Topics in string geometry:

generalised complex geometry, M5-branes & anomalies

Generalised complex geometry and sting theory

- ⋆ supergravity
- ⋆ and beyond

$$\Longleftrightarrow$$

M-theory and M5-branes

$$\Longleftrightarrow$$

Anomalies

Generalised geometry:

- is not ... "geometry generalised", but "geometry of generalised structures"
- *generalised structures* are defined on $TM \oplus \Lambda^\bullet T^*M$
  - ▷ EX: $TM \oplus \Lambda^0 T^*M$

    $$[v,w]_{\text{Lie}} \quad \mapsto \quad [v+f, w+g] = [v,w]_{\text{Lie}} + \mathcal{L}_v g - \mathcal{L}_w f + \underbrace{\imath_{v_M} \imath_{w_M} F}_{\text{twisting}}$$

  - ▷ new brackets
  - ▷ $\Lambda^n T^*M \quad \Leftrightarrow \quad (n+1)$-forms
  - ▷ new automorphisms ($F = \mathrm{d}A$, shifts by closed A)
  - ▷ extensions of the structure group (string/sugra dualities)
- $TM \oplus T^*M \quad \Rightarrow \quad$ *generalised complex geometry*
  - ▷ new integrable structures
  - ▷ for dim$(M) = 2k$ interpolation between complex and symplectic structures
  - ▷ new automorphisms $\quad \Rightarrow \quad$ $B-$field

B-field and generalised geometry for type II strings

Generalised complex structure (GCG)

- GCG $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$ $\quad$ ($\mathcal{J}^2 = -1; \quad \mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I}$)
  - ⋄ Structure group: $\quad \Rightarrow \quad U(\frac{d}{2}, \frac{d}{2})$
- GCS integrable: $\pi_+[\pi_-(v), \pi_-(w)]_{\rm Lie} = 0 \;\mapsto\; \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$ with Courant bracket:

$$[v + \xi,\, w + \eta] = [v, w]_{\rm Lie} + \Big\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \tfrac{1}{2}\, {\rm d}(\imath_v \eta - \imath_w \xi) \Big\}$$

  (Courant closes on $L_{\mathcal{J}}$ – the i-eigenbundles of $\mathcal{J}$ ($\Pi L_{\mathcal{J}} = L_{\mathcal{J}}$))

- Closed B-transform $(v_1, \rho_1) \mapsto e^B(v_1, \rho_1) = (v_1, \rho_1 + \imath_{v_1} B)$ is an automorphism of Courant : $[e^B(v_1, \rho_1), e^B(v_2, \rho_2)] \mapsto e^B[(v_1, \rho_1), (v_2, \rho_2)]$
- Twisting: $\quad {\rm d} \mapsto {\rm d} - H\wedge, \qquad [.,.]_C \mapsto [.,.]_C + \underline{\imath_v \imath_w H}$
- Two compatible GCS's: $[\mathcal{J}_1, \mathcal{J}_2] = 0$ ($U(d) \times U(d)$ structure) $\;\Rightarrow\;$ gen. metric (g, B)

Generalised tangent bundle :

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0\,,$$

Sections of E:

$$X = \begin{pmatrix} v \\ \xi \end{pmatrix} \qquad \longmapsto \qquad X' = e^{-B} X = \begin{pmatrix} \mathbb{I} & 0 \\ -B & \mathbb{I} \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ \xi - \imath_v B \end{pmatrix}\,.$$

- Note $(v,\xi) \to (v, \xi - \imath_v \mathrm{d}\Lambda)$
- Courant on $E \quad \to \quad$ *twisted* Courant on $T \oplus T^*$
- Lie bracket $\mapsto$ Courant $\Rightarrow$ generalised connection
- Courant does not define an unambiguous gen. Riemman but ...unique
  $\hat{R}_{ab} = R_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab})$ and
  $\hat{R} = R + 4\nabla^2 \phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 \qquad \leftrightarrow \qquad$ LBT: $S^{\mathrm{NS}}\epsilon = 4\left(D^a D_a - D^2\right)\epsilon$

▷ The dynamics and supersymmetry transformations of type II supergravity theories are captured by a (torsion-free) generalised connection

$\Longleftrightarrow$

LBT - Lichnerowicz-Bismut theorem

- Lichnerowicz theorem:

  $\left(\nabla^a\nabla_a - \nabla\!\!\!/^2\right)\epsilon = \frac{1}{4}R$ tensorial action!

  - ◇ ∇_a - Levi-Civita connection (no torsion)
  - ◇ $\nabla\!\!\!/ = \gamma^a\nabla_a$ - Dirac operator

- Torsion H ($\mathrm{d}H = 0$)

  $$\begin{cases} D_a\epsilon = \nabla_a\epsilon - \alpha H_{abc}\gamma^{bc}\epsilon \\ D\!\!\!/\epsilon = \left(\gamma^a\nabla_a - \beta H_{abc}\gamma^{abc}\right)\epsilon \end{cases}$$

  $\leftarrow$ torsionful Dirac operator

  - ◇ Note $D\!\!\!/ \neq \gamma^a D_a = \gamma^a\nabla_a - \alpha H_{abc}\gamma^{abc}$

- LBT: $\left(D^a D_a - D\!\!\!/^2\right)\epsilon$ **tensorial**

  $$\left(D^a D_a - D\!\!\!/^2\right)\epsilon = \frac{1}{4}\left[R - \#H^2\right]\epsilon + \gamma^{abcd}\nabla_a H_{bcd}\epsilon + (\alpha - 3\beta)\gamma^{ab}\nabla^c H_{abc}\epsilon + (\alpha - 3\beta)\gamma^{ab}H_{abc}\left(\nabla^c\epsilon\right)$$

- $\alpha = 3\beta$ ($\alpha = \frac{1}{8}$ for normalisation: $\frac{1}{12}H^2$)

Applications for string compactifications

Integrability of GCS (GCY) and Pure spinors

- i-eigenbundle of $\mathcal{J}$ $L_{\mathcal{J}}$ - max. isotropic (null space of max. dimension $\frac{d}{2}$)
- Spinor bundle $S \sim \Lambda^\bullet T^*$:
  - ◇ Clifford action on a spinor Φ: $(v+\zeta)\cdot\Phi = v^m \imath_{\partial_m}\Phi + \zeta_m \mathrm{d}x^m \wedge \Phi$
  - ◇ $L_\Phi = \{v+\zeta \in T\oplus T^* \,|\, (v+\zeta)\cdot\Phi = 0\}$ is isotropic
    $(((v+\zeta)\cdot(v+\zeta)\Phi) = -(v+\zeta, v+\zeta)\Phi)$
  - ◇ If L_Φ of max. dimension – Φ - pure spinor
  - ◇ If $L_{\mathcal{J}} = L_\Phi \quad \Rightarrow \quad \mathcal{J} \leftrightarrow$ line of pure spinors
- For $A, B \in L_\Phi$, $\quad [A,B]_{\mathrm{C}}\Phi = (AB - BA)\cdot \mathrm{d}\Phi$
  - ◇ $\mathrm{d}\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J}$ -integrable
  - ◇ $\mathrm{d}\Phi = 0$ ($[A,B]_{\mathrm{C}} \in L_\Phi = L_{\mathcal{J}}$) – GCY condition

Dilaton ϕ together with g and B needed to define ismorphism between pure spinors and forms: $\quad S^\pm(E) \simeq (\det T^*M)^{-1/2} \otimes \Lambda^{\text{even/odd}} T^*M$

Supersymmetric flux backgrounds (type II):

10D string theory with

- the metric: $ds^2_{10} = e^{2A(y)} \mathrm{d}s^2_4(M_4) + \mathrm{d}s^2_6$
- the fluxes :

$$\begin{cases} \text{NS 3-form :} \quad \mathrm{d}H = 0 \\ \text{RR (even/odd - IIA/B) :} \quad F^{(10)} = F + \mathrm{vol}_4 \wedge \lambda(*F) \qquad (\lambda(F_n) = (-1)^{\mathrm{Int}[n/2]} F_n) \end{cases}$$

Equations of motion $\Leftrightarrow$

- susy $\Leftrightarrow$ pure spinor equations

$$\mathrm{d}\left(e^{3A}\Phi_1\right) = 0 \qquad \Leftrightarrow \qquad \text{Gen. CY structure}$$

$$\mathrm{d}\left(e^{2A}\mathrm{Re}\Phi_2\right) = 0$$

$$\mathrm{d}\left(e^{4A}\mathrm{Im}\Phi_2\right) = e^{4A} e^{-B} * \lambda(F)$$

$$\langle \Phi_1, \gamma\Phi_2 \rangle = 0$$

- Bianchi identities

$$(\mathrm{d} - H\wedge)F = \delta(\mathrm{source})$$

Beyond GCG: why and how?

$\Longleftrightarrow$

From quantum gravity point of view, probing string theory at fundamental level

$\Rightarrow$ study of stringy corrections:

- $\star$ α' expansion $\Rightarrow$ higher derivative corrections to the supergravity
- $\star$ the genus expansion $\Rightarrow$ string quantum corrections in spacetime

$\Longleftrightarrow$

Generalised geometry & type II supergravity theories (vacua, symmetries....)

- Generalised complex geometry for heterotic strings (at order α')
  - $\triangleright$ first look at quantum corrections
- corrections for type II theories
  - $\triangleright$ developing GCG
  - $\triangleright$ (missing) string calculations
  - $\triangleright$ geometry for M-theory

Generalised geometry for heterotic strings

The Heterotic Bianchi Identity :

$$\mathrm{d}H_3 = \frac{\alpha'}{4}(\mathrm{tr}\, R^2(\Omega_+) - \mathrm{tr}\, F^2) + \mathcal{O}(\alpha')$$

where

$$R(\Omega_\pm) = R(\Omega^{\mathrm{LC}} \pm \frac{1}{2}\mathcal{H}) = R(\Omega) \pm \tfrac{1}{2} d\mathcal{H} + \tfrac{1}{4}\mathcal{H} \wedge \mathcal{H}, \qquad \mathcal{H}^{ab} = H_\mu{}^{ab} dx^\mu$$

- Choice of connection in heterotic BI
  - ▷ tied to a choice of field redefinitions in the higher order curvature corrections to supergravity
  - ▷ manifest (0,1) world-sheet supersymmetry: covariant Hermitian space-time metric $G \quad \leftrightarrow \quad \Omega_+ = \Omega^{\mathrm{LC}} + \frac{1}{2}\mathcal{H}$
    - ▷ (0,2) world-sheet SUSY likes Chern connection, but ... that requires G to pick up non-trivial space-time Lorentz and gauge transformations (and α' shifts in susy transformations)
  - ▷ (Narain) T-duality
  - ▷ Two types of problems associated with $\mathrm{d}H \neq 0$

Generalised geometry with a non-closed H?

- ... start with $\mathrm{d}H = 0$ and

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_{10} \\ & & \downarrow \pi \\ & & X_9 \end{array}$$

$\mathcal{L}_v g_{10} = 0 = \mathcal{L}_v H$: $\qquad \mathrm{d}e = \pi^* T \qquad (\mathcal{L}_v\, e = 0)$

$$H = \pi^* h_3 + \pi^* \tilde{T} \wedge e$$

$$\mathrm{d}H = 0 \qquad \Leftrightarrow \qquad \begin{cases} \bullet \quad \mathrm{d}h_3 = \tilde{T} \wedge T = \frac{1}{4}\left[(T_+)^2 - (T_-)^2\right] \\ \bullet \quad \mathrm{d}\tilde{T} = 0 \end{cases}$$

- $T_\pm = T \pm \tilde{T}$
- Locally $H = \mathrm{d}B = \mathrm{d}(b_2 + b_1 \wedge e) \qquad \Rightarrow \qquad \tilde{T} = \mathrm{d}b_1$ and $h_3 = \mathrm{d}b_2 - b_1 \wedge T$
- Note $b_2 \ \rightarrow \ b_2 + \mathrm{d}\lambda_1 + \lambda_0 T$, $b_1 \ \rightarrow \ b_1 + \mathrm{d}\lambda_0$ and h_3 is gauge invariant

◇ h_3 - invariant under T-duality (b_2 is not!)

- α' corrections to geometry
- HS ($\mathrm{d}H \neq 0$): $\Rightarrow$ $B \to B + \mathrm{d}\Lambda + \frac{\alpha'}{4}\mathrm{d}^{-1}\delta\mathrm{d}^{-1}X_4(\Omega_+, A)$

$\Bigg\}$ $\leftarrow$ Extensions of GTB

$X_4(\Omega_+, A) =$ tr $R^2(\Omega_+) -$ tr F^2 - two types of problems

Two simple ideas:

- Find an extended gen. tangent space with a closed 3-form! (*Global data*)

◇ Generalise $U(1)$ fibration $S^1 \hookrightarrow X \xrightarrow{\pi} M$ case:

▷ $\mathrm{d}H = 0$ on X $\Rightarrow$ $\mathrm{d}h_3 = \tilde{T} \wedge T = \frac{1}{4}\left[(T_+)^2 - (T_-)^2\right]$ on M (h_3 - invariant !)

Generalised heterotic tangent space is built as a double fibration:

$$0 \longrightarrow \mathfrak{g} \longrightarrow C \longrightarrow TM \longrightarrow 0\,,$$

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow C \longrightarrow 0$$

◇ Locally $E \simeq TM \oplus T^*M \oplus \mathrm{ad}P_G$ ($\mathrm{ad}P_G$ - the adjoint bundle with fibres in $\mathfrak{g}$ of G)

◇ $\langle V, W \rangle = \frac{1}{2}\iota_v\rho + \frac{1}{2}\iota_w\lambda + \mathrm{tr}\,\Lambda\Sigma$ for $V = v + \lambda + \Lambda$ and $W = w + \rho + \Sigma$

◇ $[\![V, W]\!] = [v, w] + \mathcal{L}_v\rho - \mathcal{L}_w\lambda - \frac{1}{2}\mathrm{d}(\iota_v\rho - \iota_w\lambda) + \mathcal{L}_v\Sigma - \mathcal{L}_w\Lambda + [\Lambda, \Sigma] + \mathrm{tr}\,(\Sigma\mathrm{d}\Lambda - \Lambda\mathrm{d}\Sigma)$

◇ ... $\mathrm{d}H \sim$ tr F^2 obstruction: $p_1(\mathfrak{g}) = 0$

gen. Lichnerowicz theorem (LBT) $\Rightarrow$ effective actions (*Local data*)

- (gen.) Lichnerowicz theorem: $\left(D^A D_A - D^2\right)\epsilon = \left[\frac{1}{4}S + \gamma^{abcd} I_{abcd}\right]\epsilon$ (S tensorial!)

◇ Heterotic effective action: $S = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2 - \frac{\alpha'}{4}\mathrm{tr}\,\hat{\mathcal{F}}^2$

◇ $I_{abcd} = \frac{1}{6}\nabla_{[a}H_{bcd]} - \frac{\alpha'}{8}\mathrm{tr}\,\hat{\mathcal{F}}_{[ab}\hat{\mathcal{F}}_{cd]} = 0$

◇ $\left.\begin{aligned} \delta\psi_a &= D_a\epsilon = \nabla_a\epsilon - \frac{1}{8}H_{abc}\gamma^{bc}\epsilon \\ \delta\zeta_\alpha &= D_\alpha\epsilon = -\frac{1}{8}\sqrt{2\alpha'}\hat{\mathcal{F}}_{ab\alpha}\gamma^{ab}\epsilon \end{aligned}\right\}$ $\leftarrow$ covariant derivative ($A = \{a, \alpha\}$)

◇ $\delta\lambda = D\epsilon = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon$ $\leftarrow$ Dirac operator

Gravitational terms (obstruction to E) ?

◇ picking a substructure of the GFB ($O(d)\times G\times O(d)$ PB) splits $E = \tilde{C}_+ \oplus \tilde{C}_{\mathfrak{g}} \oplus \tilde{C}_-$

◇ take $G \to G_{\mathrm{gauge}} \times O(d)$....

◇ reduce the structure group of E to $O(d)\times G\times O(d) \subset O(d+\dim(\mathfrak{g}))\times O(d)$

◇ Identify $O(d) \in G$ with $O(d)$ in $\tilde{C}_+$

▷ Works only for $\hat{\mathcal{A}} = \Omega_+ = \omega^{\mathrm{LC}} + \frac{1}{2}\mathcal{H}$!!! (cf susy for Ω_-!!!)

▷ For type II $G \to O(d)\times O(d)$ does **NOT** work

- ▷ Flip of the sign in $\mathcal{O}(\alpha')$ effective action wrt D_a : $\Omega_- \longrightarrow \Omega_+$!!!

◇ $$R_{mnpq}(\Omega^-) - R_{pqmn}(\Omega^+) = -12\mathrm{d}H_{mnpq}$$

- • leading to corrections all orders in α':
  - ▷ "gaugino" $\psi_{ab} \in \Gamma(\Lambda^2 C_+ \otimes S(C_-))$ for "gauge group" $O(d)_+$
    - ◇ $$\delta\psi_{O(d)ab} = \frac{1}{8}\sqrt{\alpha'} R(\Omega^+)_{\bar{a}\bar{b}ab}\gamma^{\bar{a}\bar{b}}\epsilon \,.... = D_{ab}\epsilon \ \ (?)$$
  - ▷ ψ_{ab} - *composite* "gravitino curvature"
    - ◇ $\delta\psi_{ab} = D_{ab}\epsilon + \frac{1}{8}\sqrt{\alpha'}\left(\frac{1}{8}\alpha'[\mathrm{tr}\, F\wedge F - \mathrm{tr}\, R(\Omega^+)\wedge R(\Omega^+)]_{ab\bar{a}\bar{b}}\right)\gamma^{\bar{a}\bar{b}}\epsilon \ \rightarrow \ \hat{D}_{ab}\epsilon$
  - ▷ $D_{ab} \ \rightarrow \ \hat{D}_{ab}$ in LBT $\Rightarrow$ $\mathcal{O}(\alpha'^2)$ modifications of susy for $\gamma^{\bar{a}}\hat{D}_{\bar{a}}\gamma^{\bar{b}}\hat{D}_{\bar{b}}\epsilon - \hat{D}^a\hat{D}_a\epsilon + \hat{D}^\alpha\hat{D}_\alpha\epsilon + \hat{D}^{ab}\hat{D}_{ab}\epsilon = -\frac{1}{4}S^-\epsilon + \gamma^{abcd}I_{abcd}\epsilon$
  - ▷ hierarchy of higher α' corrections (consistent with GCG)
  - ▷ $\mathcal{O}(\alpha'^3)$ agreement with literature
  - ▷ new $\mathcal{O}(\alpha'^4)$ corrections
  - ▷ iterative all order formulae ?

▷ "Tensoriality" without generalised geometry:

$$\not{D}\not{D}\epsilon - D_M D^M \epsilon - \frac{\alpha'}{64}\left(\text{tr } \not{F}\not{F}\epsilon - \text{tr } \not{R}^+\not{R}^+\epsilon\right) + 2\nabla^M \phi D_M \epsilon = -\frac{1}{4}\mathcal{L}_b \epsilon + \mathcal{O}(\alpha'^2)$$

(mod. heterotic BI: $\mathrm{d}H = \frac{\alpha'}{4}(\text{tr } F \wedge F - \text{tr } R^+ \wedge R^+)$)

▷ $\mathcal{L}_b$ - bosonic Lagrangian

▷ Multiply by $e^{-2\phi}\epsilon^\dagger$ and integrate by parts ($\epsilon^\dagger \epsilon = 1$):

$$\frac{1}{4}\mathcal{L}_b = (\not{D}\epsilon)^\dagger \not{D}\epsilon - (D_M \epsilon)^\dagger D^M \epsilon + \frac{\alpha'}{64}\left(\text{tr } \epsilon^\dagger \not{F}\not{F}\epsilon - \text{tr } \epsilon^\dagger \not{R}^+\not{R}^+\epsilon\right) + \mathcal{O}(\alpha'^2)$$

▷ The (bosonic) action

$$S_b = \int_{M_{10}} e^{-2\phi}\mathcal{L}_b = BPS^2$$

▷ Susy + BI $\Rightarrow$ solutions .... alternative to Gen. Ricci computation:

$$\Gamma^M D^-_{[N} D^-_{M]}\epsilon - \frac{1}{2}D^-_N(\mathcal{O}\epsilon) + \frac{1}{2}\mathcal{O}D^-_N\epsilon = -\frac{1}{4}\mathcal{E}_{NM}\Gamma^M\epsilon + \frac{1}{8}\mathcal{B}_{NM}\Gamma^M\epsilon + \frac{1}{48}\mathrm{d}H_{NMPQ}\Gamma^{MPQ}\epsilon$$

$\mathcal{O} = \not{\partial}\phi - \frac{1}{12}\not{H}$ and $\mathcal{E}^0_{NM}, \mathcal{B}^0_{NM}$ - EOMs for metric and B-field

M-theory & GCG

- Fields: $\{g_{mn}, C_{mnp}, \psi_m\}$
  - ◇ $S_{\mathsf{B}} = \frac{1}{2\kappa^2}\int\left(\sqrt{-g}\,R - \frac{1}{2}G\wedge *G - \frac{1}{6}C\wedge G\wedge G\right)$
  - ◇ $S_{\mathsf{F}} = \frac{1}{\kappa^2}\int\sqrt{-g}\Big(\bar{\psi}_m\gamma^{mnp}\nabla_n\psi_p + G_{p_1\dots p_4}\left(\frac{1}{96}\bar{\psi}_m\gamma^{mp_1\dots p_4 n}\psi_n + \frac{1}{8}\bar{\psi}^{p_1}\gamma^{p_2p_3}\psi^{p_4}\right)\Big)$
    - ▷ susy $\quad \delta\psi_m = \nabla_m\varepsilon + \frac{1}{288}\left(\gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2n_3n_4}\right)G_{n1\dots n_4}\varepsilon = D_m\varepsilon$
    - ▷ eom $\quad \gamma^{mnp}\nabla_n\psi_p + \frac{1}{96}\left(\gamma^{mnp_1\dots p_4}G_{p_1\dots p_4} + 12G^{mn}{}_{p_1p_2}\gamma^{p_1p_2}\right)\psi_n = 0 = L^{mn}\psi_n$

- exact sequence: $\quad S \xrightarrow{D} V\otimes S \xrightarrow{L} V\otimes S \xrightarrow{D^\dagger} S$
  - ▷ $L\circ D = 0$ follows from supersymmetry
  - ▷ reality of $\int\bar{\psi}_a L^{ab}\psi_b \Rightarrow L = -L^\dagger \Rightarrow D^\dagger\circ L = 0.$

- Lichnerowicz-type relation $\rho = \tilde{D}^a D_a\varepsilon = \gamma^{ab}D_aD_b\varepsilon \propto$ (trace of Einstein $+$ 8-form)
  - ▷ $L^{ab}\psi_b = \gamma^{abc}D_b\psi_c$ and $\tilde{D}^c = \frac{1}{9}\gamma_a L^{ac} = \gamma^{bc}D_b$
  - ▷ coefs in D_a and $\tilde{D}^a$ are *uniquely* fixed by tensoriality of rhs!

- LT $\Rightarrow \quad S_{\mathsf{B}} = \langle 1 + C, \rho\rangle = \int\sqrt{-g}(\mathcal{R} - \frac{1}{3}\frac{1}{48}G_{b_1\dots b_4}G^{b_1\dots b_4}) - \frac{1}{3}C\wedge(\mathrm{d}*G + \frac{1}{2}G\wedge G)$

Higher order (type IIA 1-loop) terms

$$D : S \to T^* \otimes S$$

$$(D\varepsilon)_a = \nabla_a \varepsilon + \alpha \left(\nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon + \beta X_{abcd} \gamma^{cd} \nabla^b \varepsilon$$

$$\tilde{D} : T^* \otimes S \to S$$

$$(\tilde{D}\psi) = \gamma^{ab} \left(\nabla_a \psi_b + \tilde{\alpha} \left(\nabla^c X_{acef}\right) \gamma^{ef} \psi_b + \tilde{\beta} X_{acef} \gamma^{ef} \nabla^c \psi_b\right)$$

where $X_{abcd} \in [0,2,0,0,0]$ and α, β etc. parametrise higher-order corrections

$$(\tilde{D}D\varepsilon) = \gamma^{ab} \nabla_a \nabla_b \varepsilon + (\alpha - \tilde{\alpha} - \beta) \left(\nabla^a X_{abcd}\right) \gamma^{cd} \nabla^b \varepsilon$$

$$- \alpha \left(\nabla^a \nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon - \left(\tilde{\beta} + \beta\right) X_{abcd} \gamma^{cd} \nabla^a \nabla^b \varepsilon + \ldots$$

$$(\text{if } \alpha - \tilde{\alpha} - \beta = 0) = -\tfrac{1}{4} \mathcal{R} \varepsilon + \tfrac{1}{2} \left(2\alpha - \tilde{\beta} - \beta\right) R^{abe}{}_{c} X_{abed} \gamma^{cd} \varepsilon$$

$$+ \tfrac{1}{4} \left(\tilde{\beta} + \beta\right) R^{abcd} X_{abcd} \varepsilon - \tfrac{1}{8} \left(\tilde{\beta} + \beta\right) R^{ab}{}_{cd} X_{abef} \gamma^{cdef} \varepsilon + \ldots$$

Ambiguities:

- ▷ $\alpha = \tilde{\alpha} = \beta = 0$ consistent: keeping susy classical and only correcting S_{F}
- ▷ $\tilde{\alpha} = 0, \alpha = \beta = \tilde{\beta}$: $\tilde{D}^a \nabla_a \varepsilon$ and $\gamma^{ab} \nabla_a D_b \varepsilon$ are separately tensorial (the fermionic action is in terms of "supercovariant" objects)

Everything that can modify susy:

Projection of R^3	Rep of $so(10,1)$	Multiplicity	multiplicity of embeddings in $\delta\psi$	of which result in $[\nabla,\nabla]R^3$	projected into form of rank
X^i	[0,2,0,0,0]	8	1	1	2
W^i	[2,0,0,0,0]	3	3	1	2
S^i	[0,0,0,0,0]	2	1	1	2
Y^i	[0,1,0,0,2]	2	1	1	6
V^i	[1,0,0,0,2]	2	4	2	$4,6$
T^i	[0,0,0,1,0]	3	5	3	$2,4,6$
Z^i	[0,1,0,1,0]	3	1	1	4
U^i	[1,0,1,0,0]	3	4	2	$2,4$
L^i	[2,1,0,0,0]	3	1	0	-
M^i	[2,0,0,1,0]	6	1	0	-

The last two lead to symmetrised ∇ and so are immediately ruled out. The other terms all admit at least one combination which corresponds to $R^3[\nabla,\nabla]\varepsilon$ in the Lichnerowicz, which thus give rise to R^4 terms.

These R^4 will appear as p-forms contracted with gamma-matrices acting on the spinor. The different terms contribute to different forms as follows:

p-form :	0	1	2	3	4	5
R^4 multiplicity:	7	0	1	2	17	0
$X^i \otimes R$	●	-	●	-	●	-
$W^i \otimes R$	-	-	-	-	-	-
$S^i \otimes R$	-	-	-	-	-	-
$Y^i \otimes R$	-	-	-	●	●	-
$V^i \otimes R$	-	-	-	-	●	-
$T^i \otimes R$	-	-	-	-	●	-
$Z^i \otimes R$	-	-	●	-	●	-
$U^i \otimes R$	-	-	●	-	●	-

!!! Nothing is possible at R^2 and R^3 order !!!

▷ Tensoriality of $\tilde{D}D$ + cancellation of 2,4,6-forms yields 2 independent invariants

◇ $x\Big(\left(t_8 t_8 - \frac{1}{8}\epsilon\epsilon\right) R^4 + \frac{1}{2}\epsilon t_8 C R^4 \Big)$

◇ $y\left(t_8 t_8 + \frac{1}{8}\epsilon\epsilon\right) R^4$ $\qquad$ ($t_8 M^4 = 24\left(\text{tr}\, M^4 - \frac{1}{4}(\text{tr}\, M^2)^2\right)$)

▷ what fixes $y = 0$?

◇ fermion terms and “actual” supersymmetry

◇ inclusion of G - **nonlinear** completion of gravity results

◇ next order $\sim R^7$ (lift from 2-loop string terms)

Will quantum corrections be a key to the (generalised) geometry of M-theory?

(How much) can generalised geometry capture the systematics of string expansion?

Plenty of open questions....

▷ Special couplings: $\qquad [C \wedge G \wedge G + C \wedge [\frac{1}{4}p_1^2(TM) - p_2(TM)]]$

MANY LIVES OF X_8 : susy, (different!) anomalies, ... , spin geometry

Euler density:

▷ $\chi = \frac{1}{4!(4\pi)^2}\ \epsilon^{a_1\cdots a_8} R_{a_1a_2} R_{a_3a_4} R_{a_5a_6} R_{a_7a_8}$

◇ Curvature two-from $\quad R_{ab} = \frac{1}{2} R_{abcd}\, e^c \wedge e^d$

Spinor density:

▷ $\hat{\chi} = \frac{1}{4!(4\pi)^2} \cdot \frac{1}{2^4}$
$\epsilon^{i_1\cdots i_8} R_{a_1a_2} \left(\Gamma^{a_1a_2}\right)^{i_1i_2} R_{a_3a_4} \left(\Gamma^{a_3a_4}\right)^{i_3i_4} R_{a_5a_6} \left(\Gamma^{a_5a_6}\right)^{i_5i_6} R_{a_7a_8} \left(\Gamma^{a_7a_8}\right)^{i_7i_8}$

Eight-forms:

▷ $\hat{\chi} = \frac{1}{16}\left(8\chi + p_1(TM)^2 - 4p_2(TM)\right)$

$$
\begin{aligned}
X_8 &= \frac{1}{48}\left(\frac{1}{4}p_1(TM)^2 - p_2(TM)\right) \\
&= \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\mathrm{tr}\, R^2)^2 + \frac{1}{192}\mathrm{tr}\, R^4\right)
\end{aligned}
$$

Supersymmetric M-theory backgrounds:

11D theory on eight-manifolds

- the metric: $ds_{10}^2 = e^{2A(y)} \mathrm{d}s_4^2(M_3) + \mathrm{d}s_8^2$ (M_8 can have holonomy $\subset Spin(7)$)
- the flux components :

$$\begin{cases} G_4^{\text{ext}} \sim \star_3 \mathrm{d}A \\ G_4^{\text{int}} \sim G \end{cases}$$

(for e.g. $SU(4)$ holonomy - G is self-dual primitive)

G_4 EOM: $\mathrm{d} * G_4 \sim G_4 \wedge G_4 + X_8 \Rightarrow$

$$\mathrm{d} *_8 \mathrm{d}A = *_8 |G|^2 + X_8 \quad \Rightarrow \quad \int |G|^2 \sim \chi(M_8)$$

hence $\Leftrightarrow$
- X_8 allows Mink$_4$ compactifications with fluxes
- Ricci-flat 8-manifols without flux are **NOT** solutions of M-theory!

A very different situation for 11D theory on seven-manifolds

- the metric: $ds^2_{10} = e^{2A(y)} \mathrm{d}s^2_4(M_4) + \mathrm{d}s^2_7$
- the flux components :

$$\begin{cases} G_4^{\text{ext}} = \mu \star_4 1 \\ G_4{}^{\text{int}} \sim G \end{cases}$$

- Classically cannot add G on M_7 of G_2-holonomy

G is compatible with susy... but requires use of nowhere-vanishing vectors on M_7

$$*_7 G \sim e^{\#A} \mathrm{d}(e^{\tilde{\#}A} \iota_{\mathbf{V}} \Phi) \qquad \Rightarrow \qquad \int |G|^2 \sim \int \mathrm{d}G \wedge \iota_{\mathbf{V}} \Phi$$

Compactification with flux $\Rightarrow \begin{cases} \bullet \quad \mathrm{d}G_4 \neq 0 \quad \ldots \quad = X_5(G_4, \nabla, R) \\ \bullet \quad X_5(G_4, \nabla, R) \text{ at least linear in } G_4 \end{cases}$

Computing $X_5(G_4, \nabla, R)$ is part of the challenge of the non-linear completion of higher-order curvature terms

M5-branes

Calssical soliton of 11D sugra

- the metric: $ds^2_{10} = e^{N_1 u(r)} \mathrm{d}s^2_6(W_{||}) + e^{N_2 u(r)} (\mathrm{d}s^2_5)_{\perp}$ (r - distance away from M5)
- the four-form : $G_4 \sim \star_{\perp} du(r)$

$$\mathrm{d}G_4 = \delta_5(r)$$

- zero-mode expansion $G_4 \to G^{(0)} + h_3 \wedge \mathrm{d}u(r) + \cdots$

$$\mathrm{d} * G_4 \sim G_4 \wedge G_4 \qquad \Rightarrow \qquad h_3 = - *_{||} h_3$$

$$\text{Theory on M5} \Leftrightarrow \begin{cases} \bullet & (2,0) \text{ tensor multiplet} \\ \bullet & (\beta^-, \psi^\alpha, x^a) \quad \alpha = 1, ..., 4; \ a = 1, ..., 5 \\ \bullet & SO(5) \text{ R-symmetry} \\ \bullet & ADE \text{ classification - non-Abelian M5} \end{cases}$$

Symmetries of the theory without M5 $[C \wedge G \wedge G + C \wedge [\frac{1}{4} p_1^2(TM) - p_2(TM)]]$

- shift: $C_3 \to C_3 + \mathrm{d}\Lambda$
- diffeomorphisms

With M5 $i : W_6 \hookrightarrow M_{11}$

- $\delta(\int_{M_{11}} C \wedge G \wedge G) \;\; \to \;\; \int_{W_6} i^*(\Lambda \wedge G)$
  - ⋆ M5 coupling $\int_{W_6} h_3 \wedge i^* C$
  - ⋆ $\delta h_3 = i^*(\mathrm{d}\Lambda)$ .... relative cohomology ($\mathrm{d}h = i^* G$)
- diffeomorphisms and $\delta \int C \wedge X_8$?
  - ⋆ X_8 - (made of) characteristic class(es)... $X_8 = \mathrm{d}X_7^{(0)}$ and $\delta X_7^{(0)} = \mathrm{d}X_6^{(1)}$
  - ⋆ assume trivial normal bundle ($p_i(TM)|_W = p_i(TW)$)
  - ⋆ $\delta \int C \wedge X_8 \;\; \to \;\; \int_{W_6} X_6^{(1)}$
  - ⋆ anomaly inflow

M5 ANOMALY

(2,0) tensor multiplet:

- Woldvolume chiral 2-form
  - $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Worldvolume fermions
  - $I_D = \frac{1}{2}\widehat{A}(TW)\,\mathrm{ch}S(N)$
  - for trivial normal bundle:

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right)$$

- Total anomaly: $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}p_1(TW)^2 - p_2(TW)\right)$
- Cancelled via inflow from a bulk coupling $\sim C_3 \wedge X_8(TM)$

$$G_4 \delta X_7^{(0)} \to \eta(M5) X_6^{(1)}(TM) \quad \leftrightarrow \quad \mathrm{d}^{-1}\delta\mathrm{d}^{-1} I_{(2,0)}$$

- Nontrivial normal bundle... $C \wedge G \wedge G$ is NOT ... diff invariant!

Non-trivial normal bundle

(single) M5 worldvolume :

- Chiral 2-form
  - ◇ $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Fermions
  - ◇ $I_D = \frac{1}{2}\widehat{A}(TW)\,\mathrm{ch}S(N)$
  - ◇ $\mathrm{ch}S(N) = 4 + \frac{1}{2}p_1(N) + \frac{1}{96}p_1(N)^2 + \frac{1}{24}p_2(N) + ...$

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) + ... = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right) + ...$$

- Total anomaly:
  $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(N)^2 - 2p_1(TW)p_1(N)) - p_2(TW) + p_2(N)\right)$
- Anomaly from the bulk (using $p_1(TM|_W) = p_1(TW) + p_1(N)$, ...)

$$I_{\text{bulk}} = -\frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(N)^2 - 2p_1(TW)p_1(N)) - p_2(TW) - p_2(N)\right)$$

- The result: $I_{\text{M5}} + I_{\text{bulk}} = \frac{p_2(N)}{24}$!!!

Non-singular p-branes

Brane worldvolume W_d $(d=p+1)$ located at $y^a=0$ $(a=1,...,D-d)$ in M_D

$S_\epsilon(W_d)$ - S^{D-d-1} sphere bundle - boundary of tubular neighbourhood of rad ϵ, $D_\epsilon(W_d)$

- Magnetic source:

  ◇ $\mathrm{d}G_{D-d-1} = 2\pi\delta(y^1)...\delta(y^{D-d})\mathrm{d}y^1\wedge ... \wedge \mathrm{d}y^{D-y} \quad \Rightarrow \quad 2\pi\Phi_{D-d}$

  Thom class of N $\Phi_{D-d} = \begin{cases} \bullet \quad \mathrm{d}(\rho(r)e_{2n-1}/2) \qquad 2n-1=D-d-1 \\ \bullet \quad \mathrm{d}\rho(r)e_{2n}/2 \qquad 2n=D-d-1 \end{cases}$

e_{D-d-1} - global angular form

- rank$(N)=2n$ - sphere bundle has fibers S^{2n-1}

  ◇ $\mathrm{d}e_{2n-1} = -\pi^*(\chi(N)) \qquad \chi(N)\in H^{2n}(M,\mathbb{Z})$

- rank$(N)=2n+1$ - sphere bundle has fibers S^{2n}

  ◇ $\mathrm{d}e_{2n}=0 \qquad (e_{2n}=\mathrm{d}e^{(0)}_{2n-1},\ \delta e^{(0)}_{2n-1}=e^{(1)}_{2n-2})$

  ◇ cohomology class e_{2n} : $\quad [e^2_{2n}]=\pi^*(p_n(N))$

  ◇ at the level of differential forms : $\quad \pi_*(e^3_{2n})=\pi_*(e_{2n}\pi^*p_n(N))=2p_n(N)$

M5 ($W_6 \hookrightarrow M_{11}$)- $SO(5)$ N bundle

$\mathbf{so(5)} \cong \Lambda^2\mathbb{R}^5$ - connection on N: $\quad \Theta^{ab} = -\Theta^{ba}$

Define $\hat{y}^a = y^a/r\,; \quad (D\hat{y})^a = \mathrm{d}\hat{y}^a - \Theta^{ab}\hat{y}^b\,; \quad F^{ab} = \mathrm{d}\Theta^{ab} - \Theta^{ac}\wedge\Theta^{ca}$

$$\left\{\begin{array}{l} \bullet \quad e_4(\Theta) = \frac{1}{64\pi^2}\epsilon_{a_1\ldots a_5}\left((D\hat{y})^{a_1}\ldots(D\hat{y})^{a_4} - 2F^{a_1a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4} + F^{a_1a_2}F^{a_3a_4}\right)\hat{y}^{a_5} \\ \bullet \quad e_3^{(0)}(\Theta) = \frac{1}{32\pi^2}\epsilon_{a_1\ldots a_5}\left(\Theta^{a_1a_2}\mathrm{d}\Theta^{a_3a_4}\hat{y}^{a_5} - \frac{1}{2}\Theta^{a_1a_2}\Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5} - 2\Theta^{a_1a_2}\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5}\right) \\ \bullet \quad e_2^{(1)}(\varepsilon, \Theta) = \frac{1}{16\pi^2}\epsilon_{a_1\ldots a_5}\varepsilon^{a_1a_2}\left(\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5} - \Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5}\right) \end{array}\right.$$

Under gauge transformations: $\delta\Theta^{ab} = (D\varepsilon)^{ab}, \quad \delta\hat{y}^a = \varepsilon^{ab}\hat{y}^b$

In the presence of M5:

- $G = \mathrm{d}C \quad \longrightarrow \quad \mathrm{d}C - 2\pi\mathrm{d}\rho\wedge e_3^{(0)}/2$
  - ◇ $\quad \delta C = -2\pi\mathrm{d}\rho\wedge e_2^{(1)}/2$
- Introduce σ_3: $\quad G_4 - 2\pi\rho e_4/2 = \mathrm{d}(C_3 - 2\pi\rho e_3^{(0)}/2) \equiv \mathrm{d}(C_3 - 2\pi\sigma_3)$
- CS couplings in presence of M5:

$$S_{\text{CS}} = \lim_{\epsilon\to 0} -\frac{1}{6(2\pi)^2}\int_{M_{11}-D_\epsilon(W_6)}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)$$

Remember $\pi_*(e_4^3) = \pi^* p_n(N)) = 2p_2(N)$

Under diffs ($SO(5)$ transformations), S_{CS} varies....

- $\delta((C_3 - 2\pi\sigma_3) = -2\pi \mathrm{d}(\rho e_2^{(1)}/2)$

$$\begin{aligned} \delta S_{\text{CS}} &= \lim_{\epsilon\to 0} \frac{1}{12\pi} \int_{M_{11}-D_\epsilon(W_6)} \mathrm{d}(\rho e_2^{(1)}/2) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \\ &= -\frac{2\pi}{6} \int_{S_\epsilon(W_6)} \frac{e_4}{2} \wedge \frac{e_4}{2} \wedge \frac{e_2^{(1)}}{2} \end{aligned}$$

Anomaly cancellation!... and a key to non-Abelian $(2,0)$ theories

- $I_{\text{M5}} + I_{\text{bulk}} + \delta S_{\text{CS}} = 0$
- Q coincident M5 - symmetry enhancement to $SU(Q)$
  - ◇ $\mathrm{d}G_4 = 2\pi Q \mathrm{d}\rho e_4/2$
  - ◇ no new ingredients in the anomaly cancellation

⋆ $I_{\text{M5}}(Q) = Q I_{\text{M5}}(Q=1) + \frac{Q^3 - Q}{24} p_2(N)$

$(2,0)$ theories have ADE symmetry enhancement

- A_{Q-1}
  - ◇ remove a centre of mass (one free (2,0) multiplet) anomaly
  - ◇ $I^{(2,0)}_{A_{Q-1}} = (Q-1) I_{\text{M5}}(Q=1) + \frac{Q^3-Q}{24} p_2(N)$
- D_Q
  - ◇ $\mathbb{R}^5/\mathbb{Z}_2$ fixed points
  - ◇ $I^{(2,0)}_{D_Q} = Q I_{\text{M5}}(Q=1) + \frac{Q(2Q-1)(2Q-2)}{24} p_2(N)$
- E
  - ◇ no direct calculation (brane picture), but ... using 5D gauge CS confirm
  - ◇ $I^{(2,0)}_{A_{Q-1}} = r_G I_{\text{M5}}(Q=1) + \frac{d_G \cdot h^\vee_G}{24} p_2(N)$
  - ◇ $r_G, d_G, h^\vee_G$ - rank, dimension, dual Coxeter

Applications

- S^4 reduction
  - ◇ vacuum configuration: $G = 2\pi Q \text{Vol}(S^4)$
  - ◇ expand to include fluctuations - flux needs to be
    - ▷ invariant under $SO(5)$ gauge transformations
    - ▷ $\mathrm{d}G = 0$
    - ▷ quantisation $\int_{S^4} G/2 = 2\pi\, Q$
  - ◇ $G = 2\pi Q e_4/2 +$ fluctuations in C_3 works!
  - ◇ $\int_{M_{11}} C \wedge G \wedge G \quad \longrightarrow \quad \sim Q^3 \int_{AdS_7} p_2^{(0)}(A)$
- Derivation of $E_8 \times E_8$ heterotic GS term also mixes $C \wedge X_8$ and $C \wedge G \wedge G$
- Anomalies in $(1,0)$ 6D theories
- "Reduction" of anomalies and calculations for 2D, 4D superconformal theories
  - ◇ holographic calculations of CFT data (e.g. central charges)
- Geometric lessons to follow?