The bundle of Weyl structures associated to an AHS-structure

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- This talk reports on joint work in progress with T. Mettler (Frankfurt). Motivated by results of M. Dunajski and T. Mettler, we study the “bundle of Weyl structures” associated to an AHS-structure (i.e. an irreducible parabolic geometry).
- On the one hand, this produces from an AHS structure a relatively small space endowed with a nice geometric structure (which in the torsion-free case includes a split-signature Einstein metric) that encodes the initial geometry. There is a natural calculus on that space that allows for efficient study of its geometric properties.
- On the other hand, this is closely connected to the study of fully nonlinear PDE that are naturally associated to the initial AHS structure. In the case of projective structure in dimension two, this provides a connection to theory of convex projective structures, representation varieties, etc.

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AHS structures are a special class of first order G-structures that are connected to certain gradings of simple Lie algebras. From the geometric point of view, they are distinguished by the fact that they are irreducible and of finite type, but have non-trivial prolongation. This means that each automorphism is locally determined by its two-jet in a point, so automorphism groups are finite dimensional. However, the one-jet in a point does not determine an automorphism locally in general, so such geometries to not determine a distinguished connection on the tangent bundle.

The best known example of such structures are conformal, projective, almost Grassmannian, and almost quaternionic structures. We will mainly work with the uniform description of such structures as Cartan geometries, so the precise description of the structures will not be very important.

The data needed to define an AHS structure is a grading of a simple Lie algebra $\mathfrak{g}$ (different from $\mathfrak{sl}_2$) of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. There is a complete classification of such gradings, which is equivalent to the classification of Hermitian symmetric spaces.

For such a grading, any group G with Lie algebra $\mathfrak{g}$ there is a closed subgroup $P \subset G$ with Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For $g \in P$, one has $\mathrm{Ad}(g)(\mathfrak{p}) \subset \mathfrak{p}$ and $\mathrm{Ad}(g)(\mathfrak{g}_1) \subset \mathfrak{g}_1$. One defines a closed subgroup $G_0 \subset P$ as consisting of those $g \in P$ such that $\mathrm{Ad}(g)(\mathfrak{g}_i) \in \mathfrak{g}_i$ for all i. Also, exp restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed subgroup $P_+ \subset P$ such that $P = G_0 \ltimes P_+$.

In particular, Ad defines an infinitesimally injective homomorphism $G_0 \to GL(\mathfrak{g}_{-1})$, so there is a well-defined notion of G_0-structures on manifolds of dimension $\dim(\mathfrak{g}_{-1})$. There also is the concept of a Cartan geometry of type (G, P) on manifolds of that dimension.

For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), $\mathcal{G} \to M$ is a principal P-bundle. Thus we can form $\mathcal{G}_0 := \mathcal{G}/P_+$ with induced projection $p_0 : \mathcal{G}_0 \to M$, which is a principal bundle with structure group $P/P_+ = G_0$. The component in $\mathfrak{g}_{-1}$ of the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, is easily seen to descend to a soldering form $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$, so we obtain an underlying G_0-structure.

Except for a specific grading in type A_n, any G_0-structure is induced by a unique (up to isomorphism) Cartan geometry for which ω satisfies a normalization condition. For the grading in the A_n-case from above, $\mathcal{G}_0$ is the full frame bundle of M and there is a similar correspondence between projective equivalence classes of linear connections on TM and normal Cartan geometries.

We will view the Cartan picture as the (given) main description of the geometry and the G_0-structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ as an underlying structure.

Weyl structures are a major tool to describe the Cartan geometry $(p : \mathcal{G} \to M, \omega)$ in terms of the underlying G_0-structure. Classically, a Weyl structure is defined as a G_0–equivariant section $s : \mathcal{G}_0 \to \mathcal{G}$ of the obvious projection. It can be shown that such sections exist globally and form an affine space modeled on $\Omega^1(M)$.

The components of ω in $\mathfrak{g}_0$ respectively in $\mathfrak{g}_1$ can be pulled back along s to obtain

- A principal connection on $\mathcal{G}_0 \to M$ (the *Weyl connection*)
- A form $\mathrm{P} \in \Omega^1(M, T^*M)$ called the *Rho-tensor* or the *Schouten tensor*.

These can be used to interpret operations coming form the Cartan geometry and to describe the curvature of the Cartan connection ω in terms of the torsion and curvature of a Weyl connection.

It is easy to see that *any* reduction of $\mathcal{G} \to M$ to the structure group $G_0 \subset P$ is given by a Weyl structure. Via this observation we can use the classical description of reductions of structure group to describe Weyl structures as smooth sections of a bundle.

- Putting $A := \mathcal{G}/G_0$ and denoting by $\pi : A \to M$ the induced projection, Weyl structures are in bijective correspondence with smooth sections of π.
- A can be identified with $\mathcal{G} \times_P (P/G_0)$, so $\pi : A \to M$ is a natural fiber bundle. It can be shown that P/G_0 is diffeomorphic to $\mathfrak{g}_1$ and in this picture the natural action of $P = G_0 \ltimes \exp(\mathfrak{g}_1)$ becomes affine.
- There are alternative explicit descriptions of A as a subset in the projectivization of a quotient of a tractor bundle $\mathcal{V} \to M$ and as the bundle of connections on the density bundle $\mathcal{E}[1]$, which will be discussed in more detail below.

From $A = \mathcal{G}/G_0$ it readily follows that $\mathcal{G} \to A$ is a principal G_0-bundle and that one may view $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ as a Cartan connection on that bundle. In particular, $TA \cong \mathcal{G} \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)$, which immediately leads to a geometric structure on A.

- As a representation of G_0, $\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$. This shows that ω induces a linear connection D on each vector bundle over A that is a associated to $\mathcal{G}$. In addition, $TA = L_- \oplus L_+$ for two natural subbundles $L_\pm \subset TA$ of rank n.
- Via the Killing form of $\mathfrak{g}$, we get $\mathfrak{g}_1 \cong \mathfrak{g}^*_{-1}$. Correspondingly, there is a non-degenerate paring $L_- \times L_+ \to M \times \mathbb{R}$ that is parallel for D.
- Skew symmetrizing this pairing, one gets a natural form $\Omega \in \Omega^2(A)$, while symmetrizing it defines a split signature metric h on A, which both a parallel for D.

We conclude that A carries a canonical *almost Bi-Lagrangean structure*, with an additional reduction to the structure group G_0. In particular, representations of G_0 gives rise to natural vector bundles on A, and on each of these there is an induced linear connection D. Let us split this as $D = D^+ \oplus D^-$ according to $T^*A = L_+^* \oplus L_-^*$.

For a representation $\mathbb{W}$ of G_0, the bundle $\mathcal{W}A := \mathcal{G} \times_{G_0} \mathbb{W} \to A$ is the pullback of $\mathcal{W}M := \mathcal{G} \times_P \mathbb{W} \to M$. Taking pullbacks of sections defines an inclusion $\Gamma(\mathcal{W}M) \to \Gamma(\mathcal{W}A)$, whose image coincides with the kernel of D^+. Specializing, we obtain $\mathfrak{X}(M) \hookrightarrow \Gamma(L_-)$ and $\Omega^1(M) \hookrightarrow \Gamma(L_+)$, and these images provide local frames for TA.

On the other hand, the torsion and curvature of D can be computed from the Cartan curvature of ω by standard methods.

Using these descriptions, one first proves:

- The tensorial map $\Lambda^2 L_- \to L_+$ induced by the Lie bracket of vector fields is induced by the $\mathfrak{g}_1$-component of the Cartan curvature.
- The form $\Omega \in \Omega^2(A)$ is symplectic if and only if the initial AHS structure on M is torsion-free.

To analyze D^-, one can use the relation of sections of $\pi : A \to M$ to Weyl structures. Let us denote by $\sigma \mapsto \tilde{\sigma}$ the inclusion $\Gamma(\mathcal{W}M) \to \Gamma(\mathcal{W}A)$ and for a section $s : M \to A$ let ∇ be the corresponding Weyl connection. Then we get:

For $\xi \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathcal{W}M)$, the section $D^-_{\tilde{\xi}}\tilde{\sigma}$ coincides, along $s(M)$, with $\widetilde{\nabla_\xi \sigma}$.

We have noted above that the Canonical connection D associated to the almost bi-Lagrangean structure on A is always metric for h. Thus the Levi-Civita connection ∇^h of h can be determined by a standard formula from the torsion of D. This can be done using that D is induced by ω, viewed as a Cartan connection on $\mathcal{G} \to A$. Using this, one can also analyze the curvature of D and, via the difference $\nabla^h - D$ computed above, the curvature of ∇^h. This leads to

Theorem

For a torsion-free AHS structure, consider the induced metric h the canonical connection D on A. Then

- The Ricci-type contraction of the curvature R^D of D is proportional to h.
- The metric h is Einstein.

Any irreducible representation of $\mathfrak{g}$ has a canonical $\mathfrak{p}$-irreducible quotient. There is a unique fundamental representation $\mathbb{V}$ of $\mathfrak{g}$, for which that quotient is one-dimensional, and we assume that $\mathbb{V}$ integrates to G. Restricting $\mathbb{V}$ to P, one obtains the *basic tractor bundle* $\mathcal{V} := \mathcal{G} \times_P \mathbb{V} \to M$ and the irreducible quotient induces the *bundle of 1-densities* $\mathcal{E}[1] \to M$. The bundle $\mathcal{V}$ carries a natural decreasing filtration by smooth subbundles $\{\mathcal{V}^i\}$. Using this, the bundle $\pi : A \to M$ can be equivalently described as:

- The open subbundle in the projectivization $\mathcal{P}(\mathcal{V}/\mathcal{V}^2)$ consisting of lines that are transversal to the hyperplane subbundle $\mathcal{V}^1/\mathcal{V}^2$.
- The bundle $Q\mathcal{E}[1] \to M$ of linear connections on the bundle of 1-densities.

The latter description nicely corresponds to the parametrization of Weyl structures by connections on a bundle of scales.

A local non-vanishing section of $\mathcal{E}[1]$ determines a flat connection on $\mathcal{E}[1]$ and thus a local smooth section s of $\pi : A \to M$. Now it turns out that there is a natural fully non-linear PDE on nowhere vanishing sections of $\mathcal{E}[1] \to M$:

- Since $\mathbb{V}$ is a fundamental representation, the corresponding first BGG operator H has order 2 and maps $\Gamma(\mathcal{E}[1])$ to $\Gamma(\circledcirc^2 T^*M \otimes \mathcal{E}[1])$ (invariant Hessian).
- The determinant induces a (non-linear) natural bundle map $S^2T^*M \to S^2\Lambda^n T^*M$, and the latter turns out to be some power $\mathcal{E}[-N]$ of $\mathcal{E}[-1] := \mathcal{E}[1]^*$.
- Thus $\sigma \mapsto \det(H(\sigma))$ is a non-linear invariant operator $\Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{E}[-(N-n)])$, and $\det(H(\sigma)) = \sigma^{-(N-n)}$ is an invariant PDE of Monge-Ampère type.

Now this nicely connects to the geometry of Weyl structures. In the picture of sections $s : M \to A$, we call a Weyl-structure *Lagrangean* if $s^*\Omega = 0$ and *non-degenerate* if s^*h is non-degenerate. Equivalently, the Rho tensor of s has to be symmetric and non-degenerate. The main connection to the invariant Monge-Ampère equation is

Theorem

Consider a locally flat AHS structure on M. Then a nowhere vanishing sections of $\Gamma(\mathcal{E}[1])$ satisfies the invariant Monge-Ampère equation if and only if it determines a non-degenerate, Lagrangean Weyl structure and for the corresponding section $s : M \to A$, $s(M) \subset A$ is a minimal submanifold.

In projective geometry, solutions of the invariant Monge-Ampère equation are related to convex projective structures, which play an important role in the study of representation varieties.