Tutorial on para-quaternionic and Grassmannian relations

Vojtěch Žádník

Masaryk University, Brno, CZ

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ALMOST PARA-QUATERNIONIC / GRASSMANIAN STRUCTURES

- EQUIVALENCE
- COMPATIBLE CONNECTIONS
- DISTINGUISHED CURVES
- TWISTOR SPACES

- MOTIVATIONS, REMARKS, ...

ALMOST PARA-QUATERNIONIC STR.

- PARA-QUATERNIONS ... $\mathbb{H}_s$

  $q = a + b\underline{i} + c\underline{j} + d\underline{k}$

  $-i^2 = -j^2 = +k^2 = -1$

  $k = ij = -ji$ ...

- THE NORM

  $|q|^2 = \bar{q}q = q\bar{q}$

  $q \in \mathbb{H}_s$

  $\underline{|q|^2 = -qq}$

  $q \in \operatorname{im} \mathbb{H}_s$

- ALMOST P-Q STR. on M

  $Q \subset \operatorname{End}(TM)$ $\quad Q \overset{loc.}{=} \langle I, J, K \rangle$

  $-I \circ I = -J \circ J = +K \circ K = -id$

  $K = I \circ J = -J \circ I$ ...

- BUNDLE METRIC

  $\underline{|A|^2 id = -A \circ A}$

  $A \in Q$

  ... sign. $(1, 2)$

- $|A|^2 = \left\{ \begin{array}{lll} 1 & \ldots \text{ almost} & \text{complex} \\ 0 & \ldots \quad \text{''} & \text{tangent} \\ -1 & \ldots \quad \text{''} & \text{para-complex} \end{array} \right\}$ str.

NOTE THAT

- $\dim M = \underline{2m} \geq 4$

- (± 1)-eigenspaces of para-compex str. from Q
  resp. ker (= im) of tangent str. . . .

  "<u>null</u>" vectors in TM

ALMOST GRASSMANNIAN STR. ... type $(2,m)$

- GRASSMANNIAN ... $Gr_2(m+2) = \{2\text{-dim subsp. in } \mathbb{R}^{m+2}\}$

- TANGENT SPACE at $\lambda \in Gr_2(m+2)$

$$\cong \underbrace{\lambda^*}_{\text{dim } 2} \otimes \underbrace{(\mathbb{R}^{m+2}/\lambda)}_{\text{dim } m}$$

- ALMOST GR. STR. on M

$$TM \cong \underbrace{E^*}_{\text{rank } 2} \otimes \underbrace{F}_{\text{rank } m}$$

- SEGRE CONE

= simple elements of $E^* \otimes F$

(= lin. maps $E \to F$ of rank 1)

- " is doubly ruled by subspaces

of rank $\underline{2}$, resp. $\underline{m}$

α-planes β-planes

NOTE THAT

- $\dim M = 2m \geq 4$
- additional $\Lambda^2 E^* \cong \Lambda^m F$

  $\rightsquigarrow$ trivialization of $\Lambda^2 E \otimes \Lambda^m F$

  $\rightsquigarrow$ ORIENTATION

EQUIVALENCE

- Almost para-quaternionic str. $Q \subset \mathrm{End}(TM)$ $\approx$ almost Grassmannian str. of type $(2, m)$ $TM \cong E^* \otimes F$

- Eigenspaces of $\overset{\text{almost,}}{\text{para-compex}}$ str. from Q (resp. kernels of almost tangent...) $\approx$ max. lin. subspaces in Segre cone (i.e. β-planes)

Ingredients:

- $\mathbb{H}_s \cong \mathrm{Mat}_{2\times 2}$ $\quad i \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad j \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad k \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

  s.t. $(\text{norm})^2 = \text{determinant}$

  $\mathrm{im}\, \mathbb{H}_s = \text{trace-free mat.}$

- pointwise: $T_xM \cong \mathbb{R}^{2*} \otimes \mathbb{R}^m \rightsquigarrow Q_x \subset \mathrm{End}(T_xM)$

  (via std. action on $\mathbb{R}^2$)

RSJ

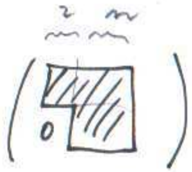
NOTE THAT

- both are G-str. with structure group

$$\underline{G_0} := GL(2,\mathbb{R}) \cdot GL(m,\mathbb{R})$$

- in part., this acts on $T_xM \cong \mathbb{R}^{2*} \otimes \mathbb{R}^m$ as
  para-quat. automorphisms
  (i.e. $f(AX) = \phi(A) f(X)$, $X \in T_xM$, $A \in Q_x$
  $\phi : Q_x \to Q_x$ lin. . .)

- notation ↑ reflects the following:

  - $Gr_2(m+2) \cong G/P$ ← parabolic 
    ↗ $PGL(m+2,\mathbb{R})$

  - Lie alg . . . $\boxed{\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1}$

    $\mathfrak{g}_{-1}$: $\mathbb{R}^{2*} \otimes \mathbb{R}^m$

    $\mathfrak{g}_0$: center ⊕ semi-simple

    $\mathfrak{g}_1$: $\mathbb{R}^2 \otimes \mathbb{R}^{m*}$

HENCE

... $|1|$-graded parabolic geometry !!!

of type $\circ\!-\!\times\!-\!\circ\!-\ \cdots$

- for $n=2$ $\rightsquigarrow$ CONFORMAL STR. of sign. $(2,2)$

INTEGRABILITY

... ex. of compatible affine torsion-free connection

... not aut. satisfied for $n>2$

NORMALIZATION

- as G_0-str. ... (choice of) G_0-inv. complement $\mathcal{C}$ to $\operatorname{im}\partial$ in $\Lambda^2\mathbb{R}^{m*}\otimes\mathbb{R}^m$ ← $m=2n$

  $\partial : \mathbb{R}^m\otimes\mathfrak{g}_0 \longrightarrow \mathbb{R}^{m*}\wedge\mathbb{R}^{m*}\otimes\mathbb{R}^m$

- compatible conn's with common torsion ... $\ker\partial$
- here $\ker\partial \cong \mathbb{R}^{m*} \leftarrow \cong \mathfrak{g}_{-1}^* \cong \mathfrak{g}_1$

- for parabolic $\mathfrak{g}$. ... natural norm. given by: curvature of Cartan connection in $\ker\partial^*$

  $$\rightleftarrows \mathfrak{g}_{-1}^*\otimes\mathfrak{g} \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} \Lambda^2\mathfrak{g}_{-1}^*\otimes\mathfrak{g} \rightleftarrows$$

- the complement

  $$\boxed{\mathcal{N} := \ker\partial^* \cap \left(\Lambda^2\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}\right)}$$

  ... harmonic homog. 1

NORMALIZATION

- [David '09] ... $\Pi : \Lambda^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m \circlearrowleft$

$$\varphi \mapsto \frac{2}{3}\left(\varphi_I^{0,2} + \varphi_J^{0,2} + \varphi_K^{0,2} \right)$$

$$\varphi_A^{0,2}(X,Y) := \frac{1}{4|A|^2}\Big(|A|^2 \varphi(X,Y) - \varphi(AX,AY) + A\varphi(AX,Y) + A\varphi(X,AY) \Big)$$

$A \in Q$, $|A|^2 \neq 0$

... $\Pi \circ \Pi = \Pi$

... $\ker \Pi = \operatorname{im} \partial$

- complement $\boxed{\mathcal{C} := \operatorname{im} \Pi \cap \{\text{trace conditions}\}}$

- compatible conn's with this torsion ... "minimal"

NORMALIZATION

• [Čap-Slovák '09]

... $\boxed{\mathfrak{n} = S^2\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^{m*} \otimes \mathbb{R}^{2*} \otimes \mathbb{R}^m \cap \{\text{trace conditions}\}}$

$\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$

LEMMA $\boxed{\mathcal{C} = \mathfrak{n}}$

(... $\mathcal{C} \subseteq \mathfrak{n}$ & both complem. to im ∂)

IN FACT this is <u>unique</u>

(... all G_0-irred. components multiplicity 1 ...)

$$\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \xrightarrow{\partial} \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$$

ALTOGETHER

- Minimal para-quatern. connections $\approx$ Weyl connections of the normal Cartan conn.
- ... integrable $\approx$ ... without torsion
- difference tensor $\Upsilon \odot \mathrm{id} + (\Upsilon \circ I) \odot I + (\Upsilon \circ J) \odot J - (\Upsilon \circ K) \odot K$ $\approx$ alg. bracket $\{\{\Upsilon, -\}, -\}$

$$\hat{\nabla} = \nabla + \Upsilon, \quad \Upsilon \in \Omega^1(M)$$

Q-planar curves

$\ldots\ \gamma : I \to M$ s.t. $\boxed{\nabla_{\dot\gamma}\dot\gamma = S(\dot\gamma)}$

some (⟹ any) comp. connection

some elem. of $\langle id \rangle \oplus Q \subset End(TM)$

- distinguish $<$ null / generic

$S \circlearrowright E \xrightarrow{\dot\gamma} F$

- under $TM \cong E^* \otimes F$ $\ldots$ $\boxed{\nabla_{\dot\gamma}\dot\gamma = \dot\gamma \circ S}$

some elem. of $End(E)$

- change $\hat\nabla = \nabla + \Upsilon$

$\rightsquigarrow$ $\hat S = S + 2\,\Upsilon \circ \dot\gamma$

$$\begin{tikzcd} E \arrow[r, bend left, "\dot\gamma"] & F \arrow[l, bend left, "\Upsilon"] \end{tikzcd}$$

LEMMA

$\boxed{\begin{array}{l} \text{generic Q-planar} = \text{geodesic of some comp. conn.} \\ \text{null Q-planar}: \ \ldots\ \text{"}\ \ldots\ \text{some}\ \ldots\ \text{"}\ \ldots \\ \qquad\qquad\qquad\qquad\quad \Updownarrow \\ \qquad\qquad\qquad \ldots\ \text{"}\ \ldots\ \text{any}\ \ldots\ \text{"}\ \ldots \end{array}}$

GRASSMANIAN CIRCLES

... distinguished curves of parabolic $\mathfrak{g}$. o—x—o— ...

- null = common geod. of all comp. connections.
- generic = solutions to 3-order ODE ...

GENERALLY

Rho tensor

- γ = disting. curve $\iff$ ex. ∇ : $\nabla_{\dot\gamma}\dot\gamma = 0$ & $P(\dot\gamma) = 0$

HERE

... in terms of Q-planar condition $\nabla_{\dot\gamma}\dot\gamma = \dot\gamma \circ S$

$$\rightsquigarrow \boxed{\nabla_{\dot\gamma} S = \frac{1}{2} S \circ S + 2\,P(\dot\gamma) \circ \dot\gamma} \quad (*)$$

cf. [Biley-Eastwood '91]

ALTOGETHER

- null Q-planar = null Gr. circle
  $\iff$ common geod. of all comp. connections

- generic Q-planar = generic Gr. circle
  $\iff$ eqn. (*) is satisfied

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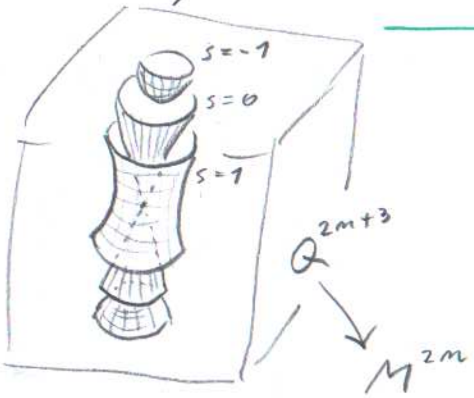
ε - TWISTOR SPACES

i.e. $A \circ A = s\,\mathrm{id}$

- for $Q \subset \mathrm{End}(TM)$ ... $\mathcal{Z}^s := \{ A \in Q : |A|^2 = -s \}$



KNOWN FACTS

- comp. connection on M ⇝ almost (para-) compex str. $\mathcal{J}^\pm$ on $\mathcal{Z}^\pm$
- minimal comp. conn. ⇝ canonical .... " ....
- Q integrable $\iff$ $\mathcal{J}^\pm$ integrable

TWISTOR CORRESPONDENCE

- for $TM \cong E^* \otimes F$ ...

$$\begin{array}{ccc} & (\mathcal{G},\omega) & \\ & \downarrow & \\ & N^{2m+1} = \mathcal{G}/Q & \\ & \times\!-\!\times\!-\!\circ\!-\cdots & \\ \swarrow & & \searrow \\ M^{2m}=\mathcal{G}/P & & \left(\begin{array}{c} X^{m+1}=\mathcal{G}/P' \\ \times\!-\!\circ\!-\!\circ\!-\cdots\end{array}\right) \\ \circ\!-\!\times\!-\!\circ\!-\cdots & & \end{array}$$

KNOWN FACTS

- $N \approx \mathbb{P}E \approx$ space of β-planes in TM
- $(\mathcal{G} \to M, \omega)$ torsion-free $\Longleftrightarrow$ $(\mathcal{G} \to N, \omega)$ regular
- In such case:
  - $M \approx$ solution space to 2-order ODE system on X
  - $N \approx \mathbb{P}TX$

"torsion-free path geom."

RELATIONS

- In general . . . $\mathbb{P}\mathcal{Z}^0 \approx N$
- In int. case . . . $\mathcal{Z}^0 \approx TX$
- Previous integrability results for $\epsilon = \pm 1$ recovered & extended for $\epsilon = 0$.

Ingredients:

- $$\begin{tikzcd} & (\mathcal{G}, \omega) \arrow[dl] \arrow[d] \\ \mathcal{Z}^\epsilon = \mathcal{G}/R^\epsilon \arrow[d] & N = \mathcal{G}/Q \arrow[dl] \\ M = \mathcal{G}/P & \end{tikzcd}$$

- Nijenhuis of $\mathcal{J}^\epsilon \approx (0,2)$-part w.r. $\mathcal{J}^\epsilon$ of torsion of $(\mathcal{G}_+ \to \mathcal{Z}^\epsilon, \omega)$

REMARKS

- Related geom. str. ... e.g. LIE CONTACT STR :
  - contact distrib. $D \subset TM$
  - para-quat. str. $Q \subset End(D)$
  - comp. with Levi bracket
    $$\mathcal{L}(AX, AY) = |A|^2 \cdot \mathcal{L}(X, Y)$$
    $$X, Y \in D, \quad A \in Q$$
  - an identif. of natural line bundles ...

- Applied in two ways :
  - reconst. of the str. from asoc. path geom. of CHAINS
  - vanishing of one of torsions $\approx$ integrability of asoc. CR structure on ϵ-TWISTOR sp.

DREAMS

$$\begin{tikzcd}
 & N \arrow[dl] \arrow[dr] & & \mathcal{Z}^{\epsilon} \arrow[dl] \\
X & & M &
\end{tikzcd}$$

(3) integrable ϵ-complex str.

(2) integrable Grass ≈ para-q. str.

(1) torsion-free path geom.

Is there any use of (3) for the study of (1)?

REFERENCES

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⋮

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(Complex)

⋮

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