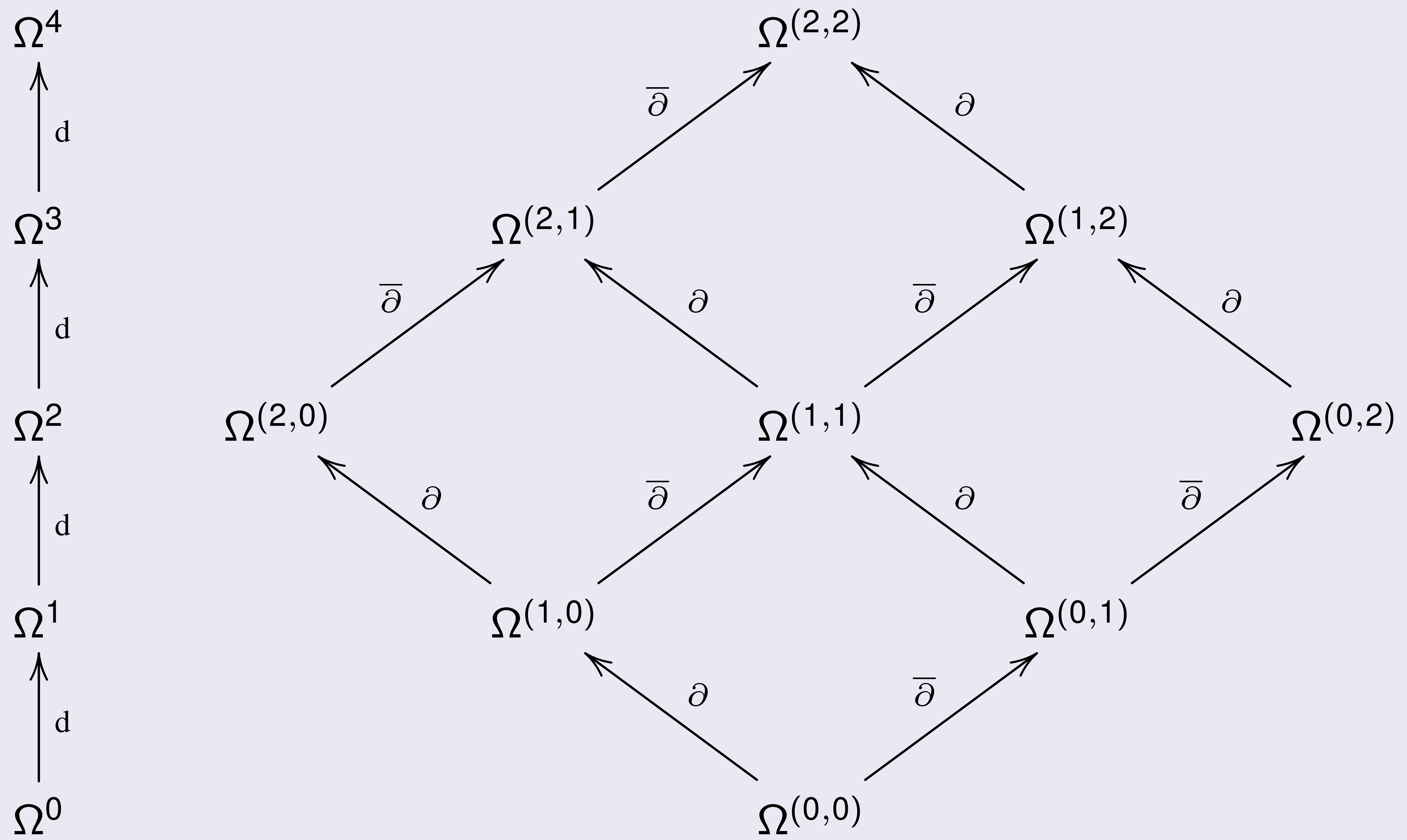
## Definition

An *almost complex structure* for a total differential $*$-calculus $\Omega^\bullet(A)$ over a $*$-algebra $A$, is an $\mathbf{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbf{N}_0^2}\Omega^{(a,b)}$ for $\Omega^\bullet(A)$ such that, for all $(a,b)\in\mathbf{N}_0^2$:

1. $\Omega^k(A) = \bigoplus_{p+q=k}\Omega^{(a,b)}$;
2. $*(\Omega^{(a,b)}) = \Omega^{(b,a)}$.

## Example

The quantum projective plane $\mathbb{C}_q[\mathbb{C}P^2]$ has such a structure



### Theorem (Newlander–Nirenberg ’57)

**Holomorphic atlases** *on a differential manifold M*

$$\equiv$$

**Complex structures** *on* $\Omega^\bullet(M)$*, that is, almost complex structures such that*

$$\mathrm{d} = \partial + \overline{\partial}.$$

## Definition

Defining two operators $\partial, \overline{\partial} : \Omega^\bullet \to \Omega^\bullet$ by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

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we say that an almost complex structure is **integrable** if

$$\mathrm{d} = \partial + \overline{\partial}.$$

We usually call an integrable almost complex structure a **complex structure**.

### Definition

We say that a noncommutative complex structure for a covariant differential calculus is **covariant** if the $\mathbf{N}_0^2$-decomposition of $\Omega^\bullet(M)$ is a $U_q(\mathfrak{g})$-decomposition.

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We say that a noncommutative complex structure for a covariant differential calculus is **covariant** if the $\mathbf{N}_0^2$-decomposition of $\Omega^\bullet(M)$ is a $U_q(\mathfrak{g})$-decomposition.

**Theorem**

*Each Heckenberger–Kolb calculus $\Omega_q^\bullet(G/L_S)$ has a unique covariant complex structure. Hence, it is a direct q-deformation of the classical complex structure of $G/L_S$.*

# Bott-Borel-Weil for the Quantum Grassmannians

**Theorem (Koszul–Malgrange)**

*A holomorphic structure for a vector bundle $\mathcal{F}$ over a compact complex manifold is equivalent to a flat $(0,1)$-connection*

$$\overline{\partial}_{\mathcal{F}} : \Gamma^{\infty}(\mathcal{F}) \to \Gamma^{\infty}(\mathcal{F}) \otimes_{C^\infty} \Omega^{(0,1)}.$$

- Thus for a differential calculus $\Omega^\bullet$ over an algebra $A$, endowed with a complex structure $\Omega^{(\bullet,\bullet)}$, and a projective right $A$-module $\mathcal{E}$, we view flat $(0,1)$-connections

$$\overline{\partial}_{\mathcal{E}} : \mathcal{E} \to \Gamma^{\infty}(\mathcal{E}) \otimes_{C^\infty} \Omega^{(0,1)}.$$

as *noncommutative holomorphic structures* for $\mathcal{E}$.

- But how to construct projective modules over the quantum flag manifolds?

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$$\begin{aligned} &\mathcal{O}_q(G)\square_{U_q(\mathfrak{l}_S)} V \\ :=&\left(\mathcal{O}_q(G) \otimes V\right)^{U_q(\mathfrak{l}_S)} \\ =&\{s \in \mathcal{O}_q(G) \otimes V \,|\, s \triangleleft X = \varepsilon(X)s, \forall X \in U_q(\mathfrak{l}_S)\}, \end{aligned}$$

where as usual $U_q(\mathfrak{l}_S)$ acts on the tensor product via the coproduct

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**Theorem (Takeuchi '79)**

*We have an equivalence of categories*

$$\mathrm{Mod}_{U_q(\mathfrak{l}_S)} \quad \simeq \quad {}^{U_q(\mathfrak{g})}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}.$$

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- Thus noncommutative line bundles are defined to be homogeneous vector bundles induced from 1-dimensional modules.

**Theorem (R. Ó B., C. Mrozinski '17)**

*The homogeneous line bundles over the quantum Grassmannians admit a unique covariant holomorphic structure. Moreover,*

$$H^0(\mathcal{E}_k) \simeq V_{k\varpi_r}.$$

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**Theorem (R. Ó B., J. Šťovíček, A.C. van Roosmallen '18)**

*For all positive line bundles $\mathcal{E}_k$, it holds that*

$$H^i(\mathcal{E}_k) = 0, \qquad \text{for all } i = 1, \ldots, r(n-r).$$

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- Jan will take up the rest of this story.

- Now back to spectral triples . . . .
- We would like to show that for each compact quantum Hermitian symmetric space, a spectral triple is given by

$$\left(\mathcal{O}_q(G/L_S), L^2(\Omega^{(0,\bullet)}, D_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^{\dagger}\right).$$

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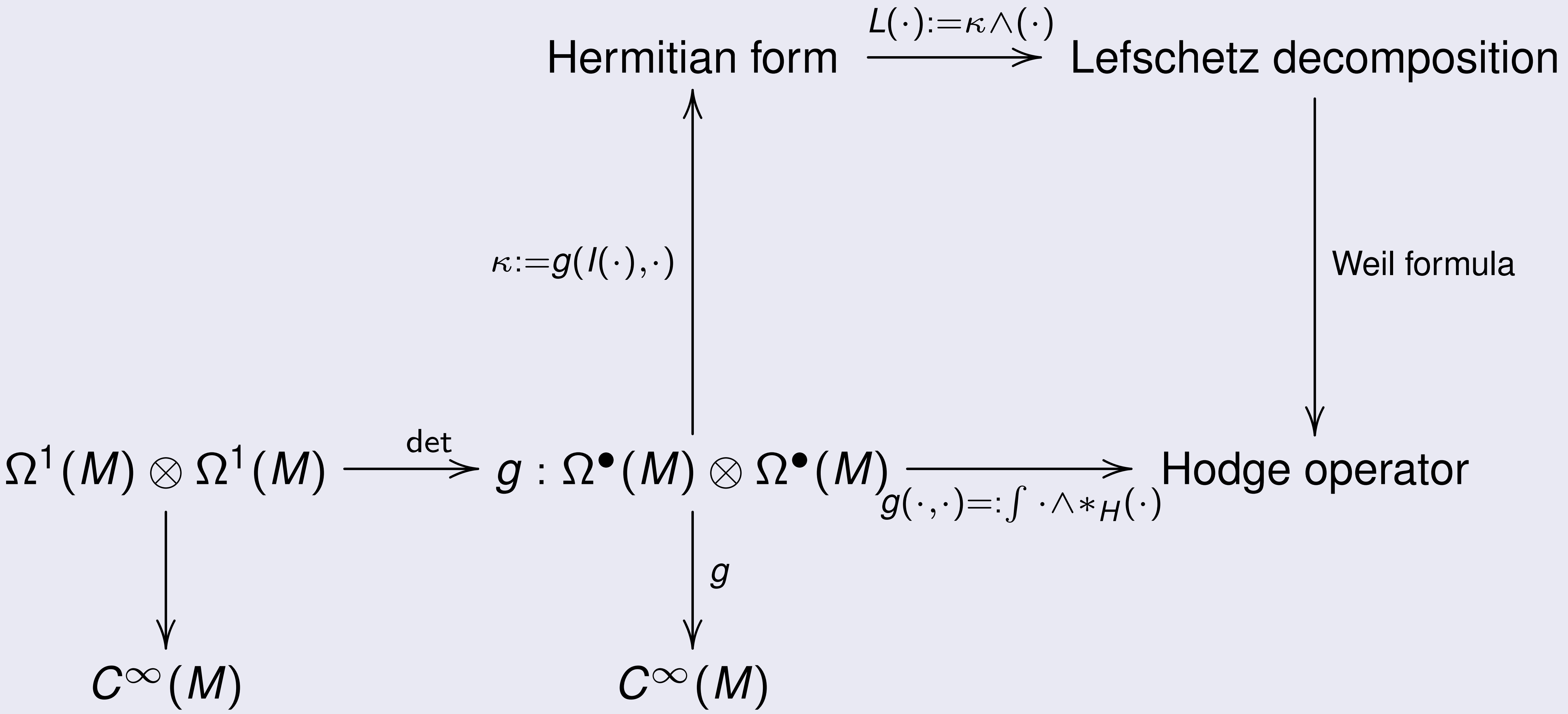
- For this we need **noncommutative Kähler structures** . . .

## Summary of Classical Hermitian Geometry

Classically we have:



## Definition (R.Ó B. '17)

An **Hermitian structure** for a differential calculus of total dimension $2n$ is a pair $(\Omega^{(\bullet,\bullet)}, \sigma)$, where

1. $\Omega^{(\bullet,\bullet)}$ is complex structure for $\Omega^{\bullet}$,
2. $\sigma \in \Omega^{(1,1)}$ is a central real form (i.e. $\kappa^* = \kappa$),
3. isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad \omega \mapsto \sigma^{n-k} \wedge \omega,$$

for all $1 \leq k < n$.

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for all $1 \leq k < n$.

### Definition (R. Ó B. '17)

A **Kähler structure** is an Hermitian structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that $\mathrm{d}\kappa = 0$.

**Theorem (R. Ó B. '17)**

*There exists a covariant Kähler structure for the Heckenberger–Kolb calculus of quantum projective space, which is unique up to real scalar multiple.*

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**Theorem (Matassa '19)**

*There exists a covariant Kähler structure for the Heckenberger–Kolb calculus of each compact quantum Hermitian symmetric space $\mathcal{O}_q(G/L_S)$, which is unique up to real scalar multiple.*

# Quantum Flag Manifolds: From Quantum Groups to Noncommutative Geometry III

Réamonn Ó Buachalla

Université Libre de Bruxelles

*39th Winter School Geometry and Physics 2019 - Srní*

# Very Quick Recap

- The Gelfand–Naimark Theorem:

  Compact Hausdorff Spaces $\longleftrightarrow$ Commutative $C^*$-algebras

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# Very Quick Recap

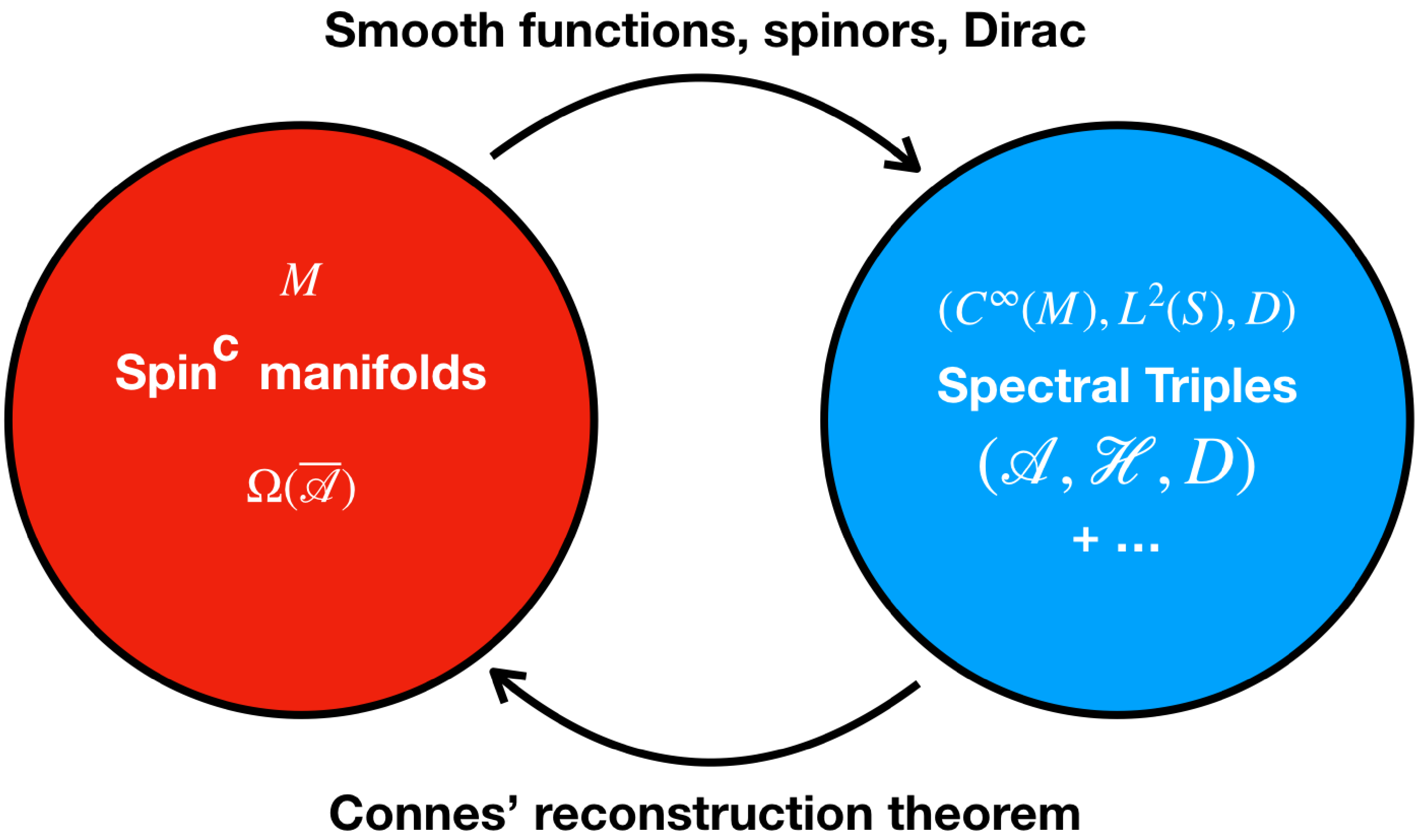
- The Gelfand–Naimark Theorem:

  Compact Hausdorff Spaces $\longleftrightarrow$ Commutative $C^*$-algebras

- Woronowicz's Theorem:

  Compact topological Groups $\longleftrightarrow$ Commutative Compact Quantum Groups

- **Question** Can we express differential structures on a compact Hausdorff space in terms of some $C^*$-algebraic differential structure on $C(X)$?


Smooth functions, spinors, Dirac
$M$
Spin$^c$ manifolds
$\Omega(\overline{\mathscr{A}})$
$(C^\infty(M), L^2(S), D)$
Spectral Triples
$(\mathscr{A}, \mathscr{H}, D)$
+ ...
Connes' reconstruction theorem

## Definition

A **spectral triple** is a triple $(A, \mathcal{H}, D)$, where

- $A$ is a dense $*$-subalgebra of a $C^*$-algebra,
- $\mathcal{H}$ is a Hilbert space with a faithful $*$-representation $\rho : A \to \mathcal{B}(\mathcal{H})$
- $D$ is a densely defined unbounded self-adjoint operator $D : \mathrm{dom}(D) \to \mathcal{H}$, such that

$$[D, a] \in \mathcal{B}(\mathcal{H}), \text{ for all } a \in A, \quad \text{and} \quad (1 - D^2)^{-1} \in \mathcal{K}(\mathcal{H}).$$

- $\mathcal{K}(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$, i.e. the norm closure of the finite rank operators

## Example

For a compact Riemannian spin manifold $M$, we have a spectral triple

$$(C^\infty(M), L^2(\mathbf{S}), D),$$

where $L^2(\mathbf{S})$ is the space of square integrable sections of the spinor bundle of $M$, and $D$ is the Dirac operator.

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## Example

For a compact Hermitian manifold $M$, we have a spectral triple

$$\left(C^\infty(M), L^2(\Omega^{(0,\bullet)}), D_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^\dagger\right),$$

where $\mathrm{d} = \partial + \overline{\partial}$, and $\overline{\partial}^\dagger$ is the adjoint of $\overline{\partial}$.


Connes
Jimbo
Drinfeld

- For $S$ a subset of simple roots, we have the *quantum Levi subalgebra*

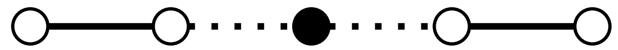
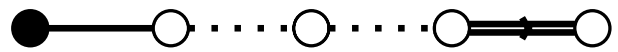
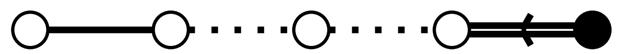
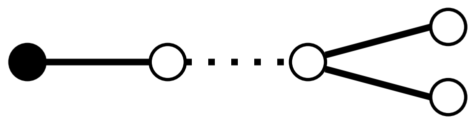
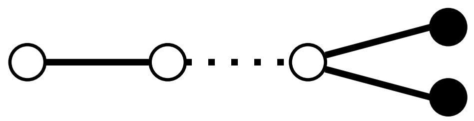
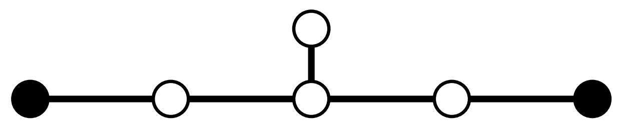
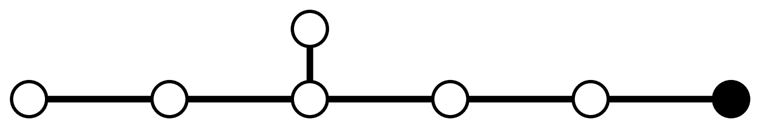
$$U_q(\mathfrak{l}_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \rangle$$

## Definition

For $S$ a subset of simple roots of $\mathfrak{g}$, the corresponding *quantum flag manifold* is the invariant subspace

$$\begin{aligned} \mathcal{O}_q(G/L_S) :=& \mathcal{O}_q(G)^{U_q(\mathfrak{l}_S)} \\ =& \{g \in \mathcal{O}_q(G) | g \triangleleft X = \varepsilon(X)g, \forall X \in U_q(\mathfrak{l}_S)\}. \end{aligned}$$

## Compact Quantum Hermitian Symmetric Spaces

| | | | |
|---|---|---|---|
| $A_n$ |  | $\mathcal{O}_q(\mathrm{Gr}_{n,r})$ | quantum Grassmanian |
| $B_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n+1})$ | odd quantum quadric |
| $C_n$ |  | $\mathcal{O}_q(\mathrm{L}_n)$ | symmetric q.-Lagrangian Grassmannian |
| $D_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n})$ | even quantum quadric |
| $D_n$ |  | $\mathcal{O}_q(\mathrm{S}_n)$ | quantum spinor variety |
| $E_6$ |  | $\mathcal{O}_q(\mathbb{OP}^2)$ | quantum Cayley plane |
| $E_7$ |  | $\mathcal{O}_q(\mathrm{F})$ | quantum Freudenthal variety |

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**Theorem (Heckenberger, Kolb '06)**

*For each compact quantum Hermitian symmetric flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique equivariant differential calculus $\Omega^\bullet_q(G/L)$ of classical dimension.*

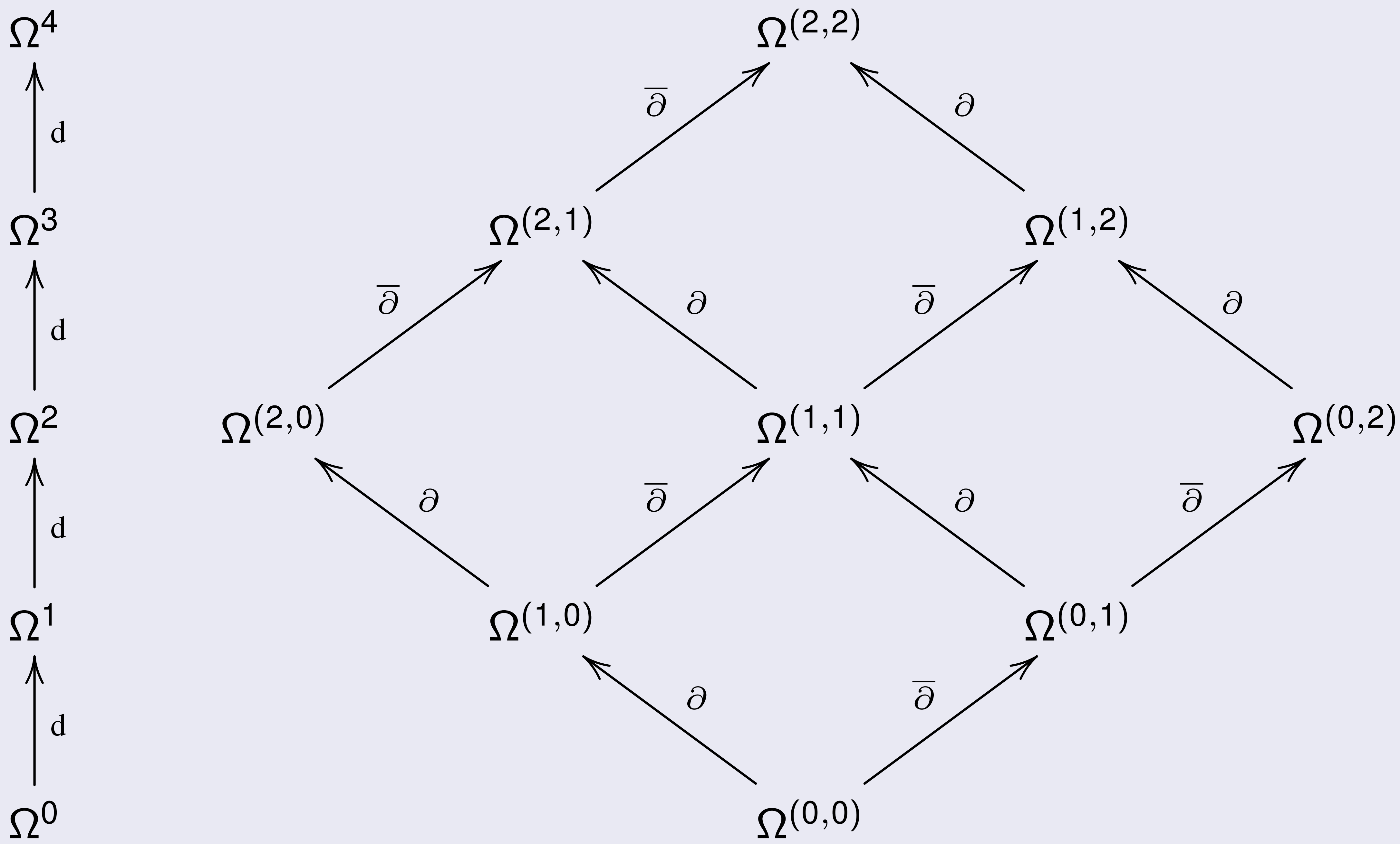
## Definition

An *almost complex structure* for a total differential $*$-calculus $\Omega^\bullet(A)$ over a $*$-algebra $A$, is an $\mathbf{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbf{N}_0^2}\Omega^{(a,b)}$ for $\Omega^\bullet(A)$ such that, for all $(a,b)\in\mathbf{N}_0^2$:

1. $\Omega^k(A) = \bigoplus_{a+b=k}\Omega^{(p,q)}$;
2. $*(\Omega^{(a,b)}) = \Omega^{(b,a)}$.

## Example

The quantum projective plane $\mathcal{O}_q(\mathbb{C}P^2)$ has such a structure



## Definition

Defining two operators $\partial, \overline{\partial} : \Omega^\bullet \to \Omega^\bullet$ by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

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we say that an almost complex structure is **integrable** if

$$\mathrm{d} = \partial + \overline{\partial}.$$

## Theorem

*Each Heckenberger–Kolb calculus $\Omega^\bullet_q(G/L_S)$ has a unique covariant complex structure. Hence, it is a direct q-deformation of the classical complex structure of $G/L_S$.*

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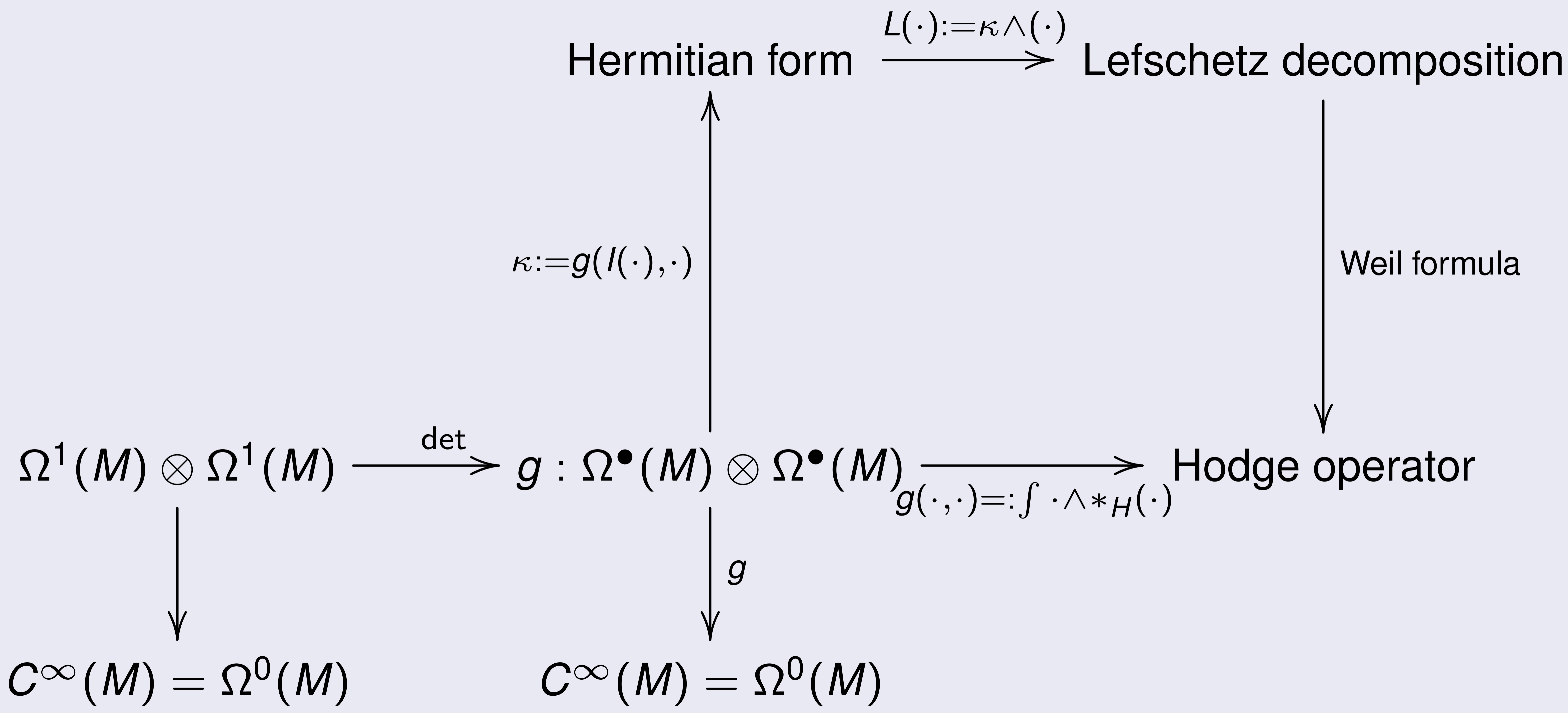
## Summary of Classical Hermitian Geometry

Classically we have:



# Nichols Algebras

- A braided vector space is a pair $(V, \sigma)$, where $V$ is a vector space, and $\sigma : V \otimes V \to V \otimes V$ is a linear map satisfying the *Yang–Baxter equation*

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\mathrm{id} \otimes \sigma) = (\mathrm{id} \otimes \sigma) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}).$$

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$$s : \mathbb{S}_n \to \mathbb{B}_n.$$

- With respect to this splitting, for any $\pi \in S_n$, we have a well-defined map $s(\pi) : V^{\otimes n} \to V^{\otimes n}$ which is independent of the choice of reduced expression for $\pi$.

- With respect to the Matsumoto lift, we can define a *braided anti-symmetrizer*

$$A_{\sigma,k} := \sum_{\pi \in \mathbb{S}_n} s(\pi) : V^{\otimes k} \to V^{\otimes k}.$$

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**Definition**

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- When $\sigma$ is the flip map, we get back the usual exterior algebra of $V$.

## Theorem (A. Krutov, R. Ó B., K. Strung '18)

*For the quantum Grassmannians, endowed with their Heckenberger–Kolb calculus, there exist (Yetter–Drinfeld) braidings*

$$\sigma^+ : V^{(0,1)} \otimes V^{(0,1)} \to V^{(0,1)} \otimes V^{(0,1)},$$
$$\sigma^- : V^{(1,0)} \otimes V^{(1,0)} \to V^{(1,0)} \otimes V^{(1,0)}.$$

*Denoting by* $\sigma^\pm$ *the induced braidings,*

$$B\left(\Phi(\Omega^{(1,0)}), \sigma^+\right) \simeq V^{(\bullet,0)},$$
$$B\left(\Phi(\Omega^{(0,1)}), \sigma^-\right) \simeq V^{(0,\bullet)}.$$

## Definition (R.Ó B. '17)

An **Hermitian structure** for a differential calculus of total dimension $2n$ is a pair $(\Omega^{(\bullet,\bullet)}, \sigma)$, where

1. $\Omega^{(\bullet,\bullet)}$ is complex structure for $\Omega^\bullet$,
2. $\sigma \in \Omega^{(1,1)}$ is a central real form (i.e. $\kappa^* = \kappa$),
3. isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad \omega \mapsto \sigma^{n-k} \wedge \omega,$$

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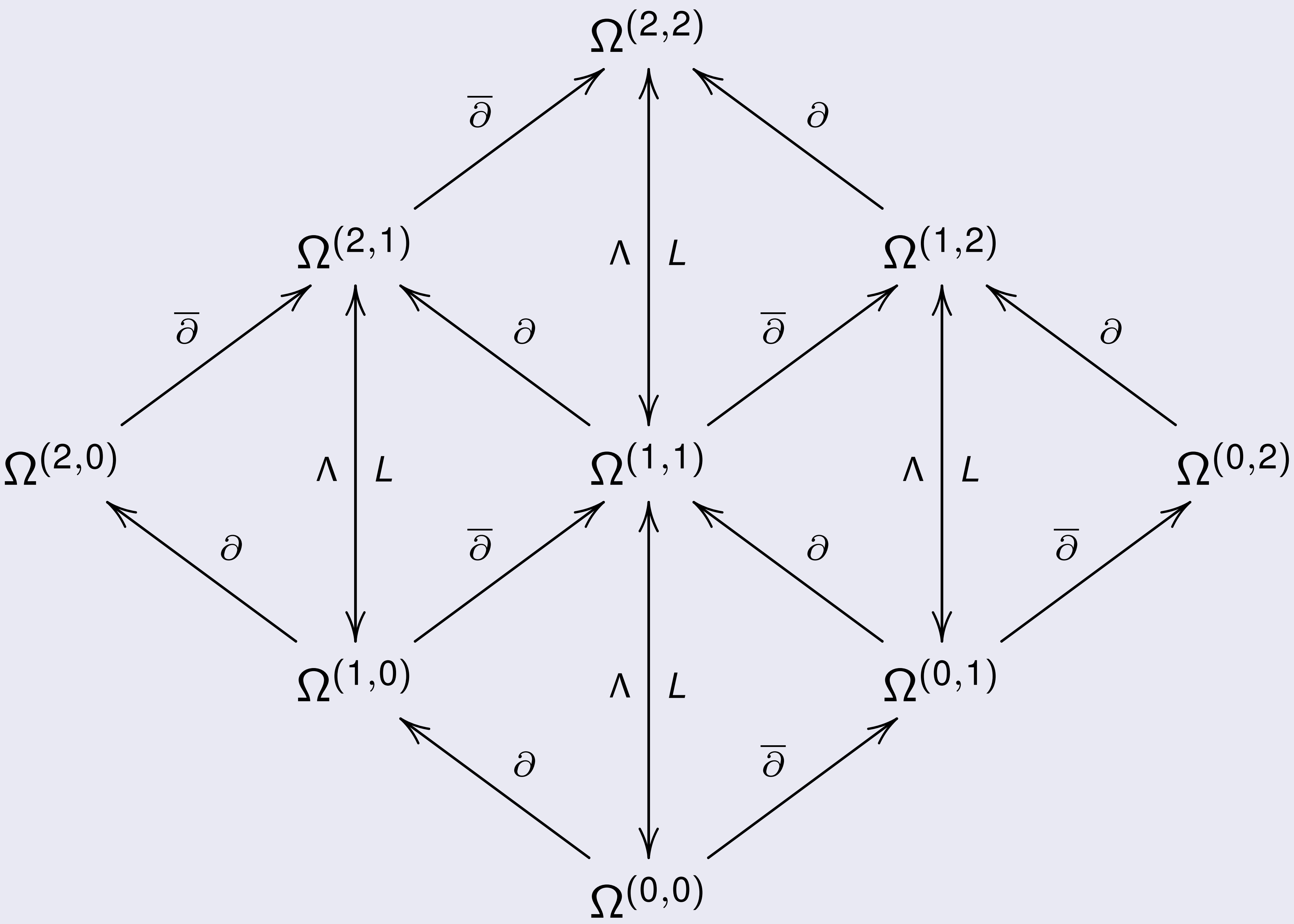
for all $1 \leq k < n$.

## Definition

A **Kähler structure** is an Hermitian structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that $\mathrm{d}\kappa = 0$.

## Example

For the quantum projective plane $\mathcal{O}_q(\mathbb{C}P^2)$, we have



**Theorem (R. Ó B. '17)**

*There exists a covariant Kähler structure for the Heckenberger–Kolb calculus of quantum projective space, which is unique up to real scalar multiple.*

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**Theorem (Matassa '19)**

*There exists a covariant Kähler structure for the Heckenberger–Kolb calculus of each compact quantum Hermitian symmetric space $\mathcal{O}_q(G/L_S)$, which is unique up to real scalar multiple.*

## Lemma (Lefschetz Decomposition)

*For any equivariant Hermitian structure for $\Omega^\bullet(M)$, we have the* Lefschetz decomposition*:*

$$\Omega^{(a,b)}(M) := \bigoplus_{i=0}^{\min\{a,b\}} L^i(P^{(a-i,b-i)}),$$

*where we have denoted*
$P^{(a,b)} := \ker\big(L^{n-(a+b)+1} : \Omega^{(a,b)}(M) \to \Omega^{(n-b+1,n-a+1)}(M)\big).$

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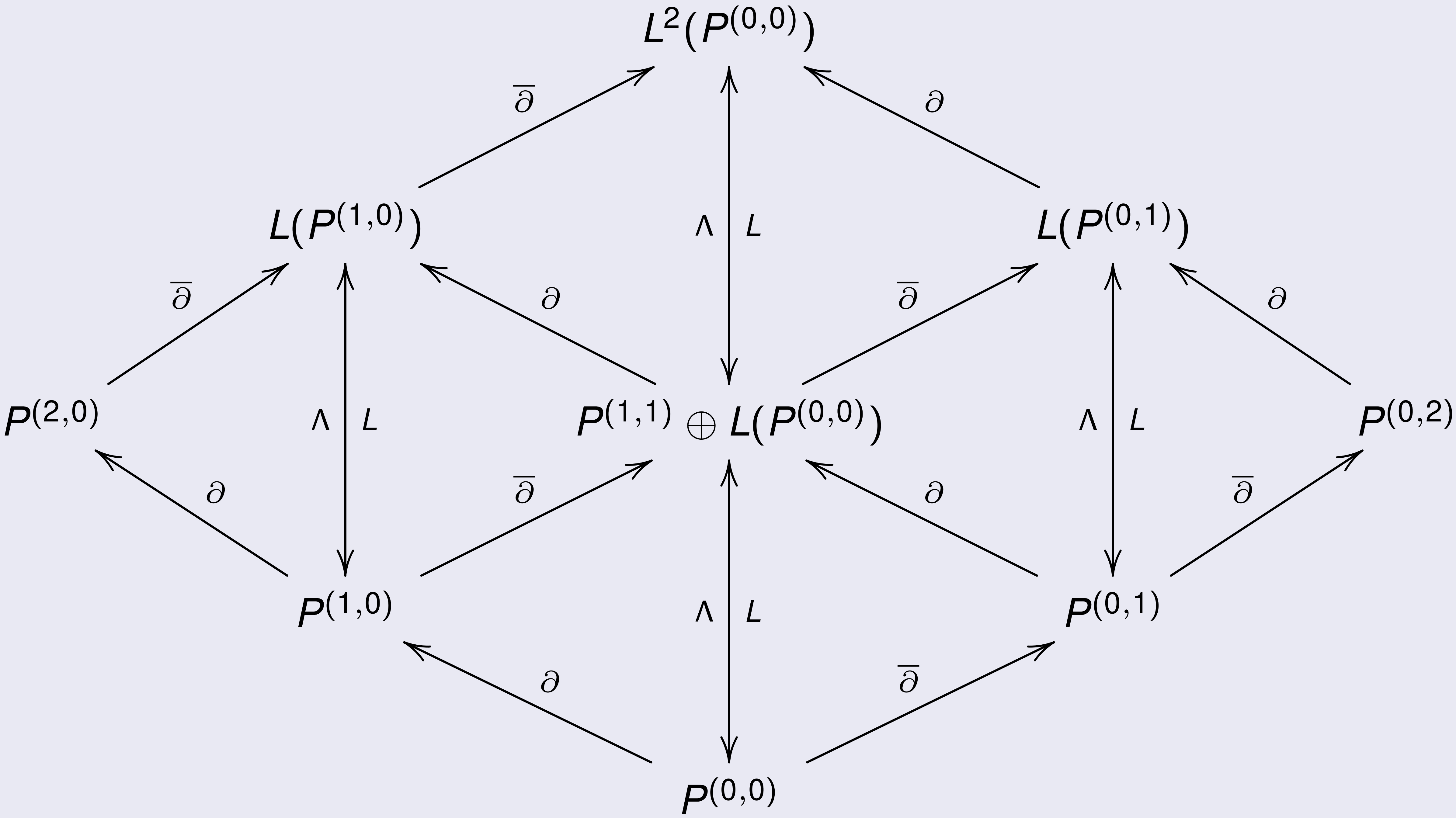

## Definition (Weil Formula)

The *Hodge map* associated to an Hermitian structure is the morphism uniquely defined, for $\omega \in P^{(p,q)}(M)$, and $k = p + q$, by

$$*_H(L^j(\omega)) := i^{p-q}(-1)^{\frac{k(k+1)}{2}} \frac{j!}{(N-k-j)!} L^{N-j-k}(\omega).$$

- The Hodge map is not unique in any obvious way.

## Definition (Weil Formula)

The *h-Hodge map* associated to an Hermitian structure is the morphism uniquely defined, for $\omega \in P^{(p,q)}(M)$, and $k = p + q$, by

$$*_H(L^j(\omega)) := i^{p-q}(-1)^{\frac{k(k+1)}{2}} \frac{[j]_h!}{[N-k-j]_h!} L^{N-j-k}(\omega),$$

where the *quantum integer* and *quantum factorial* are the scalars

$$[k]_h := \frac{1-h^k}{1-h}, \qquad [k]_h! := [k]_h[k-1]_h \cdots [2]_h.$$

- the **associated metric** is the map

$$g(\omega, \nu) := \mathrm{vol}(\omega \wedge *_H(\nu^*)),$$

## Summary

Noncommutatively we define:

Hermitian form $\xrightarrow{L(\cdot):=\kappa\wedge(\cdot)}$ Lefschetz decomposition

Lefschetz decomposition $\xrightarrow{\text{Weil formula}}$ Hodge operator

Hodge operator $\xrightarrow{g(\cdot,\cdot):=\int \cdot\wedge *_H(\cdot)}$ $g : \Omega^\bullet(M) \otimes \Omega^\bullet(M) \to \Omega^0$

## Lemma

*It holds that*

1. $*_H(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)}$;
2. $*_H^2 = (-1)^k$;
3. $[*_H, *] = 0$;
4. *the complex structure* $\mathbf{N}_0^2$*-decomposition is orthogonal with respect to g;*
5. *the Lefschetz decomposition is orthogonal with respect to g;*
6. $*_H$ *is a unitary operator;*
7. $g(\omega, \nu) = g(\nu, \omega)^*$.

## Lemma

*The maps* $L, \mathrm{d}, \partial, \overline{\partial}$ *are adjointable with respect to g, and*

1. $L^* = \Lambda = *_H^{-1} \circ L \circ *_H,$
2. $\mathrm{d}^* = - *_H \circ \mathrm{d} \circ *_H,$
3. $\partial^* = - *_H \circ \overline{\partial} \circ *_H,$
4. $\overline{\partial}^* = - *_H \circ \partial \circ *_H.$

# Hodge Decomposition and Cohomology

## Definition

The $\mathrm{d}$, $\partial$, and $\overline{\partial}$ **Laplacians** are respectively the operators

$$\Delta_{\mathrm{d}} = (\mathrm{d} + \mathrm{d}^*)^2, \qquad \Delta_{\partial} = (\partial + \partial^*)^2, \qquad \Delta_{\overline{\partial}} = (\overline{\partial} + \overline{\partial}^*)^2.$$

We denote their respective kernels by $\mathcal{H}_{\mathrm{d}}$, $\mathcal{H}_{\partial}$, and $\mathcal{H}_{\overline{\partial}}$, and call them the $\mathrm{d}$,$\partial$, and $\overline{\partial}$ **harmonic forms**.

**Theorem (R. Ó B. '17)**

*For the Heckenberger–Kolb calculi, with their unique Kähler structure, we have the three decompositions*

1. $\Omega^\bullet(M) = \mathcal{H}_{\mathrm{d}} \oplus \mathrm{d}(\Omega^\bullet(M)) \oplus \mathrm{d}^*(\Omega^\bullet(M))$,
2. $\Omega^\bullet(M) = \mathcal{H}_{\partial} \oplus \partial(\Omega^\bullet(M)) \oplus \partial^*(\Omega^\bullet(M))$,
3. $\Omega^\bullet(M) = \mathcal{H}_{\overline{\partial}} \oplus \overline{\partial}(\Omega^\bullet(M)) \oplus \overline{\partial}^*(\Omega^\bullet(M))$.

## Definition

1. $\int := \textbf{haar} \circ *_H$,
2. $< \cdot, \cdot > := h \circ g = \int *_H(\bar{\cdot}) \wedge (\cdot)$.

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## Corollary

*If $< \cdot, \cdot >$ is positive definite, then we have isomorphisms:*

$$\mathcal{H}^k_{\mathrm{d}} \to H^k_{\mathrm{d}}; \qquad \mathcal{H}^{(p,q)}_{\partial} \to H^{(p,q)}_{\partial}; \qquad \mathcal{H}^{(p,q)}_{\overline{\partial}} \to H^{(p,q)}_{\overline{\partial}}.$$

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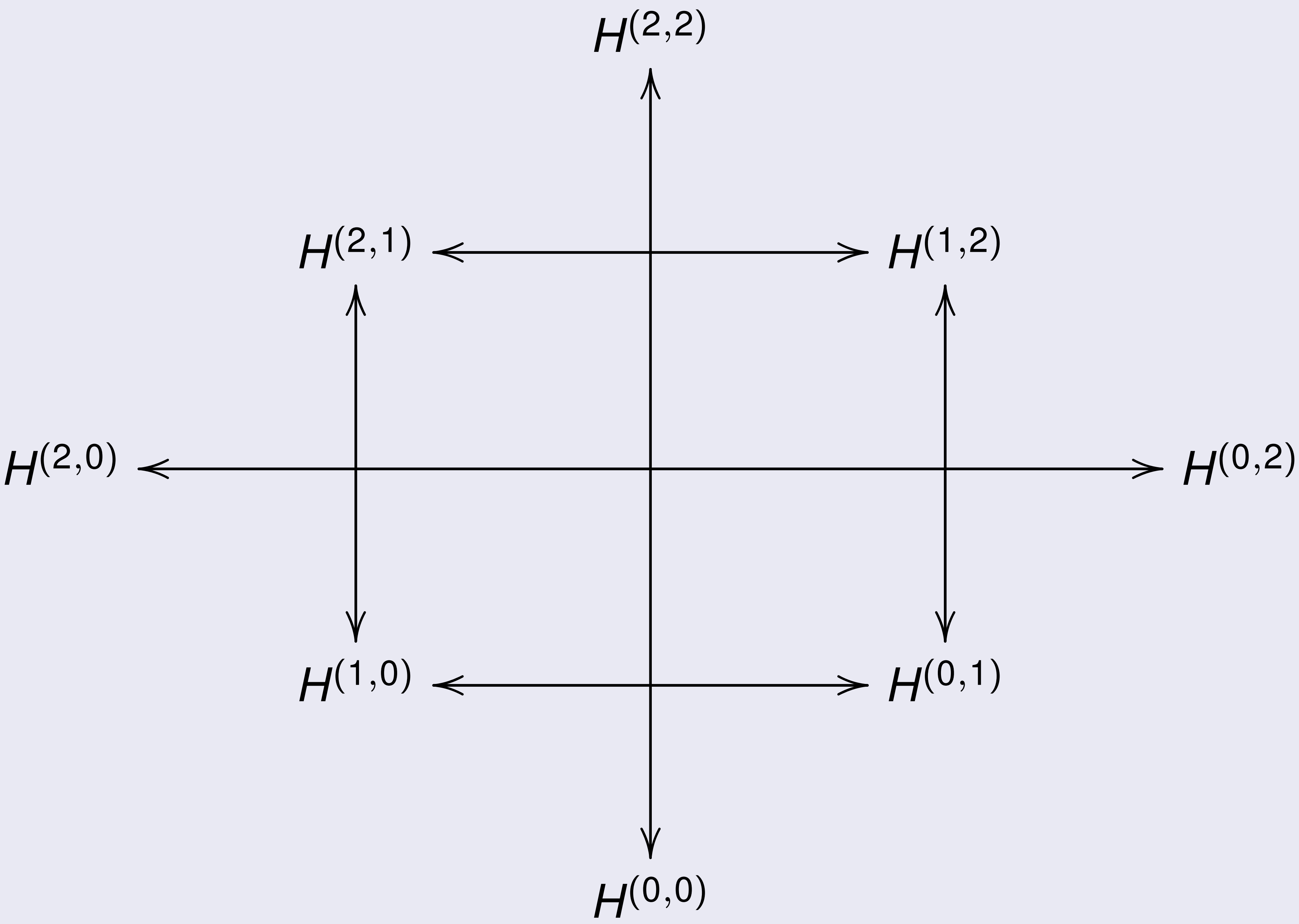
## Corollary

*The $*$ and Hodge maps descend to isomorphisms on the cohomology groups.*

## Example

For $\mathcal{O}_q(\mathbb{C}P^2)$, we have isomorphisms



## Proposition

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*For every compact quantum Hermitian symmetric space, the cohomology rings of the Heckenberger–Kolb calculi have at least classical dimension.*

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*For a covariant Kähler structure on a calculus $\Omega^\bullet$, every left $U_q(\mathfrak{g})$-invariant form is harmonic, and hence, gives a cohomology class.*

## Theorem

*For every compact quantum Hermitian symmetric space, the cohomology rings of the Heckenberger–Kolb calculi have at least classical dimension.*

- Contrast this with cyclic cohomology, the usual analogue of de Rham cohomology in noncommutative geometry, where the dimension of the cyclic cohomology of $\mathcal{O}_q(S^2)$ is less than in the classical case.

## Theorem (The Kähler Identities)(R. Ó B '17)

*For any Kähler structure* $(\Omega^{(\bullet,\bullet)}, \kappa)$, *we have the following relations:*

$$[L, \overline{\partial}] = 0, \qquad [L, \partial] = 0, \qquad [\Lambda, \partial^*] = 0, \qquad [\Lambda, \overline{\partial}^*] = 0,$$
$$[L, \partial^*] = i\overline{\partial}, \qquad [L, \overline{\partial}^*] = -i\partial, \qquad [\Lambda, \partial] = i\overline{\partial}^*, \qquad [\Lambda, \overline{\partial}] = -i\partial^*.$$

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## Corollary

*The Frölicher spectral sequence terminates on the first page:*

$$H^k = \bigoplus_{k=p+q} H^{(p,q)}.$$

# Spectral Triples

## Theorem (B. Das, R. Ó B., P. Somberg)

*For any covariant Hermitian structure on a compact quantum Hermitian symmetric space $\mathcal{O}_q(G/L_S)$, with positive definite inner product, a pair of spectral triples, which we call a* Dolbeault–Dirac pair, *is given by*

$$\Big(\mathcal{O}_q(G/L_S), L^2\big(\Omega^{(0,\bullet)}\big), D_{\overline{\partial}}\Big), \quad \Big(\mathcal{O}_q(G/L_S), L^2\big(\Omega^{(\bullet,0)}\big), D_{\partial}\Big),$$

*if and only if the Laplace operator $\Delta_{\overline{\partial}} = D^2_{\overline{\partial}}$ (which is automatically diagonalisable) has eigenvalues*

1. *of finite multiplicity*
2. *tending to infinity.*

**Theorem (B. Das, R. Ó B., P. Somberg '18)**

*For quantum projective space $\mathcal{O}_q(\mathbb{C}P^{N-1})$, endowed with its Heckenberger–Kolb calculus and its unique Kähler structure, the eigenvalues of the Laplacian $\Delta_{\overline{\partial}} = D_{\overline{\partial}}^2$ have finite multiplicity and tend to infinity.*

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## Corollary

*A Dolbeault–Dirac pair of spectral triples is given by*

$$\Big(\mathcal{O}_q(\mathbb{C}P^{N-1}), L^2\big(\Omega^{(\bullet,0)}\big), D_{\partial}\Big), \quad \Big(\mathcal{O}_q(\mathbb{C}P^{N-1}), L^2\big(\Omega^{(0,\bullet)}\big), D_{\overline{\partial}}\Big).$$

- Explicitly, the integers of the classical spectrum get replaced by $q^2$-integers!

### Theorem (B. Das, R. Ó B., P. Somberg '18)

*For quantum projective space $\mathcal{O}_q(\mathbb{C}P^{N-1})$, endowed with its Heckenberger–Kolb calculus and its unique Kähler structure, the eigenvalues of the Laplacian $\Delta_{\overline{\partial}} = D^2_{\overline{\partial}}$ have finite multiplicity and tend to infinity.*

### Corollary

*A Dolbeault–Dirac pair of spectral triples is given by*

$$\left(\mathcal{O}_q(\mathbb{C}P^{N-1}), L^2\left(\Omega^{(\bullet,0)}\right), D_{\partial}\right), \quad \left(\mathcal{O}_q(\mathbb{C}P^{N-1}), L^2\left(\Omega^{(0,\bullet)}\right), D_{\overline{\partial}}\right).$$

- Explicitly, the integers of the classical spectrum get replaced by $q^2$-integers!

- So how does one go about calculating the spectrum?

- So how does one go about calculating the spectrum?
- The essential simplifying assumption is that the left $U_q(\mathfrak{g})$-module

$$\overline{\partial}\Omega^{(0,k)}, \qquad \text{for all } k \in \mathbf{N}_0,$$

is **mutiplicity free**.

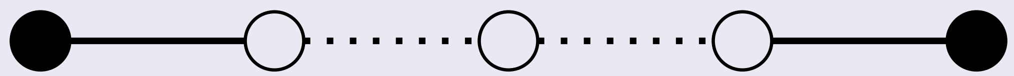
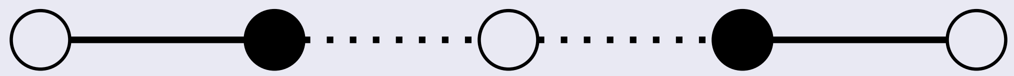
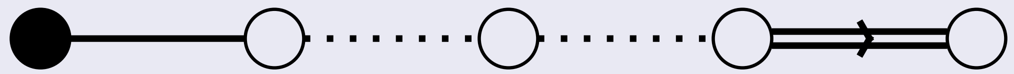
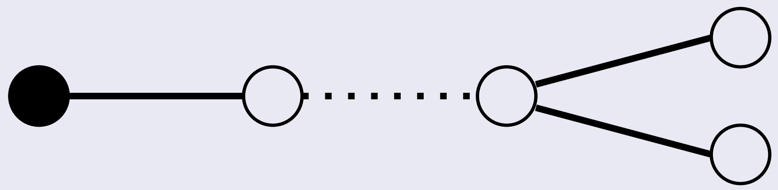
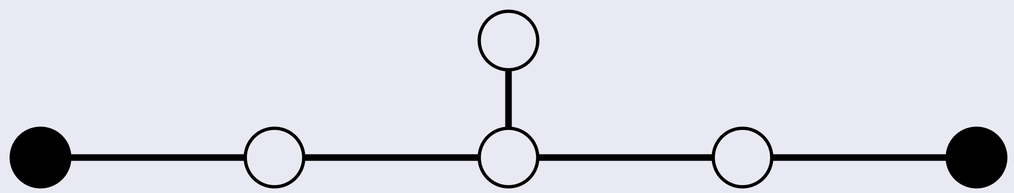
**Theorem**

*The compact quantum Hermitian spaces for which $\overline{\partial}\Omega^{(0,k)}$ is multiplicity free are precisely those in the following two diagrams.*

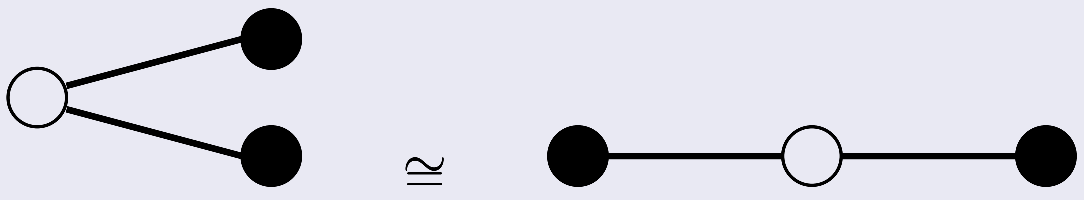
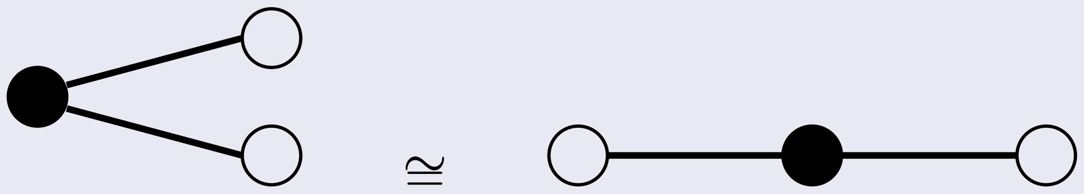
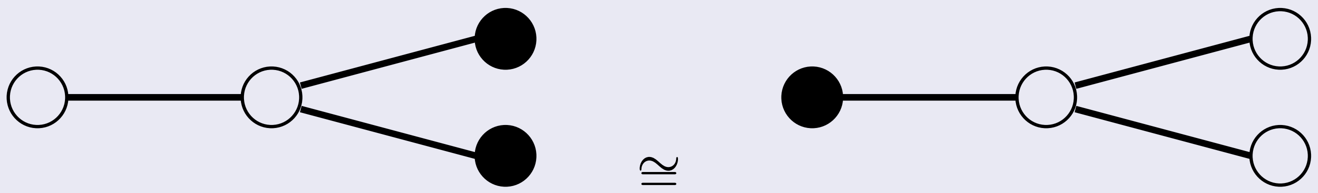
## Theorem

*The compact quantum Hermitian spaces for which $\overline{\partial}\Omega^{(0,k)}$ is multiplicity free are precisely those in the following two diagrams.*

*The first identifies four countable families:*

| | | |
|---|---|---|
| $A_n$ |  | $\mathcal{O}_q(\mathbb{CP}^n)$ |
| $A_n$ |  | $\mathcal{O}_q(\mathrm{Gr}_{n+1,2})$ |
| $B_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n+1})$ |
| $D_n$ |  | $\mathcal{O}_q(\mathrm{Q}_{2n})$ |
| $E_6$ |  | $\mathcal{O}_q(\mathbb{OP}^2)$ |

The second diagram identifies four isolated examples, arising from low dimensional redundancies in the table of compact quantum Hermitian spaces given above.

| | | |
|---|---|---|
| $B_3$ |  | $\mathcal{O}_q(\mathrm{L}_2) \simeq \mathcal{O}_q(\mathrm{Q}_5)$ |
| $D_3$ |  | $\mathcal{O}_q(\mathrm{S}_3) \simeq \mathcal{O}_q(\mathbb{CP}^3)$ |
| $D_3$ |  | $\mathcal{O}_q(\mathrm{Q}_6) \simeq \mathcal{O}_q(Gr_{4,2})$ |
| $D_4$ |  | $\mathcal{O}_q(\mathrm{S}_4) \simeq \mathcal{O}_q(\mathrm{Q}_8)$ |

## Conjecture

For all compact quantum Hermitian symmetric spaces $\mathcal{O}_q(G/L_S)$ appearing in the above list, a Dolbeault–Dirac pair of spectral triples is given by

$$\Big(\mathcal{O}_q(G/L_S), L^2\big(\Omega^{(\bullet,0)}\big), D_{\partial}\Big), \quad \Big(\mathcal{O}_q(G/L_S), L^2\big(\Omega^{(0,\bullet)}\big), D_{\overline{\partial}}\Big).$$

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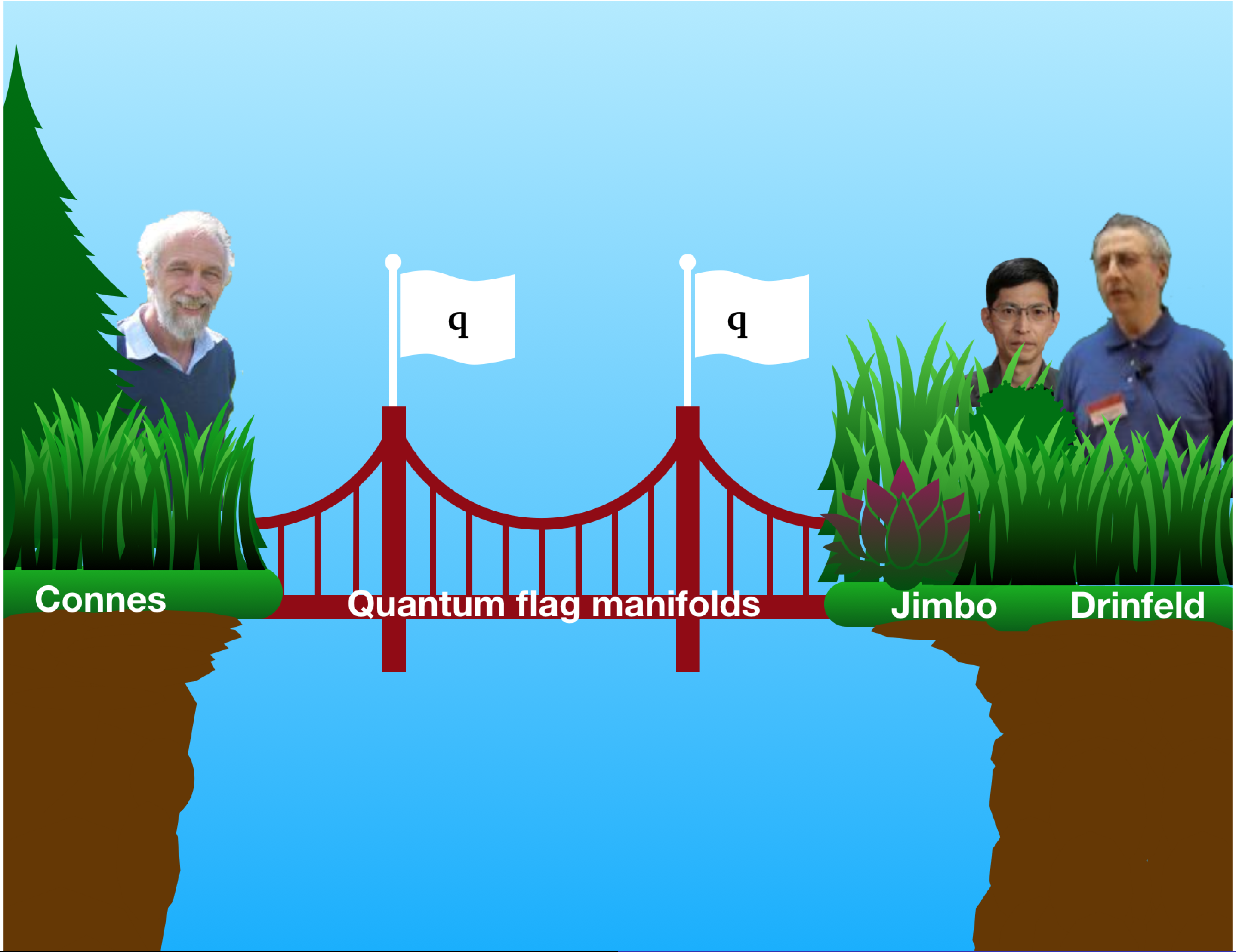
q
q
Connes
Quantum flag manifolds
Jimbo
Drinfeld

# Operator $K$-theory

For a $C^*$-algebra $\mathcal{A}$, let $(V(\mathcal{A}), \oplus)$ be the abelian semigroup of isomorphism classes of finitely-generated projective $\mathcal{A}$-modules with direct sum.

# Operator $K$-theory

For a $C^*$-algebra $\mathcal{A}$, let $(V(\mathcal{A}), \oplus)$ be the abelian semigroup of isomorphism classes of finitely-generated projective $\mathcal{A}$-modules with direct sum. Then

$$K_0(\mathcal{A}) := \{x - y \mid x, y \in V(A)\}$$

is the Grothendieck group of $(V(A), \oplus)$, that is, $x - y = z - w$ if and only if there is $r \in V(A)$ such that $x + w + r = z + y + r$.

# $K$-homology

Let $A$ by a $*$-algebra dense in a $^*$-algebra $\mathcal{A}$. A Fredholm module over $A$ consists of a $*$-representation of $A$ on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator $F$, of square 1 and such that the commutator

$$[F, a] \in \mathcal{K}(\mathcal{H}), \qquad \text{for all } a \in A.$$

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The $K^0(\mathcal{A})$ consists of homotopy equivalence classes of even Fredholm modules over $\mathcal{A}$.

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$$K^0(\mathcal{A}) \times K_0(\mathcal{A}) \to K^0(\mathcal{A})$$

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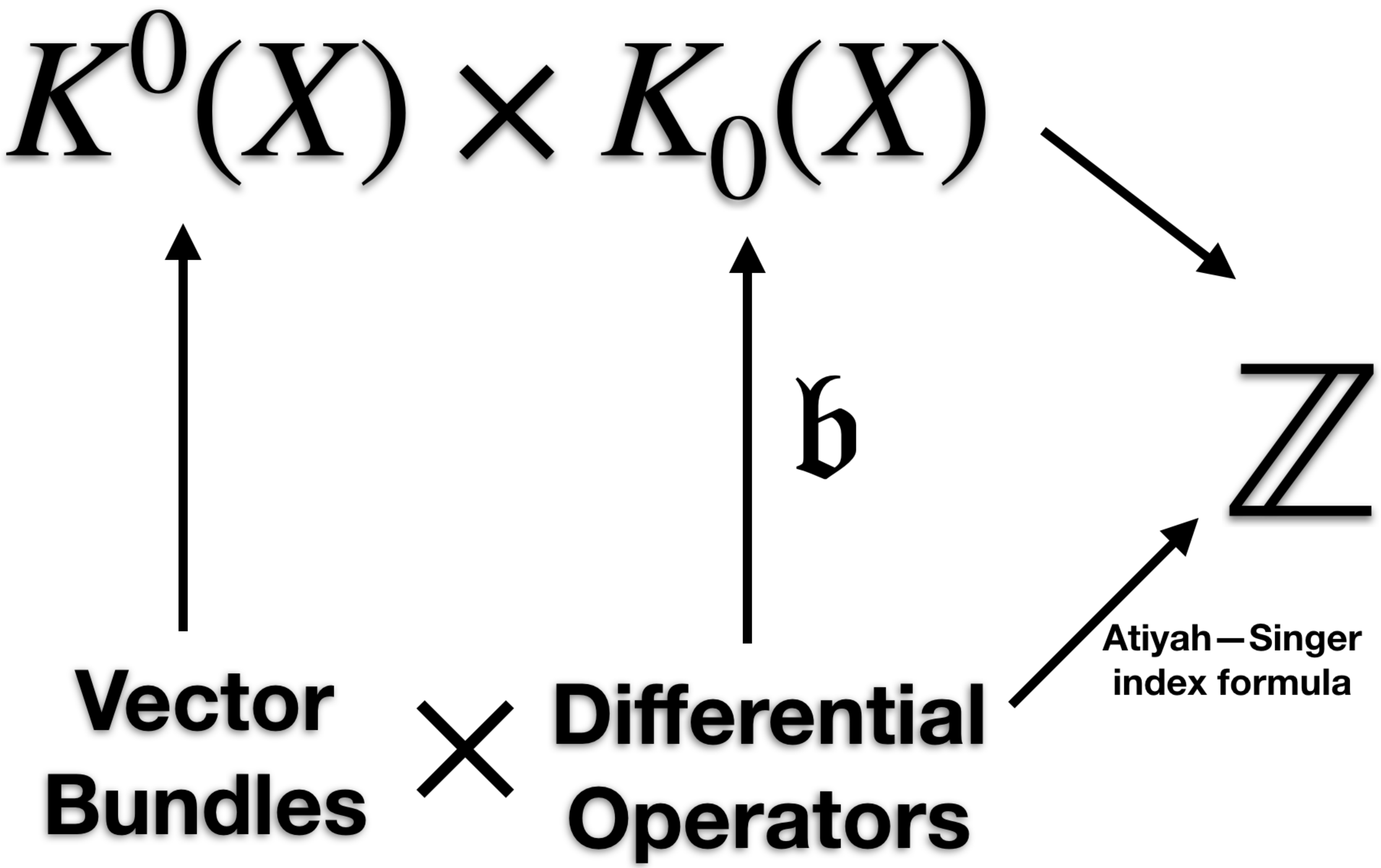
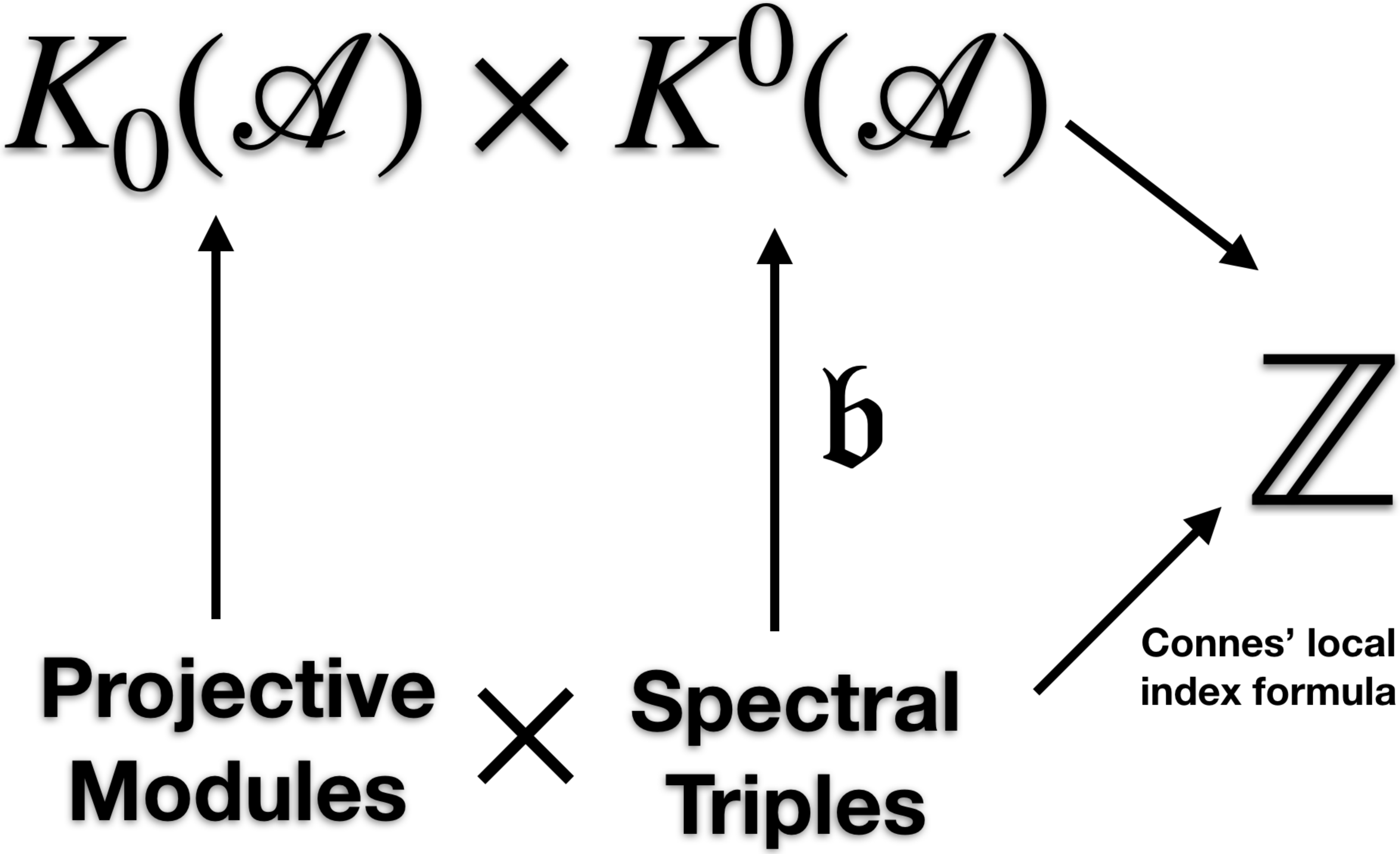
- $K_0(X) \times [\mathfrak{b}(D)] = K^0(\mathcal{A})$


$K^0(X) \times K_0(X)$
$\mathfrak{b}$
$\mathbb{Z}$
Atiyah—Singer
index formula
Vector
Bundles
$\times$
Differential
Operators


$K_0(\mathscr{A}) \times K^0(\mathscr{A})$
$\mathfrak{b}$
$\mathbb{Z}$
Projective Modules
×
Spectral Triples
Connes' local index formula

## Theorem

*For any Dolbeault–Dirac spectral triple*

$$(\mathcal{O}_q(G/L_S), L^2(\Omega^{(0,2)}), D_{\overline{\partial}}),$$

*and a noncommutative homogeneous vector bundle* $\mathcal{F} = \mathcal{O}_q(G)\Box_{U_q(\mathfrak{l}_S)} V$, *with a noncommutative holomorphic structure* $\overline{\partial}_{\mathcal{F}}$, *it holds that*

$$\langle(\mathcal{F}, \overline{\partial}_{\mathcal{F}}), D_{\overline{\partial}}\rangle = \sum_{k=0}^{n} (-1)^k H^{(0,k)}_{\mathcal{F}}.$$