Overdetermined systems of PDEs: formal theory and applications

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Lecture 1: Formal Theory
(d'apres D. Spencer, S. Sternberg, D. Quillen, V. Guillemin, J.-P. Serre, M. Kuranishi, H. Goldschmidt, B. Malgrange, J.-F. Pommaret, A. Vinogradov, V. Lychagin et al)

Jet Spaces

For a bundle $\pi : E \to M$ consider the equivalence relation on $\Gamma(E)$: $s \simeq s'$ iff $s - s' \in \mathfrak{m}_x^{k+1} \cdot \Gamma(E)$, where $\mathfrak{m}_x \subset C^\infty(M)$ is the max. ideal of $x \in M$. The equiv. classes $j^k_x s \equiv$ the jet-space $J^k\pi$.

In local coordinates (x^i, u^j) on E the Taylor expansion of sections $s : x \mapsto u(x)$ yields coordinates (x^i, u^j_σ) with $|\sigma| \leq k$. This defines the jet-lift $j^k : \Gamma(E) \to \Gamma(J^k\pi)$, $s \mapsto j^k s$. We have the projections

$$J^\infty\pi \to \dots \to J^k\pi \to J^{k-1}\pi \to \dots \to J^0\pi = E \to M.$$

Choose a point $a_k \in J^k\pi$ with $\pi_{k,l}(a_k) = a_l$ for $k > l$, $a_0 = a$, $\pi(a) = x$. Let $F = \mathrm{Ker}(d_a\pi) \subset T_aE$, $T = T_xM$. Then we identify $V(a_k) = \mathrm{Ker}(d_{a_k}\pi_{k,k-1}) = S^kT^* \otimes F$.

Compactification: a manifold E and jets of m-dim submanifolds $M \subset E$. The last projection $\pi : J^0 = E \to M$ does not exist and $\mathrm{Ker}(d_{a_k}\pi_{k,k-1}) = S^kT^* \otimes F$, where $T = T_aM$, $F = T_aE/T_aM$.

Cartan distribution is given by the formula

$$\mathcal{C}(a_k) = \mathrm{span}\{L(a_{k+1}) : a_{k+1} \in \pi^{-1}_{k+1,k}(a_k)\} \subset T_{a_k} J^k\pi,$$

where $L(a_{k+1}) = T_{a_k}(j^k s)$ for $a_{k+1} = j^{k+1}_x s$, $a_k = j^k_x s$. In local coordinates we have:

$$\mathcal{C}(a_k) = \langle D_i^{(k)} = \partial_{x^i} + \sum_{|\sigma|<k} u^j_{\sigma+1_i} \partial_{u^j_\sigma}, \partial_{u^j_\tau} : |\tau| = k \rangle.$$

We have: $\mathcal{C}(a_k) = L(a_{k+1}) \oplus V(a_k)$ and $d_{a_k}\pi_{k,k-1}\mathcal{C}(a_k) = L(a_k)$.

On $J^\infty\pi$ we have integrable distribution

$$\mathcal{C}(a_\infty) = L(a_\infty) = \langle D_i = \partial_{x^i} + \sum u^j_{\sigma+1_i} \partial_{u^j_\sigma} \rangle,$$

where $D_i = D_i^{(\infty)}$ is the operator of total derivative along x^i.

Geometrically differential equation of order k is a submanifold $\mathcal{E}_k \subset J^k\pi$ that submerses on $J^{k-1}\pi$. We let $\mathcal{E}_i = J^i\pi$ for $i < k$ and $\mathcal{E}_i = \mathcal{E}_k^{(i-k)} \subset J^i\pi$ be the $(i-k)$-th prolongation. The s-th prolongations of $\mathcal{E}_k = \{f = 0\}$ has defining equations $\mathcal{D}_\tau f = 0$, $|\tau| = s$, where $\mathcal{D}_\tau = D_{i_1} \cdots D_{i_s}$ for $\tau = (i_1, \dots, i_s)$.

Thus a differential equation is a co-filtered manifold $\mathcal{E} = \{\mathcal{E}_i\}_{i=0}^{\infty}$. It is called compatible (sometimes said involutive) if $\pi_{i+1,i} : \mathcal{E}_{i+1} \to \mathcal{E}_i$ is a submersion for $i \geq k$. Cartan distribution of $\mathcal{E}$ is the distribution $\mathcal{C}_\mathcal{E} = T\mathcal{E} \cap \mathcal{C}$ for each jet-level i.

$\mathbb{E}$x. Complete system of equations of order k:

$$u_\sigma = f_\sigma(j^{k-1}u), \quad |\sigma| = k.$$

For this system $\mathcal{E}$ the Cartan distribution $\mathcal{C}_\mathcal{E}$ is horizontal, and the compatibility is equivalent to its Frobenius-integrability, i.e. $D_i f_{\tau+1_j} = D_j f_{\tau+1_i}$ for all $i < j$ and τ with $|\tau| = k-1$.

Symbolic module

Let $\mathbf{1}_M = M \times \mathbb{R}$ be the trivial one-dimensional bundle. If $\mathcal{E}$ is a linear system, then $\mathcal{E}^* = \mathrm{Hom}(\mathcal{E}, \mathbf{1}_M)$ is a D-module, i.e. a module over the algebra of differential operators $\mathrm{Diff}(\mathbf{1}_M, \mathbf{1}_M)$.

Both the algebra and the module are filtered. Passing to the graded objects at a point get: the ring $ST = \oplus_{i=0}^{\infty} S^i T$ of polynomial functions on T^*M and the symbolic module $\mathrm{gr}(\mathcal{E}^*)$.

For nonlinear systems the symbols are bundles over $\mathcal{E}$ defined as $g_i = \mathrm{Ker}(d\pi_{i,i-1} : T\mathcal{E}_i \to T\mathcal{E}_{i-1}) \subset S^i T^* \otimes F$.
In fact, starting from the symbol g_k of $\mathcal{E}_k$ define the symbolic system $\{g_i\}_{i=0}^{\infty}$ by: $g_i = S^i T^* \otimes F$ for $i < k$ and

$$g_i = g_k^{(i-k)} := (S^{i-k} T^* \otimes g_k) \cap (S^i T^* \otimes F).$$

The dual module $g^* = \oplus g_i^*$ is naturally a module over $\mathcal{R} = ST$, and is called the symbolic module $\mathcal{M}_{\mathcal{E}}$.

Spencer δ-cohomology

For a module $\mathcal{M}$ over the ring ST Koszul complex is defined by

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{M} \otimes T \leftarrow \mathcal{M} \otimes \Lambda^2 T \leftarrow \mathcal{M} \otimes \Lambda^3 T \leftarrow \dots$$

Dualizing it over $\mathbb{R}$ (but not over $\mathcal{R}$!) we get the Spencer δ-complex, in each gradation

$$0 \to g_k \to g_{k-1} \otimes T \to g_{k-2} \otimes \Lambda^2 T^* \to g_{k-3} \otimes \Lambda^3 T^* \to \dots$$

Its cohomology at the term $g_i \otimes \Lambda^j T^*$ is denoted by $H^{i,j}(g)$, and also $H^{i,j}(\mathcal{E})$ if g is the symbolic system of $\mathcal{E}$.

The Spencer cohomology $H^{i,j}(g)$ are dual to the Koszul homology $H_{i,j}(g^*)$, and if $g^* = ST/I$ for an ideal I this also equals to $H_{i-1,j+1}(I)$.

Iterpretation of δ-cohomology

Cohomology $H^{*,0}(g)$ is supported in gradation 0: $H^{0,0}(g) = F$ and $H^{i,0}(g) = 0$ for $i > 0$.

Cohomology $H^{*,1}(g)$ counts generators of the module g^*. For the equation $\mathcal{E}$: $H^{i,1}(\mathcal{E})$ is the number of defining differential equations of order i, so in our setup it is non-zero only for $i = k$.

Cohomology $H^{*,2}(g)$ counts compatibility conditions of $\mathcal{E}$. In fact, the curvature (structure tensor) of the distribution $\mathcal{C}_{\mathcal{E}}$ at a point a_k, that splits pointwise $\mathcal{C}_{\mathcal{E}}(a_k) = L(a_{k+1}) \oplus g_k(a_k)$, is an element $\Xi \in V(a_{k-1}) \otimes \Lambda^2 \mathcal{C}^*_{\mathcal{E}}(a_k)$. Its restriction to a horizontal plane $L(a_{k+1}) \subset \mathcal{C}_{\mathcal{E}}(a_k)$ gives $\Xi_{L(a_{k+1})} \in g_{k-1} \otimes \Lambda^2 T^*$.

Now change of $a_{k+1} \in \pi^{-1}_{k+1,k}(a_k)$ results in change of this tensor by $\mathrm{Im}(\delta : g_k \otimes T^* \to g_{k-1} \otimes \Lambda^2 T^*)$, whence $W_k(a_k) \in H^{k-1,2}(\mathcal{E})$.

System $\mathcal{E}$ of order k is involutive if $W_k \equiv 0$, $H^{i,2}(\mathcal{E}) = 0 \ \forall\ i \geq k$.

Variations of Spencer cohomology

When the system is not compatible ($\equiv$ involutive after prolongation), this is achieved by prolongation-projection, resulting in a smaller equation $\tilde{\mathcal{E}} \subset \mathcal{E}$ with the same amount of solutions.

Consider the partial case of Lie equation $\mathcal{E}$: the linear system equations $L_X(q) = 0$ for some geometric structure q. In other words, $\mathcal{E}$ describes the infinitesimal symmetries of q.

In this case for classical geometries, after re-numeration $\mathfrak{g}_i = g_{i+1}$ we get (after prolongation-projection) the Lie algebra of symmetries $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots$ as a symbolic system.

A generalization of this is related to filtered geometries, when $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \dots \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots$. A partial case of this is given by the class of parabolic geometries.

Another generalization arises in super-geometry. The Spencer δ-cohomology is defined as the cohomology of the super-complex:

$$0 \to \mathfrak{g}_k \to \mathfrak{g}_{k-1} \otimes T \to \mathfrak{g}_{k-2} \otimes \Lambda^2 T^* \to \mathfrak{g}_{k-3} \otimes \Lambda^3 T^* \to \dots$$

Characteristic variety

From computational viewpoint let $f = (f^1, \dots, f^r)$ be the defining (non-linear) operators of order k for $\mathcal{E}$, and let

$$\ell(f^i)(v) = \sum_{|\sigma|\le k} \sum_j P_j^{i,\sigma} v_\sigma^j$$

be the linearization of the components, $P_j^{i,\sigma}$ depending on a_k.

For $p = (p_1, \dots, p_n) \in T^*M$ denote $P_j^i(p) = \sum_{|\sigma|=k} P_j^{i,\sigma} p_\sigma$, where $p_\sigma = p_{i_1} \cdots p_{i_k}$ for a multi-index $\sigma = (i_1, \dots, i_k)$. The symbol of f at a_k is

$$\mathrm{smbl}_f(p) = \begin{pmatrix} P_1^1(p) & \cdots & P_m^1(p) \\ \vdots & \ddots & \vdots \\ P_1^r(p) & \cdots & P_m^r(p) \end{pmatrix}$$

Assume $\mathcal{E}$ is (over)determined, in naïve terms: $\dim F = m < r$. The affine/projective/complex characteristic variety is

$$\mathrm{Char}(\mathcal{E}; a_k) = \{p \in T_x^*M : \mathrm{rank}(\mathrm{smbl}_f(p)) < m\}.$$

Relation to the symbolic module

Recall that $\mathcal{M}_\mathcal{E} = g^*$ is the symbolic module. In the language of commutative algebra $\mathrm{Char}(\mathcal{E}) = \mathrm{supp}(\mathcal{M}_\mathcal{E})$.

Every projective module over the Noetherian ring $\mathcal{R} = ST$ can be realized as a module of sections of some sheaf. For the module $\mathcal{M}_\mathcal{E}$ the characteristic sheaf $\mathcal{K}$ is defined in such a way and evaluation at $p \in T^*$ of the stalk is the kernel of the operator $\mathrm{smbl}_f(p)$ considered as a map from $\mathbb{R}^m$ to $\mathbb{R}^r$. For $p \in \mathrm{Char}(\mathcal{E})$ the rank of evaluation belongs to $[1, m]$.

Covector $p \in T^*$ is characteristic iff $p^k \otimes v \in g_k \setminus 0$ for some $v \in F$.

In particular, the system $\mathcal{E}$ is of finite type $\dim g < \infty$ iff $\mathrm{Char}^\mathbb{C}(\mathcal{E}) = 0$ (in projective setting $\emptyset$).

Dimensions of the solution spaces

Cartan's theory of characters gives Cauchy data that uniquely determines solutions: in the analytic context of the Cartan-Kähler theorem the general local solution depends on s_d functions of d arguments, ..., s_1 functions of 1 arguments, s_0 constants.

These numbers have no invariant meaning except for the Cartan genre d (maximal i with $s_i \neq 0$) and Cartan integer $\sigma = s_d$. In modern terms these can be defined as follows:

$$d = \dim \mathrm{Char}^{\mathbb{C}}_{\mathsf{aff}}(\mathcal{E}), \qquad \sigma = \sum d_\epsilon \cdot \deg \Sigma_\epsilon,$$

where $\mathrm{Char}^{\mathbb{C}}_{\mathsf{proj}}(\mathcal{E}) = \cup_\epsilon \Sigma_\epsilon$ is the decomposition of the projective characteristic variety into irreducible components.

Theorem

General local solution of $\mathcal{E}$ depends on d functions of σ arguments.

An example: systems of complete intersection type

Consider a scalar system $\mathcal{E}$: $f_1 = \cdots = f_r = 0$, with the unique dependent variable u. In this case, $\mathrm{smbl}_{f_1}(p), \dots, \mathrm{smbl}_{f_r}(p)$ are homogeneous polynomials of degrees $k_1, \dots, k_r$.

The system is called a complete intersection if their loci intersect transversally, i.e. $\operatorname{codim} \mathrm{Char}^{\mathbb{C}}(\mathcal{E}) = r$. In this case the compatibility conditions are

$$\{f_i, f_j\} = 0 \bmod \mathcal{J}_{k_i+k_j-1}(f), \quad i < j,$$

where $\mathcal{J}_t(f)$ is the differential ideal generated by components of f up to jet-order t and

$$\{f, h\} = \ell(f) \circ h - \ell(h) \circ f$$

is the higher Jacobi-Mayer bracket. The Spencer δ-cohomology is

$$H^{*,j}(\mathcal{E}) \simeq \Lambda^j F.$$

If the system is compatible, $\mathrm{Sol}_{\mathsf{loc}}(\mathcal{E})$ depends on $k_1 \cdots k_r$ functions of $n - r$ variables.

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Lecture 2: Application to Monge-Ampère equations
(based on joint work with E. Ferapontov and V. Novikov)

A Monge-Ampère equation in a region $M \subset \mathbb{R}^n$ is a nonlinear scalar second order PDE of the form

$$\sum a_\sigma \det U_\sigma = 0,$$

where $a_i = a_i(x, u, \partial u)$ is a function on the first jets J^1M, $U = \mathrm{Hess}(u) = \big(u_{ij}\big)$ is the Hessian matrix and σ encodes all minors of U of sizes $0 \le |\sigma| \le n$.

Such equations arise in a variety of applications, for instance self-dual gravity, special Lagrangian and Kähler geometry, gas dynamics and non-linear accoustic.

More invariantly, for any n-dimensional manifold M a Monge-Ampère equation is given by a choice of n-form $\omega \in \Omega^n(J^1M)$, namely the equation is $\mathcal{E} : \omega|_{j^1u} = 0$. n-forms that differ by the contact ideal give identical equations, so ω can be normalized to be an effective n-form.

Symplectic Monge-Ampère equations

Monge-Ampère equations with $a_i = \mathrm{const}$ are called symplectic. This class is $\mathbf{Sp}(2n, \mathbb{C})$-invariant. It is a subclass of Hirota type equations, given by $f(\partial^2 u) = 0$.

Geometrically Hirota type equations $\mathcal{E}$ correspond to hypersurfaces X in Lagrangian Grassmanian Λ. Note that Λ corresponds to a compactification of J^2M and $\dim \Lambda = d(n) := \frac{n(n+1)}{2}$.

Symplectic Monge-Ampère equations are hyperplane sections of the Plücker embedding Λ into $\mathbb{P}^{p(n)-1}$, where $p(n) := \frac{2(2n+1)!}{n!(n+2)!}$ is the number of independent minors of a symm $n \times n$ matrix.

In applications it is important to characterize Monge-Ampère equations by differential relations. The following is based on this:

Theorem (E. Ferapontov, BK, V. Novikov)

For dim $n \geq 4$, the integrability of a non-degenerate Hirota type equation by the method of hydrodynamic reductions implies the symplectic Monge-Ampère property.

Characterization: geometric form

To characterize Monge-Ampère equations, it is useful to change the implicit form of equation (this contains a reparametrization of the defining function) to an explicit one, which is convenient to write with $n \mapsto n+1$, so that local coordinates are $x^0, \dots, x^n$

$$u_{00} = f(u_{01}, \dots, u_{0n}, u_{11}, u_{12}, \dots, u_{nn}).$$

In this non-symmetric form f depends on $n(n+3)+1$ arguments and uniquely defines the embedding

$$X \subset \Lambda \hookrightarrow \mathbb{P}^{p(n+1)-1}.$$

The following gives the projective-invariant characterization.

Theorem

Equation $u_{00} = f$ is of Monge-Ampère type if and only if $d^2 f$ belongs to the span of the second fundamental forms of the Plücker embedding of Λ restricted to the hypersurface X.

Characterization: analytic form (MAE)

This leads to a system of PDEs for f for distinct indices $i \neq j \neq k \neq l \in \{1, \dots, n\}$

$$f_{u_{ii}} f_{u_{0i}u_{0i}} + f_{u_{ii}u_{ii}} = 0, \quad \tfrac{1}{2} f_{u_{0i}} f_{u_{0i}u_{0i}} + f_{u_{0i}u_{ii}} = 0,$$

$$\begin{gathered}
\tfrac{1}{2} f_{u_{0j}} f_{u_{0i}u_{0i}} + f_{u_{0i}} f_{u_{0i}u_{0j}} + f_{u_{0i}u_{ij}} + f_{u_{0j}u_{ii}} = 0, \\
\tfrac{1}{2} f_{u_{ij}} f_{u_{0i}u_{0i}} + f_{u_{ii}} f_{u_{0i}u_{0j}} + f_{u_{ii}u_{ij}} = 0, \\
\tfrac{1}{2} f_{u_{jj}} f_{u_{0i}u_{0i}} + \tfrac{1}{2} f_{u_{ii}} f_{u_{0j}u_{0j}} + f_{u_{ij}} f_{u_{0i}u_{0j}} + f_{u_{ii}u_{jj}} + \tfrac{1}{2} f_{u_{ij}u_{ij}} = 0,
\end{gathered}$$

$$\begin{gathered}
f_{u_{0k}} f_{u_{0i}u_{0j}} + f_{u_{0j}} f_{u_{0i}u_{0k}} + f_{u_{0i}} f_{u_{0j}u_{0k}} + f_{u_{0i}u_{jk}} + f_{u_{0j}u_{ik}} + f_{u_{0k}u_{ij}} = 0, \\
\tfrac{1}{2} f_{u_{jk}} f_{u_{0i}u_{0i}} + f_{u_{ik}} f_{u_{0i}u_{0j}} + f_{u_{ij}} f_{u_{0i}u_{0k}} + f_{u_{ii}} f_{u_{0j}u_{0k}} + f_{u_{ii}u_{jk}} + f_{u_{ij}u_{ik}} = 0,
\end{gathered}$$

$$\begin{aligned}
f_{u_{kl}} f_{u_{0i}u_{0j}} + f_{u_{jl}} f_{u_{0i}u_{0k}} + f_{u_{jk}} f_{u_{0i}u_{0l}} + f_{u_{il}} f_{u_{0j}u_{0k}} + f_{u_{ik}} f_{u_{0j}u_{0l}} \\
+ f_{u_{ij}} f_{u_{0k}u_{0l}} + f_{u_{ij}u_{kl}} + f_{u_{ik}u_{jl}} + f_{u_{il}u_{jk}} = 0.
\end{aligned}$$

Criterion for Monge-Ampère property

Theorem

Hirota type equation is of Monge-Ampère type iff the rhs f satisfies the above overdetermined system of PDEs.

This will be proved by the formal theory of differential equations. Note that every differential system can be described by either its defining relations (PDEs) or jets of its solutions. In the latter case the system is compatible (involutive), but we do not have control over the defining relations. In the former case we have control but do not know compatibility a priori.

To demonstrate that these two descriptions coincide we first prove that Monge-Ampère equations have defining relations of the second order only, then by dimensional reasons we conclude that these must coincide with the above system of relations (MAE).

Proof modulo symbolic involutivity

Let $\mathcal{E} \subset J^\infty(\mathbb{R}^{\bar{d}})$ denote the system of PDEs for f. Here $\bar{d} = d(n+1) - 1$, $\mathbb{R}^{\bar{d}}$ is the space of independent arguments of f and $\mathcal{E}_0 = J^0 = \mathbb{R}^{\bar{d}+1} \subset \Lambda$. Locally $X = \text{graph}(f) \subset J^0$ for f from (MAE) and k-jets of these define $\mathcal{E}_k$. Clearly, $\mathcal{E}_1 = J^1$. We claim that $\mathcal{E}_2$ is generated by (MAE).

The symbols of $\mathcal{E}$ are $g_k \subset S^k\tau^*$ where $\tau = T_o\mathbb{R}^{\bar{d}}$, and they can be interpreted as the space of linearly independent minors of A of size k. Thus $g_0 = \mathbb{R}$, $g_1 = \tau^*$ and in general

$$\dim g_k = b(k, n+1)$$

for all $k \in [0, n+1]$ with the exception of $k = 1$, in which case $\dim g_1 = b(1, n+1) - 1 = \bar{d}$ due to the relation $u_{00} = f$.

Here $b(k,n) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}$ is the number of independent $k \times k$ minors of a symm $n \times n$ matrix.

Proof modulo ... cont'd

For $k > n+1$ the symbol of $\mathcal{E}$ vanishes, $g_k = 0$, signifying that this system is of finite type. The solution space $S = \mathcal{E} \cap \pi_{n+2}^{-1}(o)$ has dimension

$$\dim S = \sum_{k=0}^{\infty} \dim g_k = \sum_{k=0}^{n+1} b(k, n+1) - 1 = p(n+1) - 1.$$

Next, $\mathcal{E}_2$ is contained in the locus of relations (MAE), and the number of relations is $N(n) = \binom{d(n+1)}{2} - b(2, n+1)$ which is the codimension of $g_2 \subset S^2T^*X$. This count along with the quasi-linearity of (MAE) implies the claim.

To finish the proof we observe that the higher symbols coincide with the prolongations: $g_{k+2} = g_2^{(k)}$ for $k > 0$.

Use of A_n-equivariance

To compute prolongations of the symbols used above we exploit the subalgebra $A_n = \mathfrak{sl}_{n+1}$ in the Lie algebra $C_{n+1} = \mathfrak{g}$ of the equivalence group $G = \mathbf{Sp}(2n+2, \mathbb{C})$: in the $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ corresponding to the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_{n+1}$ we have $\mathfrak{g}_0 = \mathfrak{gl}_{n+1} = \mathfrak{sl}_{n+1} \oplus \mathbb{R}$ and this naturally acts on the tangent space to the Lagrangian Grassmannian $\Lambda = G/P$. Thus the tangent and derived spaces are all A_n-modules.

Denote by $\hat{\mathcal{E}}$ the equation describing implicit Monge-Ampère equations. As specified by its solutions, this equation is involutive. A-priori, it can have PDE-generators of different orders.

Proposition

The defining equations of $\hat{\mathcal{E}}$ have second order: $\hat{\mathcal{E}}_{i+2} = \hat{\mathcal{E}}_2^{(i)}$.

The Lie algebra A_n acts naturally on $\hat{\mathcal{E}}_k$ and hence its symbols $\hat{g}_k$ are A_n-modules.

Proof of the proposition

The statement is equivalent to the successive identities $\hat{g}_{k+1} = \hat{g}_k^{(1)}$, $k \geq 2$, i.e. $H^{1,k}(\hat{g}) = 0$ or $H_{1,k}(\hat{g}^*) = 0$ for the Koszul homology complex

$$0 \leftarrow \hat{g}_{k+1}^* \xleftarrow{\partial} T \otimes \hat{g}_k^* \xleftarrow{\partial} \Lambda^2 T \otimes \hat{g}_{k-1}^* \longleftarrow \dots \qquad (1)$$

The symbols, considered as A_n-modules, are

$$\hat{g}_k = S^k S^2 V_n^* \cap S^2 \Lambda^k V_n^* = \Gamma_{2\lambda_{n-k+1}},$$

where $V_n = \Gamma_{\lambda_1} = \mathbb{R}^n$ is the standard irep and $V_n^* = \Gamma_{\lambda_n}$ its dual. Dualising the symbol we get $\hat{g}_k^* = \Gamma_{2\lambda_k}$. We also get $T = \Gamma_{2\lambda_1}$.

We work over $\mathbb{C}$ since the complexification commutes with passing to (co)homology. The main advantage of passing to dual is that the computations with the Littlewood-Richardson rule are n-independent and yields the following tensor decompositions.

Proof of the proposition cont'd

For the second term of Koszul's complex and $0 < k \leq n$:

$$\Gamma_{2\lambda_1} \otimes \Gamma_{2\lambda_k} = \Gamma_{2\lambda_{k+1}} + \Gamma_{\lambda_1+\lambda_k+\lambda_{k+1}} + \Gamma_{2\lambda_1+2\lambda_k}.$$

For the third term, using the plethysm $\Lambda^2 T = \Gamma_{2\lambda_1+\lambda_2}$, we have for $k \geq 2$ ($\lambda_0 = \lambda_{n+1} = 0$; for $k = 2$ and $k = n+1$ a modification of terms is required):

$$\Gamma_{2\lambda_1+\lambda_2} \otimes \Gamma_{2\lambda_{k-1}} = \Gamma_{\lambda_2+2\lambda_k} + \Gamma_{\lambda_1+\lambda_k+\lambda_{k+1}} + (\Gamma_{\lambda_1+\lambda_2+\lambda_{k-1}+\lambda_k} + \Gamma_{2\lambda_1+2\lambda_k}) + \Gamma_{2\lambda_1+\lambda_{k-1}+\lambda_{k+1}} + (\Gamma_{2\lambda_1+\lambda_2+2\lambda_{k-1}} + \Gamma_{3\lambda_1+\lambda_{k-1}+\lambda_k}).$$

Similarly, using futher plethysms, Shur's lemma and Young symmetrisers we compute the Koszul homology: $H_{1,k}(\hat{g}^*) = 0$ and $H_{2,k}(\hat{g}^*) = 0$ for $k \geq 2$. Hence the Spencer cohomology also vanish: $H^{1,k}(\hat{g}) = 0$, $H^{2,k}(\hat{g}) = 0$ for $k \geq 2$.

The last part of the proof of the main theorem

Lemma

For $k > 2$ the following holds: $g_k = g_2^{(k-2)}$.

Proof. The symbols of the two considered equations agree $\hat{g}_k = g_k$ for $k \neq 1$. However they form symbolic complexes over different vector spaces: T for $\hat{g}$ and τ for g, related by

$$0 \longrightarrow \tau \longrightarrow T \longrightarrow \mathbb{R} \longrightarrow 0$$

or by embedding $X \hookrightarrow \Lambda$, with the normal bundle $\mathbb{R}$.

This unites the Spencer δ-complexes for $\hat{g}$ and g into a commutative diagram of three complexes with exact vertical sequences that, in view of the preceding proposition, implies the claim by the standard diagram chase.

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Overdetermined systems of PDEs: formal theory and applications

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Lecture 3: Application to paraconformal structures
(based on joint work with E. Ferapontov)

Paraconformal structures

Paraconformal or $GL(2)$ geometry on an n-dimensional manifold M is defined by a field of rational normal curves of degree $n-1$ in the projectivised cotangent bundle $\mathbb{P}T^*M$. Equivalently, for a coframe $\{\omega_i\}$ on M it can be viewed as a field of 1-forms

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^{n-1}\omega_{n-1}.$$

This field and the parameter λ are defined up to transformations $\lambda \mapsto \frac{a\lambda+b}{c\lambda+d}$, $\omega(\lambda) \mapsto r(c\lambda+d)^{n-1}\omega(\lambda)$, where a, b, c, d, r are arbitrary smooth functions on M with $ad-bc=1$, $r \neq 0$.

Conventionally, a $GL(2)$ geometry is defined by a field of rational normal curves in the projectivised *tangent* bundle $\mathbb{P}TM$. Both pictures are projectively dual: the equation $\omega(\lambda) = 0$ defines a one-parameter family of hyperplanes that osculate a dual rational normal curve $\tilde{\omega}(\lambda) \subset \mathbb{P}TM$.

$GL(2)$ geometry is known to arise on solution spaces of ODEs with vanishing Wünschmann (Doubrov-Wilczynski) invariants. $GL(2)$ structures also arise in the theory of bi-Hamiltonian integrable systems as Veronese webs, in the context of exotic holonomy in four dimensions, in the geometry of submanifold of the Grassmannians, in the deformation theory of rational curves in compact complex surfaces X with positive normal bundle.

Jointly with E. Ferapontov we established that dispersionless integrable hierarchies of PDEs, such as the dispersionless Kadomtsev-Petviashvili, Adler-Shabat and universal hierarchies, provide $GL(2)$ geometry as characteristic varieties on the solutions.

In this way we obtain torsion-free $GL(2)$ structures of Bryant as well as totally geodesic $GL(2)$ structures of Krynski. The latter possess a compatible affine connection (with torsion) and a two-parameter family of totally geodesic α-manifolds, making them a natural generalisation of the Einstein-Weyl geometry.

Theorem 1 (E. Ferapontov, BK): Paremetrization

Every involutive $GL(2)$ structure is locally of the form

$$\omega(\lambda) = \sum_{i=1}^{n} \Big[\prod_{j \neq i} \Big(\lambda - \frac{u_j}{v_j} \Big) \Big] u_i dx^i.$$

Here u and v are functions of $(x^1, \dots, x^n)$, and subscripts denote partial derivatives: $u_i = u_{x^i}$, $v_i = v_{x^i}$. These functions satisfy a system of second-order PDEs, 2 equations for each quadruple of indices $1 \le i < j < k < l \le n$: $E_{ijkl} = 0$, $F_{ijkl} = 0$ with

$$E_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l) \left(\frac{2u_{ij} - (a_i + a_j) v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l) v_{kl}}{u_k u_l} \right)$$

$$F_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (b_i - b_j)(b_k - b_l) \left(\frac{2v_{ij} - (b_i + b_j) u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l) u_{kl}}{v_k v_l} \right)$$

where $a_i = \frac{u_i}{v_i}$, $b_i = \frac{v_i}{u_i}$, and $\mathfrak{S}$ denotes cyclic summation.

Proof of Thm 1 (sketch)

The space of α-manifolds is parametrised by 1 arbitrary function of 1 variable. Choose n 1-parameter family of α-manifolds $\equiv$ (local) foliations of M given by $\lambda = a_i(x)$ and rectify them: $\omega(a_i) = f_i dx^i$ (no summation). In this coordinate system:

$$\omega(\lambda) = \sum_{i=1}^{n} \Big[\prod_{j\neq i} \frac{\lambda - a_j}{a_i - a_j} \Big] f_i dx^i.$$

Choose two extra 1-parameter families of α-manifolds: $\omega(a_{n+1}) = f_{n+1} du$ and $\omega(a_{n+2}) = f_{n+2} dv$, i.e.

$$f_i \prod_{j\neq i} \frac{a_{n+1} - a_j}{a_i - a_j} = f_{n+1} u_i, \quad f_i \prod_{j\neq i} \frac{a_{n+2} - a_j}{a_i - a_j} = f_{n+2} v_i.$$

Using the coordinate freedom send $a_{n+1} \to \infty$ and $a_{n+2} \to 0$ and use conformal freedom of $\omega(\lambda)$ to get its required formula.

The above overdetermined PDE system (EF) is obtained from the integrability condition $d\omega(\lambda) \wedge \omega(\lambda) = 0$.

Proof of Thm 1 - end of the argument

Indeed, collecting coefficients at $dx^i \wedge dx^j \wedge dx^k$ we obtain

$$\underset{(jkl)}{\mathfrak{S}} \frac{\lambda - a_i}{u_i} \left(\frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \lambda_i + S_{ijk} = 0, \qquad (\dagger)$$

where $\lambda_i = \lambda_{x^i}$ and S_{ijk} is given by

$$u_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_j} \right) + u_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_k} \right) + u_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda}{\lambda - a_j} + \frac{\lambda}{\lambda - a_k} \right)$$
$$- v_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_j}{\lambda - a_j} \right) - v_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_k}{\lambda - a_k} \right) - v_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda a_j}{\lambda - a_j} + \frac{\lambda a_k}{\lambda - a_k} \right).$$

Denote by T_{ijk} the left-hand side of $(\dagger)$. For four distinct indices $i \neq j \neq k \neq l$ there are only two non-trivial linear combinations that do not contain derivatives of λ:

$$T_{ikj} + T_{ijl} + T_{ilk} + T_{jkl} \quad \text{and} \quad \frac{1}{\lambda - a_l} T_{ikj} + \frac{1}{\lambda - a_k} T_{ijl} + \frac{1}{\lambda - a_j} T_{ilk} + \frac{1}{\lambda - a_i} T_{jkl}.$$

The first linear combination is equal to zero identically, while the second combination vanishes iff relations (EF) are satisfied.

Thus system (EF) governing general involutive $GL(2)$ structures results on elimination of the derivatives of λ from equations $(\dagger)$.

For every value of n, the following holds:

- The characteristic variety of system (EF) is the tangential variety of the rational normal curve $\mathbb{P}^1 \ni \lambda \mapsto \omega(\lambda) \in \mathbb{P}^{n-1}$.
- The characteristic variety has degree $2n-4$, and the rational normal curve can be recovered as its singular locus.
- System (EF) is in involution and its general solution depends on $2n-4$ functions of 3 variables (for analytic/formal case).

Note that although the PDE system (EF) formally consists of $2\binom{n}{4}$ equations, only $2\binom{n-2}{2}$ of them are linearly independent: we can restrict to equations $E_{12kl}=0$ and $F_{12kl}=0$ for $3 \leq k < l \leq n$ since all other equations are their linear combinations.

For $n=4$ system (EF) is determined: it consists of 2 second-order PDEs for 2 functions u and v of 4 independent variables, so the claim is instant, and this implies the count of Bryant.

Proof of Thm 2 - part 1

We parametrize the rational normal curve as

$$\lambda \mapsto [p_1 : \cdots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i}, \quad a_i = \frac{u_i}{v_i}, \qquad (\dagger)$$

so that its tangential variety is given by

$$(\lambda, \mu) \mapsto [p_1 : \cdots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i} + \frac{u_i \mu}{(\lambda - a_i)^2}. \quad (\ddagger)$$

The symbol of $\mathcal{E} = \{E = 0,\ F = 0\}$ is given by the matrix

$$\ell_{\mathcal{E}}(p) = \begin{bmatrix} \ell_E^u(p) & \ell_E^v(p) \\ \ell_F^u(p) & \ell_F^v(p) \end{bmatrix},$$

where $\ell_E^u(p) =$ is the symbol of u-linearization of E is given by

$$\ell^u_{E_{ijkl}}(p) = \sum_{a \le b} \frac{\partial E_{ijkl}}{\partial u_{ab}} p_a p_b = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\left(\frac{p_i p_j}{u_i u_j} + \frac{p_k p_l}{u_k u_l}\right)$$

and similarly for other entries. Substitution of $(\dagger)$ yields $\ell_{\mathcal{E}}(p) = 0$, while substitution of $(\ddagger)$ outside $(\dagger)$ yields $\operatorname{rank}\big(\ell_{\mathcal{E}}(p)\big) = 1$.

Proof of involution: A. Projecitve modules

If an involutive PDE system $\mathcal{E}$ is linear, its symbolic module $\mathcal{M}_{\mathcal{E}} = g^*$ over the ring $\mathcal{R} = ST$ is projective (locally free).

Let $\mathcal{E}$ be defined by a k-th order differential operator $\Delta : \Gamma(\pi) \to \Gamma(\nu)$, corresponding to morphisms $\psi^{\Delta}_{k+i} : J^{k+i}\pi \to J^i\nu$, i.e. $\mathcal{E}_{k+i} = \mathrm{Ker}(\psi^{\Delta}_{k+i})$ for $i \geq 0$.

We construct a minimal free resolution of the symbolic module:

$$\cdots \to \mathcal{R} \otimes \varpi^* \xrightarrow{\psi^*} \mathcal{R} \otimes \nu^* \xrightarrow{\sigma^*_\Delta} \mathcal{R} \otimes \pi^* \longrightarrow \mathcal{M}_{\mathcal{E}} \to 0$$

and applying the functor $* = \mathrm{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$ get the exact sequence

$$0 \to g \hookrightarrow ST^* \otimes \pi \xrightarrow{\sigma_\Delta} ST^* \otimes \nu \xrightarrow{\psi} ST^* \otimes \varpi \to \dots$$

from which the compatibility conditions of $\mathcal{E} = \{\Delta = 0\}$ are $\Psi \circ \Delta|_{\mathcal{E}} = 0$ for $\Psi \in \mathrm{Diff}(\nu, \varpi)$ with the symbol ψ at x.

For nonlinear equations, apply the linearisation operator on a solution instead of Δ. Its symbol yields a syzygy, and hence compatibility operators, and Ψ is an operator in total derivatives.

Proof of involution: B. Explicit differential syzygies

The symbol $\ell_{\mathcal{E}}$ of the nonlinear vector-operator defining $\mathcal{E}$ in new coordinates $\xi_i = \frac{p_i}{u_i}$ on T_x^*M has components

$$\ell^u_{E_{ijkl}}(\xi) = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\big(\xi_i\xi_j + \xi_k\xi_l\big),$$

$$\ell^v_{E_{ijkl}}(\xi) = - \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\big((a_i + a_j)\xi_i\xi_j + (a_k + a_l)\xi_k\xi_l\big),$$

$$\ell^u_{F_{ijkl}}(\xi) = - \underset{(jkl)}{\mathfrak{S}} \frac{(a_i - a_j)(a_k - a_l)}{a_i a_j a_k a_l}\big((a_i + a_j)\xi_i\xi_j + (a_k + a_l)\xi_k\xi_l\big),$$

$$\ell^v_{F_{ijkl}}(\xi) = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\Big(\frac{\xi_i\xi_j}{a_k a_l} + \frac{\xi_k\xi_l}{a_i a_j}\Big),$$

in the basis e_u, e_v of $\mathcal{R}^2$ and basis $e_{E_{ijkl}}, e_{F_{ijkl}}$ of $\mathcal{R}^{2\binom{n-2}{2}}$, where we restrict to indices $i = 1, j = 2$, $2 < k < l \leq n$.

This means that the homomorphism $\ell_{\mathcal{E}}$ maps $f(\xi)e_u$ to $f(\xi)\sum_{k<l}(\ell^u_{E_{12kl}}(\xi)e_{E_{12kl}} + \ell^u_{F_{12kl}}(\xi)e_{F_{12kl}})$ and similarly for $h(\xi)e_v$.

Proof of involution: B — cont'd

Now we resolve $\ell_{\mathcal{E}}$ by a homomorphism $\mathcal{C} = \mathcal{C}_{\mathcal{E}}$. The image $\mathcal{C}(\xi)(w)$ for $w = \sum_{i<j}(w_{E_{12ij}} e_{E_{12ij}} + w_{F_{12ij}} e_{F_{12ij}})$ has the following components $(2 < i < j < k \leq n)$:

$$\mathcal{C}^{I}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \big((a_2 - a_k)\xi_1 + (a_k - a_1)\xi_2 + (a_1 - a_2)\xi_k\big) w_{E_{12ij}}$$

$$\begin{aligned}\mathcal{C}^{II}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \Big[&\big((a_1 - a_2)(a_2 - a_k)a_1\xi_1 + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 \\ &+ ((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2)\xi_k\big) w_{E_{12ij}} \\ &+ 2a_1 a_2 a_i a_j (a_1 - a_k)(a_2 - a_k)\xi_k w_{F_{12ij}}\Big]\end{aligned}$$

$$\begin{aligned}\mathcal{C}^{III}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \Big[&2(a_1 - a_k)(a_2 - a_k)\xi_k w_{E_{12ij}} \\ &+ \big((a_1 - a_2)(a_2 - a_k)a_1\xi_1 + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 \\ &+ ((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2)\xi_k\big) a_i a_j w_{F_{12ij}}\Big]\end{aligned}$$

$$\mathcal{C}^{IV}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \big((a_2 - a_k)a_1^2\xi_1 + (a_k - a_1)a_2^2\xi_2 + (a_1 - a_2)a_k^2\xi_k\big) a_i a_j w_{F_{12ij}}$$

Proof of involution: B — finish'd

One verifies that with these homomorphisms the following sequence is exact:

$$\mathcal{R}^2 \xrightarrow{\ell_\mathcal{E}} \mathcal{R}^{2\binom{n-2}{2}} \xrightarrow{\mathcal{C}_\mathcal{E}} \mathcal{R}^{4\binom{n-2}{3}}.$$

In other words, $\mathcal{C}_\mathcal{E}$ is the first syzygy for the module

$$\mathcal{M}_\mathcal{E}^\star = \mathrm{Ker}(\ell_\mathcal{E}) = \mathrm{Hom}_\mathcal{R}(\mathcal{M}_\mathcal{E}, \mathcal{R}).$$

Therefore, the differential syzygies for $\mathcal{E}$ are enumerated by 5 different indices $(12ijk)$, $2 < i < j < k \leq n$.

Consequently to verify compatibility conditions for each of these 5-tuples one can work in the corresponding 5-dimensional space, which is verified straightforwardly. And this yields involutivity.

Compatibility via free resolution

For a homomorphism $\varphi : \mathcal{R}^{n-2} \to \mathcal{R}^2$ the following sequence is known as the Eagon-Northcott complex ($\star = \mathrm{Hom}_{\mathcal{R}}(\cdot, \mathcal{R})$)

$$\cdots \to S^3\mathcal{R}^{\star 2} \otimes \Lambda^5\mathcal{R}^{n-2} \xrightarrow{\partial} S^2\mathcal{R}^{\star 2} \otimes \Lambda^4\mathcal{R}^{n-2} \xrightarrow{\partial} \mathcal{R}^{\star 2} \otimes \Lambda^3\mathcal{R}^{n-2} \xrightarrow{\partial} \Lambda^2\mathcal{R}^{n-2} \xrightarrow{\epsilon} \mathcal{R}.$$

It is exact when the Fitting ideal $I(\varphi)$, generated by 2×2 determinants of φ, contains a regular sequence of length $(n-3)$.

For the system $\mathcal{E}$ the map $\ell_{\mathcal{E}}$ split: $\ell_{\mathcal{E}}(e_u)$ and $\ell_{\mathcal{E}}(e_v)$ generate two complementary submodules $\Lambda^2\mathcal{R}^{n-2} \subset \mathcal{R}^{2\binom{n-2}{2}}$. Therefore two copies of the $\star$-dual Eagon-Northcott complex yield the following resolution of the $\star$-dual symbolic module:

$$0 \to \mathcal{M}_{\mathcal{E}}^{\star} \to \mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^2 \otimes \Lambda^2\mathcal{R}^{n-2} \xrightarrow{\mathcal{C}_{\mathcal{E}}} \mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^3\mathcal{R}^{n-2} \xrightarrow{\partial^\star} S^2\mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^4\mathcal{R}^{n-2} \to \dots$$

The Fitting condition corresponds to codimension $n-3$ of the zero set of $I(\ell_{\mathcal{E}})$ is the tangential variety to the rational normal curve.

That only 5-tuples of distinct indices enter the compatibility conditions we read off $\Lambda^3\mathcal{R}^{n-2}$: triples (ijk) yield 5-tuples $(12ijk)$.

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