IBL_∞-structure and string topology conjecture

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IBL_∞-algebra

- C ... graded vector space.
- $\mathrm{E}_k C := C[1]^{\otimes k}/\mathbb{S}_k$ ... k-th exterior power.

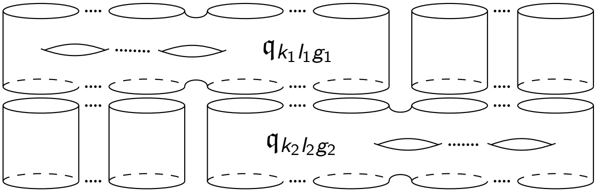
Definition (IBL_∞-algebra). Collection of linear homogenous maps $\mathfrak{q}_{klg} : \mathrm{E}_k C \to \mathrm{E}_l C$ for k, $l \geq 1$, $g \geq 0$ satisfying IBL_∞-relations — it is an ∞-version of an *involutive Lie bialgebra*. We denote it by $\mathrm{IBL}_\infty(C)$.

- Here, $\mathfrak{q}_{110} : C[1] \to C[1]$ is a *boundary operator*, $\mathfrak{q}_{210} : \mathrm{E}_2 C \to \mathrm{E}_1 C$ a *product* and $\mathfrak{q}_{120} : \mathrm{E}_1 C \to \mathrm{E}_2 C$ a *coproduct*, such that $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ is an involutive Lie bialgebra on the homology of $\mathfrak{q}_{110}$.
- If $\mathfrak{q}_{110}$, $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ are the only non-zero $\overset{def}{\Longleftrightarrow}$ dIBL-*algebra* $\mathrm{dIBL}(C)$.
- If $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ are the only non-zero $\overset{def}{\Longleftrightarrow}$ IBL-*algebra* $\mathrm{IBL}(C)$.

IBL_∞-*relations:*

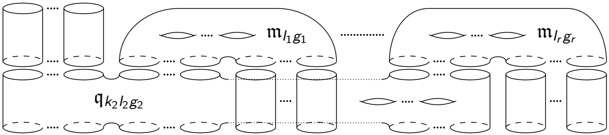
$$\sum_{h=1}^{g+1} \sum_{\substack{k_1+k_2=k+h \\ l_1+l_2=l+h \\ g_1+g_2=g+1-h}} \mathfrak{q}_{k_2 l_2 g_2} \circ_h \mathfrak{q}_{k_1 l_1 g_1} = 0 \quad \text{for all } k, l \geq 1, g \geq 0.$$

The composition $\circ_h$ is graphically represented as:



Elements of the IBL_∞-theory

- *Morphisms* $\mathfrak{f} = (\mathfrak{f}_{klg} : \mathrm{E}_k C \to \mathrm{E}_l C') : \mathrm{IBL}_\infty(C) \to \mathrm{IBL}_\infty(C')$ and IBL_∞*-homotopies.*
- *Maurer-Cartan element* $\mathfrak{m} = (\mathfrak{m}_{lg} \in \mathrm{E}_l C) \Longrightarrow$ deformation theory.
- *Twisted* IBL_∞*-algebra* $\mathrm{IBL}^{\mathfrak{m}}_\infty(C) = (C, \mathfrak{q}^{\mathfrak{m}}_{klg} : \mathrm{E}_k C \to \mathrm{E}_l C)$.



- Twisted morphism $\mathfrak{f}^{\mathfrak{m}} : \mathrm{IBL}^{\mathfrak{m}}_\infty(C) \to \mathrm{IBL}^{\mathfrak{n}}_\infty(C')$, where $\mathfrak{n} = \mathfrak{f}_* \mathfrak{m}$ is the *pushforward Maurer-Cartan element.*
- *Canon.* dIBL*-str.* and *canon. Maurer-Cartan el.* on cyclic cochains

$$C(V) := \mathrm{span}\Big\{ \psi : V[1]^{\otimes k} \to \mathbb{R} \mid \text{lin., homog., cyclic sym.}, k \geq 1 \Big\}.$$

of a finite dimensional cyclic differential graded algebra V.

Canonical dIBL-algebras on cyclic cochains of $\mathrm{H}_{\mathrm{dR}}(M)$

Fact: For M closed oriented n-manifold, the quadruple $(\mathrm{H}_{\mathrm{dR}}(M), \mathrm{d} = 0, \wedge, \int_M \alpha_1 \wedge \alpha_2)$ is a cyclic dga.

- $\Longrightarrow$ *canonical* dIBL*-structure* $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$:

$$\mathfrak{q}_{110} = 0, \quad \mathfrak{q}_{210}, \quad \mathfrak{q}_{120}.$$

- $\Longrightarrow$ *canonical Maurer-Cartan element* $\mathfrak{m} = \mathfrak{m}_{10} \in C(\mathrm{H}_{\mathrm{dR}}(M))$:

$$\mathfrak{m}_{10}(\alpha_1, \alpha_2, \alpha_3) = \pm \int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \quad \text{for } \alpha_1, \alpha_2, \alpha_3 \in \mathrm{H}_{\mathrm{dR}}(M).$$

- $\Longrightarrow$ *Canonical twisted* dIBL*-algebra* $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$:

$$\mathfrak{q}^{\mathfrak{m}}_{110} = \mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}, \quad \mathfrak{q}^{\mathfrak{m}}_{210} = \mathfrak{q}_{210}, \quad \mathfrak{q}^{\mathfrak{m}}_{120} = \mathfrak{q}_{120}$$

Let $(e_i) \subset \mathrm{H}_{\mathrm{dR}}$ be a basis, $(e^i) \subset \mathrm{H}_{\mathrm{dR}}$ the dual basis s.t. $\int_M e_i \wedge e^j = \delta_i^j$, and $T^{ij} := \pm \int_M e^i \wedge e^j$. Then for α_i, $\alpha_{ij} \in \mathrm{H}_{\mathrm{dR}}[1]$, ψ, $\psi_i \in C(\mathrm{H}_{\mathrm{dR}})$:

$$\mathfrak{q}_{210}(\psi_1 \otimes \psi_2)(\alpha_1 \ldots \alpha_k) = \sum_{\substack{c=1,\ldots,k_1+k_2 \\ i,j=1,\ldots,m}} \pm T^{ij} \psi_1(e_i \alpha_c \ldots \alpha_{c+k_1-2}) \psi_2(e_j \alpha_{c+k_1-1} \ldots \alpha_{c+k_1+k_2-3}),$$

$$\mathfrak{q}_{120}(\psi)(\alpha_{11} \ldots \alpha_{1l_1} \otimes \alpha_{21} \ldots \alpha_{2l_2}) = \sum_{\substack{c_1=1,\ldots,l_1 \\ c_2=1,\ldots,l_2 \\ i,j=1,\ldots,m}} \pm T^{ij} \psi(e_i \alpha_{1,c_1} \ldots \alpha_{1,c_1+l_1-1} e_j \alpha_{2,c_2} \ldots \alpha_{2,c_2+l_2-1}),$$

$$\mathfrak{q}_{110}^{\mathfrak{m}}(\psi)(\alpha_1 \ldots \alpha_k) = \underbrace{\sum_{i=1}^{k-1} \pm \psi(\alpha_1 \ldots (\alpha_i \wedge \alpha_{i+1}) \ldots \alpha_k) \pm \psi((\alpha_k \wedge \alpha_1) \ldots \alpha_{k-1}).}_{\text{dual of Hochschild differential}}$$

- $\Longrightarrow$ get an IBL-structure on $\mathrm{H}_*(C(\mathrm{H}_{\mathrm{dR}}), \mathfrak{q}_{110}^{\mathfrak{m}}) \simeq \mathrm{H}_\lambda^*(\mathrm{H}_{\mathrm{dR}}, \wedge)$, which is the *cyclic cohomology* of $(\mathrm{H}_{\mathrm{dR}}(M), \wedge)$.

“Formal pushforward” Maurer-Cartan element

$=$ Another Maurer-Cartan element $\mathfrak{n} = (\mathfrak{n}_{lg})$ for $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ constructed by picking a Riemannian metric and a Green kernel G and computing integrals associated to trivalent ribbon graphs with G as a propagator and with harmonic forms $\alpha \in \mathcal{H}(M)$ at exterior vertices.

Definition (Green kernel). A form $G \in \Omega^{n-1}(M \times M \backslash \Delta)$ is called a *Green kernel* if the following conditions are satisfied:

- (G1) G extends smoothly to a blow-up $\mathrm{Bl}_\Delta(M \times M)$.
- (G2) For $\mathcal{G} : \Omega^*(M) \to \Omega^{*-1}(M)$ given for $\alpha \in \Omega(M)$ by $\mathcal{G}(\alpha)(y) = \int_x G(x,y)\alpha(x)$, we have $\mathrm{d}\,\mathcal{G} + \mathcal{G}\,\mathrm{d} = \mathbb{1} - \pi_{\mathcal{H}}$, where $\pi_{\mathcal{H}} : \Omega(M) \to \mathcal{H}(M)$ is the harmonic projection.
- (G3) $\tau^* G = (-1)^n G$, where $\tau(x,y) = (y,x)$.

Theorem (in prep. by K. Cieliebak & E. Volkov). Let M be a closed oriented Riemannian manifold and G a Green kernel. The formula

$$\mathfrak{n}_{lg}(\omega_1 \otimes \cdots \otimes \omega_l) := \frac{1}{l!} \sum_{[\Gamma] \in \overline{\mathrm{RG}}^{(3)}_{klg}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{L_1, L_3^b} \pm I(G, \Gamma, L; \omega_1, \ldots, \omega_l),$$

where $\omega_i = \alpha_{i1} \ldots \alpha_{is_i}$, $\alpha_{ij} \in \mathcal{H}(M) \simeq \mathrm{H}_{\mathrm{dR}}(M)$, $l \geq 1$, $g \geq 0$ defines a Maurer-Cartan element $\mathfrak{n} = (\mathfrak{n}_{lg})$, $\mathfrak{n}_{lg} \in \mathrm{E}_l C(\mathrm{H}_{\mathrm{dR}}(M))$ for the canonical dIBL-algebra $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$.

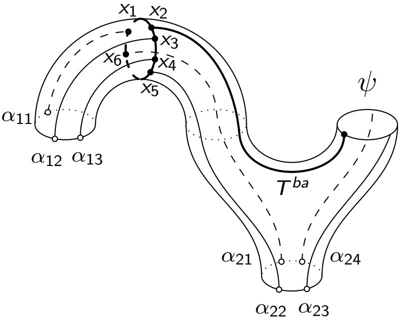
- *Conjecture:* If M_1 and M_2 are homotopy equivalent, then $\mathrm{dIBL}^{\mathfrak{n}_1}(C(\mathrm{H}_{\mathrm{dR}}(M_1)))$ and $\mathrm{dIBL}^{\mathfrak{n}_2}(C(\mathrm{H}_{\mathrm{dR}}(M_2)))$ are IBL_∞-homotopy equivalent.

- The *twisted* IBL_∞*-algebra* $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ consists of:

  (basic op.) $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$, $\mathfrak{q}^{\mathfrak{n}}_{210} = \mathfrak{q}_{210}$, $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$.

  (higher op.) $\mathfrak{q}^{\mathfrak{n}}_{1lg} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}$ for $(l,g) \neq (1,0), (2,0)$, $l \geq 1$, $g \geq 0$.

- *Example:* For α_{11}, α_{12}, α_{13}, α_{21}, α_{22}, α_{23}, $\alpha_{24} \in \mathcal{H}(M)$ and $\psi : \mathcal{H}(M)[1]^{\otimes 3} \to \mathbb{R}$, the contribution of $\Gamma \in \overline{\mathrm{RG}}^{(3)}_{6,2,0}$ below to $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\psi)(\alpha_{11}\alpha_{12}\alpha_{13} \otimes \alpha_{21}\alpha_{22}\alpha_{23}\alpha_{24})$ is:



$$\left.\begin{array}{l}\sum_{a,b}\sum_{c=1}^{4} \pm T^{ab}\psi(e_a\alpha_{2,c+2}\alpha_{2,c+3}) \\ \Big(\int_{x_1x_2x_3x_4x_5x_6} G(x_1,x_2)G(x_2,x_3)G(x_3,x_4) \\ G(x_4,x_5)G(x_5,x_6)G(x_6,x_1)\big(\alpha_{11}(x_1)\alpha_{12}(x_3) \\ \alpha_{13}(x_4)\big)\big(e_b(x_2)\alpha_{2,c}(x_6)\alpha_{2,c+1}(x_5)\big)\Big)\end{array}\right\} I(\Gamma,L)$$

$+\sum$ labelings L^b_3, L^v_1

The twist is often trivial.

Theorem. M closed or. Riem. n-mfd. There is a Green kernel G s.t.:

(basic op.:)
1. $\mathrm{H}^1_{\mathrm{dR}}(M) = 0 \Longrightarrow \mathfrak{n}_{20} = 0$, hence $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120}$.
2. M geom. formal $\Longrightarrow \mathfrak{n}_{10} = \mathfrak{m}_{10}$, hence $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$.

(higher op.:) $\mathfrak{q}^{\mathfrak{n}}_{1lg} = 0$ for all $(l, g) \neq (1, 0), (2, 0)$ with the possible exceptions of surfaces and 3-manifolds with $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$.

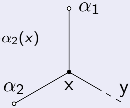
In particular, $(1)\&(2)\&(n \neq 2) \Longrightarrow \mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}})) = \mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}))$

- $\mathbb{S}^2$: unsolved, lots of graphs 0.
- $\mathbb{S}^1$: $\mathfrak{n}_{20} \neq 0$ and $\mathfrak{q}^{\mathfrak{n}}_{120} \neq \mathfrak{q}_{120}$, but agree on $\mathrm{H}(C(\mathrm{H}_{\mathrm{dR}}(\mathbb{S}^1), \mathfrak{q}^{\mathfrak{n}}_{110}))$.

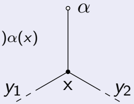
Idea of the proof.

- $\Gamma \in \overline{\mathrm{RG}}^{(3)}_{klg}, \Gamma \neq Y \implies$ int. vertices of types A, B, C:

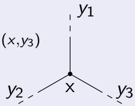
$A_{\alpha_1,\alpha_2}(y) = \int_x G(x,y)\alpha_1(x)\alpha_2(x)$



$B_\alpha(y_1,y_2) = \int_x G(x,y_1)G(x,y_2)\alpha(x)$



$C(y_1,y_2,y_3) = \int_x G(x,y_1)G(x,y_2)G(x,y_3)$



- $\alpha = \alpha_1 = 1 \in \mathcal{H}^0(M)$: $A_{1,\alpha_2} = 0 \iff \mathcal{G} \circ \pi_{\mathcal{H}} = 0 \quad (G4)$

$$\iff A_{\alpha_1,\alpha_2} = 0 \quad M \text{ geom. formal}$$

$$B_1 = 0 \iff \mathcal{G} \circ \mathcal{G} = 0 \quad (G5)$$

- A $\mathcal{G}$ satisfying (G1)–(G5) exists.
- Fubini: $\int_{x,y,\ldots}(\cdots) = \int_{y,\ldots}\left(\left[\int_x G(x,y)\alpha_1(x)\alpha_2(x)\right]\cdots\right)$
- Degree arguments using $nk = D + (n-1)e$. □

Chen's iterated integrals

Let $\mathrm{L}M = \{\gamma : \mathbb{S}^1 \to M\}$ be the *free loop space* of M.

Definition. Consider $I_\lambda : \bigoplus_{k\geq 1} \Omega(M)^{\otimes k} \to C^*_{\mathrm{sing}}(\mathrm{L}M)$ defined for $\alpha_1, \ldots, \alpha_k \in \Omega(M)$ and $\sigma : K_\sigma \to \mathrm{L}M$ by

$$I_\lambda(\alpha_1 \ldots \alpha_k)(\sigma) = \int_{K(\sigma)\times\Delta^k_{\mathrm{cyc}}} \mathrm{ev}^*_\sigma(\pi^*_1\alpha_1 \wedge \ldots \wedge \pi^*_k\alpha_k)$$

where $\Delta^k_{\mathrm{cyc}} = \{\vec{t} \in [0,1]^k \mid t_1 \leq t_2 \leq \cdots \leq t_k \leq t_1\}$ and

$$\mathrm{ev}_\sigma : K_\sigma \times \Delta^k_{\mathrm{cyc}} \to M^{\times k},$$
$$\mathrm{ev}_\sigma(p, t_1, \ldots, t_k) = (\sigma(p)(t_1), \ldots, \sigma(p)(t_k)).$$

This map is called the *Chen's iterated integrals map.*

String topology conjecture

Formally, we have the following diagram:

$$
\begin{array}{ccccc}
\mathrm{dIBL}(C_\infty(\Omega(M))) & & \text{“}\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega(M)))\text{”} & \xleftarrow{\;I_\lambda^*\;} & (C_*^{\mathrm{sing}}(\mathrm{L}M), \partial) \\
\Big\downarrow \mathfrak{f} & \xrightarrow{\text{twist by } \mathfrak{m}} & \Big\downarrow \mathfrak{f}^{\mathfrak{m}} = (\mathfrak{f}^{\mathfrak{m}}_{klg}) & \swarrow & \\
\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M))) & & \mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M))),\ \mathfrak{n} = \mathfrak{f}_*\mathfrak{m} & &
\end{array}
$$

- The first arrow is the homotopy equivalence $\mathfrak{f} = (\mathfrak{f}_{klg})$ with $\mathfrak{f}_{110} = \iota^*$, $\iota : \mathrm{H}_{\mathrm{dR}}(M) \simeq \mathcal{H}(M) \to \Omega(M)$ from the *homotopy transfer theorem for* IBL_∞*-algebras.*
- The component $\mathfrak{f}^{\mathfrak{m}}_{110}$ of $\mathfrak{f}^{\mathfrak{m}} = (\mathfrak{f}^{\mathfrak{m}}_{klg})$ expands as

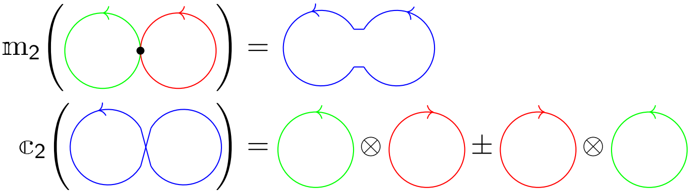
$$
\mathfrak{f}^{\mathfrak{m}}_{110} = \mathfrak{f}_{110} + \mathfrak{f}_{210} \circ_1 \mathfrak{m}_{10} + \frac{1}{2!}\mathfrak{f}_{310} \circ_{1,1} (\mathfrak{m}_{10}, \mathfrak{m}_{10}) + \cdots .
$$

Theorem (in prep. by K. Cieliebak & E. Volkov). Let M be a closed or. Riem. manifold and G a Green kernel. Then the composition

$$\mathfrak{f}^{\mathfrak{m}}_{110} \circ I^*_\lambda : (C^{\mathrm{sing}}_*(M), \partial) \longrightarrow (C(\mathrm{H}_{\mathrm{dR}}(M)), \mathfrak{q}^{\mathfrak{n}}_{110})$$

is a chain map which if $\pi_1(M) = 1$ satisfies the following:

- It induces an iso. $\mathrm{H}^{\mathbb{S}^1}_*(\mathrm{L}M, \{*\}; \mathbb{R}) \simeq \mathrm{H}_*(C_{\mathrm{red}}(\mathrm{H}_{\mathrm{dR}}(M)), \mathfrak{q}^{\mathfrak{n}}_{110})$.
- The IBL-structure on $\mathrm{H}^{\mathbb{S}^1}_*(\mathrm{L}M, \{*\}; \mathbb{R})$ induced by $\mathfrak{q}_{210}$, $\mathfrak{q}^{\mathfrak{n}}_{120}$ is compatible with the Chas-Sullivan operations $\mathbb{m}_2$, $\mathbb{c}_2$.



Questions about $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$

1. M formal, $\pi_1(M) = 1 \overset{?}{\Longrightarrow} \mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ and $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ homotopy equivalent IBL_∞-algebras.
2. Computation of $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ for surfaces $M = \Sigma_g$.
3. Is the *standard Green operator* $\mathcal{G}_{\mathrm{std}} := \mathrm{d}^* \mathcal{G}_\Delta$, $\Delta \mathcal{G}_\Delta = \mathbb{1} - \pi_{\mathcal{H}}$ a canonical Green operator satisfying (G1)–(G5)?
4. Define a "weak, non-reduced IBL_∞-algebra" based on $\Omega(M; \mathfrak{g})$ and study its relation to perturbative Chern-Simons theory with a gauge group G within the BV-formalism.

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4. Define a "weak, non-reduced IBL_∞-algebra" based on $\Omega(M;\mathfrak{g})$ and study its relation to perturbative Chern-Simons theory with a gauge group G within the BV-formalism.
5. IBL_∞-algebra on cyclic cochains of H_{dR}, in symplectic field theory (K. Cieliebak, J. Latschev) and in open-closed string field theory (K. Münster, I. Sachs). Are they related and how?

Thank you for your attention.