Deformation Theory

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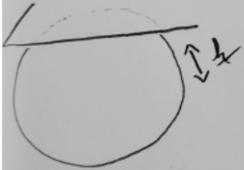
Plan

I Deformation theory and dgLa

II How to obtain dgLa

III Homotopy Structures

Classic	Quantum
$\mathbb{R}^{2n}$	$\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$
$\mathcal{C}^\infty(\mathbb{R}^{2n}), \cdot, \{\,\}$	$\mathcal{O}(\mathcal{H}), \circ, [\,,\,]$
$\dfrac{d\mathcal{O}}{dt} = \{h, \mathcal{O}\}$	$\dfrac{d\Theta}{dt} = [H, \Theta]$



quantisation

$$p \rightsquigarrow \hat{p} : \Psi \longmapsto -i\hbar \frac{\partial \Psi}{\partial x}$$

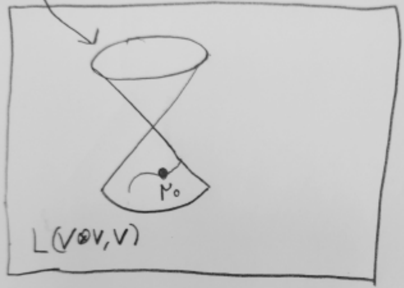
$$q \rightsquigarrow \hat{q} : \Psi \longmapsto x\Psi$$

$$f *_{\hbar} g := \left(\hat{f} \circ \hat{g}\right)^{\wedge -1}$$

Thm (Moyal - Groenewald)

$$*_h = \cdot + \hbar B_1(,) + \hbar^2 B_2(,) + \dots$$

$$Ass(V) = \{\mu \ / \ \mu(\mu(a,b),c) = \mu(a,\mu(b,c))\}$$



$$\mu_t = \mu_0 + t\mu_1 + \cdots$$

$$\mu_t(a,b) \in V \otimes K[[t]]$$

order n

$$\mu_0(\mu_n(a,b),c) + \sum_{\substack{i+j=n \\ i,j\neq 0}} \mu_i(\mu_j(a,b),c) + \mu_n(\mu_0(a,b),c)$$

$$= \mu_0(a,\mu_n(b,c)) + \sum_{\substack{i+j=n \\ i,j\neq 0}} \mu_i(a,\mu_j(b,c)) + \mu_n(a,\mu_0(b,c))$$

$$d\mu_n + \frac{1}{2} \sum_{\substack{i+j=n \\ i,j\neq 0}} [\mu_i,\mu_j] = 0$$

$$d\tilde{\mu}_t + \frac{1}{2}[\tilde{\mu}_t,\tilde{\mu}_t] = 0$$

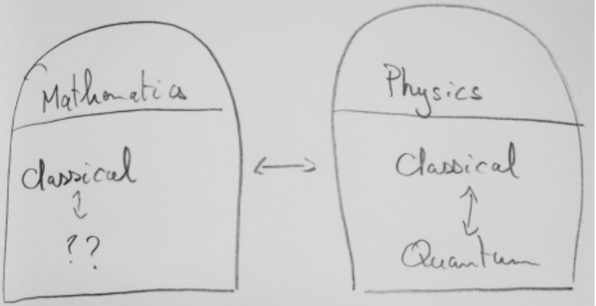
$$[a,b]_i = \frac{1}{2}\left(\mu_i(a,b) - \mu_i(b,a)\right)$$

Prop

$[\ ]_1$ is Poisson

$\forall\ (\cdot, \{\})$ $\quad \exists ?\ \mu_t$

A : "Any Poisson manifold is quantizable"

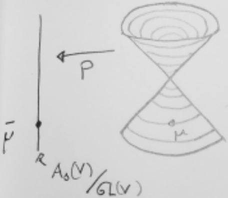


Quantum groups

$$G \hookrightarrow \mathcal{C}^\infty(G)$$

$$\cdot \qquad \Delta$$

$$1_e \qquad \varepsilon$$



$$GL(V) \curvearrowright L(V\otimes V, V)$$

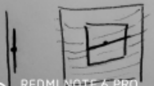
$$(g\cdot\mu)(a,b) \triangleq g(\mu(g^{-1}a, g^{-1}b))$$

<u>Theorem</u>:

$$T_{\bar{\mu}} \mathcal{A}_s = H^1_{Hoch}$$

pf:

$$T_{\bar{\mu}}(A_s/G) = T_\mu A_s / T_\mu G\cdot\mu$$



$C^n = L(\underbrace{V \otimes \cdots \otimes V}_{n}, V)$

$C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2}$

δ_n ?

$[\ ]$?

Def : gLa

$L = \bigoplus L^n$

$[\ ] : L \otimes L \longrightarrow L$

$[A,[B,C]] = [[A,B],C] + (-1)^{AB}[B,[A,C]]$

$[A,B] = -(-1)^{AB}[B,A]$

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Exemple:

$$[\mu,\nu]_{Ger} = \mu\circ\nu - (-1)^{\mu\nu}\,\nu\circ\mu$$

$$\mu\circ\nu = \sum \pm$$

Prop:

$[\ ]_{Ger}$ is gLa

$$A_s = \{\mu \in L^1 \,/\, [\mu,\mu]_{Ger} = 0\}$$

$$d_\mu^2 = 0 \qquad d_\mu :\hat{=} [\mu, -]$$

$$L^0 = \mathrm{End}(V)$$

$$\text{"}e^{L^0}\text{"} = GL(V)$$

$$L^0 \xrightarrow{\mathrm{ad}} L \quad \rightsquigarrow \quad GL(V) \hookrightarrow L(V \otimes V, V).$$

E graded $\qquad L(E,E), [\ ]$

$$[A,B] = A \circ B - (-1)^{AB} B \circ A$$

$E = \mathcal{A}$ algebra

$$\mathrm{Der}(\mathcal{A}) = \left\{ \partial \ / \ \partial(ab) = (\partial a)b + (-1)^{a\partial} a \partial b \right\}$$

Lemma

$$[\mathrm{Der}(\mathcal{A}), \mathrm{Der}(\mathcal{A})] \subset \mathrm{Der}(\mathcal{A})$$

$\mathcal{A} = TV = \bigoplus V \otimes \dots \otimes V$

Prop

$$\mathrm{Der}\, TV \longleftrightarrow \{\varphi : V \longrightarrow TV\}$$

Cor

$$\mathrm{Der}\, TV^* \simeq L = \bigoplus L(V \otimes \dots \otimes V, V)$$

Prop

$$(\mathrm{Der}\, TV^*[1], [\ ]) \longleftrightarrow (C^\bullet_{Hoch}, [\ ]_{Ger})$$

Prop

$$\mathrm{Der}\ S\,V^*[1],\ [\ ] \longleftrightarrow C^{\bullet}_{CE},\ [\ ]_{NR}$$

Prop

$$MC(L_{NR}) \triangleq \{\alpha \in C^1_{CE} \,/\, [\alpha,\alpha]=0\}$$
$$= \mathrm{Lie}(V)$$

$$\mathrm{As}(V) = MC(\dots T\dots)$$

$$\mathrm{Lie}(V) = MC(\dots S\dots)$$

$$\mathcal{O}(V) = MC(\dots ?\dots)$$

Koszul duality !

$$\mathrm{As}^! = \mathrm{As}$$

$$\mathrm{Lie}^! = \mathrm{Com}$$

Ex:

$$\mathcal{E}nd\, V = \bigoplus L(V^{\otimes i}, V)$$

$$f \circ_i g = f(\cdots, \underset{\uparrow i}{g(\cdots)}, \cdots)$$

Ex: $\langle a,b \rangle = \{ a, b, ab, aab, \ldots \}$

$$\langle 1, \xi \rangle = \{ 1, \xi, \overset{\xi}{1}, \overset{\xi}{\overset{1}{1}}, \ldots \}$$

$$\langle Y \rangle = \{ Y, \ldots \}$$

$$A_s = \langle Y \rangle / (Y - Y)$$

Prop:

$$As(V) = \{ \rho : As \longrightarrow End(V) \}$$

$$\rho(f \circ_i g) = \rho(f) \circ_i \rho(g).$$

$$F_{As}(V) = \left\{ v_1 ,\ \overset{v_1\ v_2}{Y} ,\ \overset{v_1\ v_2\ v_3}{Y} ,\ \dots \right\}$$

$$\uparrow$$

$$((v_1 \otimes v_2) \otimes v_3)$$

$$Y \longmapsto \left(\left(\overset{v_1\ v_2}{Y} \right) \left(v_3 \right) \right) \overset{``Y"}{\longmapsto} \overset{v_1\ v_2\ v_3}{Y}$$

$$``Y" \in End(F_{As}(V))$$

M manifold $\qquad q_i$

$T^*M \qquad q_i \; p_i$

$\{ \; \} \qquad \dfrac{\partial}{\partial q_i} \wedge \dfrac{\partial}{\partial p_i}$

$$C^\infty(T^*[1]M) \simeq \Gamma(\Lambda^\bullet TM) = \mathfrak{X}^\bullet(M)$$

$$\{ \; \} \longleftrightarrow [\;]_{sch}$$

Prop :

$[\pi, \pi] = 0 \iff \pi$ poisson

rem:

$$\phi : L, [\ ], d \longrightarrow L', [\ ]', d$$

$$\mathcal{MC}(L) \longmapsto \mathcal{MC}(L')$$

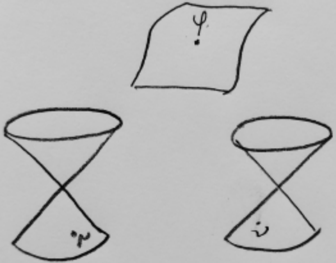
Iso if ϕ quasi.

If $L = \mathcal{X}^\bullet, [\ ]_{sch}$ $\quad L' = C^\bullet_{Hoch}, [\ ]_{Ger}, d_{Hoch}$

$$\phi \text{ quasi} \Leftrightarrow \{\text{Poisson}\}/_{\text{equ}} \simeq \{\text{deformations}\}/_{\text{equ}}$$

Konts: ϕ morp L_∞

$$(U, \mu) \xrightarrow{\varphi} (V, \nu)$$



$$\nu(\varphi(a)\, \varphi(b)) = \varphi\, \mu(a, b)$$

$$\alpha = \begin{pmatrix} \mu \\ \nu \\ \varphi \end{pmatrix}$$

$$d\alpha + \frac{1}{2}\{\alpha,\alpha\} + \frac{1}{3!}\{\alpha,\alpha,\alpha\} + \cdots = 0$$

<u>$L_\infty[1]$ algebra</u>

$$\{\ldots\}_n : \Lambda^n V \longrightarrow V \quad 2-n$$

$$S^n V[1] \longrightarrow V[1] \quad 1$$

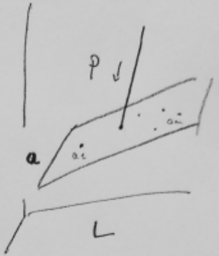
$L \quad [\;]$

$a \quad [a,a]=0$

$p \; L \to a \quad p^2 = p$

$[\mathrm{Im}\, p, \ker p] \subset \ker p$

$[\Delta,\Delta]=0 \quad \Delta \in L^1$



Def: $a \quad L_\infty[1]$

$$\{a_1, \dots a_n\} = [\dots[[\Delta, a_1]\, a_2] \dots a_n]$$

Prop:

$L[1] \oplus \mathfrak{a}$ $\quad L_\infty[1]$ alg

$$d(x[1], a) = -(Dx)[1], P(x + Da)$$

with $D = [\Delta, -]$

$$\{x[1], y[1]\} = [x, y][1]$$

$$\{a_1, \dots a_n\} = P[\dots [Da_1, a_2] \dots a_n]$$

$$\{x[1], a_1, \dots a_n\} = P[\dots [x, a_1], a_2] \dots a_n]$$

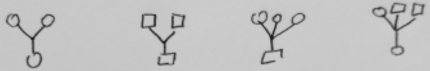
$$(L[1] \oplus a)^P_\Delta$$

Application:

$$L = L^\bullet(U+V, U+V)$$

$$= \bigoplus L((U+V)\otimes \cdots \otimes (U+V), U+V)$$

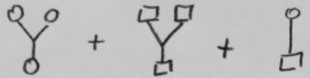
$$= \bigoplus L^\bullet(U,U) \oplus L^\bullet(V,V) \oplus L^\bullet(U,V) \oplus \text{Rest}$$



$$a = L^\bullet(U,V)$$

$$L' = L^\bullet(U,U) + L^\bullet(V,V)$$

$\alpha \in (L'[1]+a)^0$:

$\alpha =$ 

$$= \begin{pmatrix} \mu \\ \nu \\ \varphi \end{pmatrix}$$

Prop

$\ulcorner$ $\alpha \in MC(L'[1]+a)$ $\urcorner$

$\llcorner$ $\Leftrightarrow$ $\mu \xrightarrow{\varphi} \nu$ morphism $\lrcorner$

Prop:

$\mathfrak{g}$ L_∞, $\alpha \in MC(\mathfrak{g})$

$\Rightarrow \mathfrak{g}^\alpha$ L_∞:

$$d^\alpha = d + \{\alpha, -\}$$

$$\{,, \dots ,\}_n = \{\dots\}_n + \{\alpha, \dots\}_{n+1} + \dots + \frac{1}{k!}\{\underbrace{\alpha, \dots \alpha}_{k}, \dots\}_{n+k}$$

Prop:

$$\alpha+\alpha' \in MC(\mathfrak{g}) \iff \alpha' \in MC(\mathfrak{g}^{\alpha})$$

Prop:

$$\alpha+\alpha' \in MC(\mathfrak{g}) \Leftrightarrow \alpha' \in MC(\mathfrak{g}^{\alpha})$$

Cor:

$(L' \oplus a)^{\mu,\nu,\varphi}$ governs simultaneous

deformations of $\mu \xrightarrow{\varphi} \nu$

For arbitrary $\mathcal{O}$ Koszul:

$\mu \xrightarrow{\varphi} \nu$:

Consider $\quad L = \mathrm{Der}\, \mathcal{O}^!(U+V)$

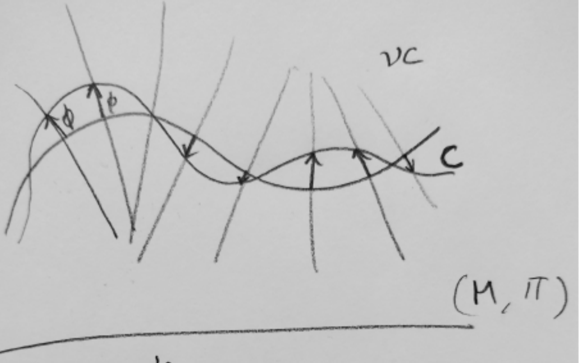
- geometry:

Lie $\xrightarrow{\text{KKS}}$ Poisson

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "\varphi"] & \mathfrak{g}^* \\ \mathfrak{h} & \mathfrak{h}^* \arrow[u, "\varphi^*"'] \end{tikzcd}$$

$$(M,\pi) \xrightarrow{\varphi} (M',\pi')$$

Poisson $\Leftrightarrow$ $\Gamma(\varphi)$ coisotropic



$$\overset{\nu C}{\parallel}$$

$$L = \left(\mathcal{X}^{\bullet}(M),\ [\ ]_{sch} \right)$$

$$a = \Gamma(\wedge \nu_c)$$

$$P = \text{restriction}$$

$$\Delta = \circlearrowleft$$

Prop :

$$(\pi, \phi) \in MC(L[1] + \mathfrak{a})$$

$$\iff \begin{cases} \pi & \text{Poisson} \\ (\phi) & \text{coisotropic} \end{cases}$$

generalisation:

- generalized complex & dirac
- arbitrary diagrams
- quantization of algebraic varieties / schemes

(abelian category of quasi-coherent sheaves)