Multi-centered higher spin solutions and $\mathcal{W}_N$ conformal blocks

Introduction to higher spin gravity

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3d gravity as a toy model

- Dynamics of general gravitational theories can be complicated.
- 3*d* offers drastic simplification as the theory is topological
- Still has enough complexity do study phenomena such as black holes and their entropy, etc.

3d gravity and string-theoretic embedding

- Tensionless limit of string theory in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$ exhibits a huge amount of gauge symmetry.
- The massless art of the spectrum is governed by Chern Simons Higher spin theory
- There are however scalars coupled to the HS CS theory too.

$$m^2 \sim \frac{1}{\alpha'} \to 0 \qquad m^2 \sim \frac{1}{\ell}$$

Metric formulation

The metric formulation of gravity is defined via action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^3 x \sqrt{-g}\,(R + 2\Lambda)$$

basic dynamical field is the metric g.

- Doesn't have straightforward higher spin generalization.
- It's hard to quantize.

We are going to restrict ourselves to three dimensional asymptotically AdS solutions which have $\Lambda < 0$ and metric can be cast in the following form

$$g = \mathrm{d}\rho^2 - \mathrm{e}^{-2\rho}\mathrm{d}x_+\mathrm{d}x_- + T\mathrm{d}x_+^2 + \bar{T}\mathrm{d}x_-^2 - \mathrm{e}^{2\rho}T\bar{T}\mathrm{d}x_+\mathrm{d}x_-$$

Chern-Simons formulation

In $3d$ gravity is classically equivalent to chern Simons theory for special choice of the strucure group. For $\Lambda < 0$ we get

$$S = \frac{k}{4\pi} \int \mathrm{Tr}\left[A \wedge \mathrm{d}A + A \wedge A \wedge A\right] - \frac{k}{4\pi} \int \mathrm{Tr}\left[\bar{A} \wedge \mathrm{d}\bar{A} + \bar{A} \wedge \bar{A} \wedge \bar{A}\right]$$

with structure group $\mathrm{SL}(2, \mathbb{R})$.
Different topologies and choices of cosmological constant are reflected in the choice of the structure group.

- The connection to metric formulation $k = \frac{\Lambda}{4G}$
- Classical equivalence is apparent from Einstein EoM being equivalent to flatness condition on $F, \bar{F}$.
- Quantum mechanical equivalence is still unclear?

Structure group

Different choices of signature and cosmological constant have different structure group.

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
L	$\mathrm{SO}(2,2)$	$\mathrm{ISO}(1,2)$	$\mathrm{SO}(1,3)$
E	$\mathrm{SO}(1,3)$	$\mathrm{ISO}(3)$	$\mathrm{SO}(4)$

Comparison of two formulations

Once can go back and forth between two formulations and connect the fundamental fields and EoM's.

$$A = J_a A^a_m \mathrm{d}x^m \qquad g_{mn} = \mathrm{tr}(A_m - \bar{A}_m)(A_n - \bar{A}_n)$$
$$\bar{A} = \bar{J}_a \bar{A}^a_m \mathrm{d}x^m$$

$$F = 0 \quad \bar{F} = 0 \qquad R_{mn} - \frac{1}{2} R g_{mn} + \Lambda g_{mn} = 0$$

Going the other way is exactly representing the metric in terms of vielbein and further combination with spin connection.

3d gravity and higher spin generalization

Having defined Chern Simons theory for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ one can straightforwardly generalize to the case $\mathrm{SL}(N,\mathbb{R}) \times \mathrm{SL}(N,\mathbb{R})$ which describes three dimensional theory with fields of higher spin. To single out a gravitational subsector one needs to specify an embedding

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(N,\mathbb{R}) \times \mathrm{SL}(N,\mathbb{R})$$

There is a way to describe infinitely many fields when algebra $hs[\lambda]$ is being used instead of $\mathrm{SL}(N,\mathbb{R})$.

$$\frac{U(\mathfrak{sl}_2(\mathbb{R}))}{\langle C_2 - \lambda(\lambda - 1)\rangle}$$

Higher spin metric-like variables

There is similar relation with connection like fields and metric like fields in higher spin case.

$$g_{mn} = \mathrm{tr}(A_m - \bar{A}_m)(A_n - \bar{A}_n)$$
$$\phi_{mnr} = \mathrm{tr}(A_m - \bar{A}_m)(A_n - \bar{A}_n)(A_r - \bar{A}_r)$$

It's useful to write the gauge connection as

$$A = J_a A^a_m \mathrm{d}x^m + A^{l,m}_m W_{l,m} \mathrm{d}x^m$$

where we have used the decomposition of the structure group with respect to its embedded $\mathrm{SL}(2,\mathbb{R})$.

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \left(\bigoplus_l V_l \right)$$

Boundary conditions

In coordinates (ρ, x_+, x_-) we can describe asymptotically AdS_3 solutions we need to prescribe following boundary conditions

$$A_-| = 0 \qquad \bar{A}_+| = 0$$

and furthermore

$$A = A_{\mathrm{AdS}} + \mathcal{O}(r^0)$$
$$\bar{A} = \bar{A}_{\mathrm{AdS}} + \mathcal{O}(r^0)$$

Highest-weight gauge

One can use residual gauge freedom to cast the connection into following form

$$A = bab^{-1} + \mathrm{d}bb^{-1}$$

where

$$a = \begin{pmatrix} 0 & T\mathrm{d}x^{+} & 2W\mathrm{d}x^{+} \\ 2\mathrm{d}x^{+} & 0 & 2T\mathrm{d}x^{+} \\ 0 & \mathrm{d}x^{+} & 0 \end{pmatrix}$$

and

$$b = \mathrm{e}^{\rho L_0}$$

this gauge choice has an advantage of direct connection to the CFT data via the currents.
This gauge choice gives rise to the metric mentioned previously.

Toda gauge

In our investigation of multi-particle solutions another gauge choice turns out to be useful. In pure gravity case

$$\begin{pmatrix} \frac{i}{2}\partial\phi \mathrm{d}z - \frac{i}{2}\bar{\partial}\phi \mathrm{d}\bar{z} & e^{-\phi}\mathrm{d}z \\ e^{-\phi}\mathrm{d}\bar{z} & -\frac{i}{2}\partial\phi \mathrm{d}z + \frac{i}{2}\bar{\partial}\phi \mathrm{d}\bar{z} \end{pmatrix}$$

and in $\mathrm{SL}(3,\mathbb{R})$ case

$$\begin{pmatrix} *\mathrm{d}\phi_1 & e^{-2\phi_1+\phi_2}\mathrm{d}z & 0 \\ e^{-2\phi_1+\phi_2}\mathrm{d}\bar{z} & -*\mathrm{d}\phi_1 + *\mathrm{d}\phi_2 & e^{\phi_1-2\phi_2}\mathrm{d}z \\ 0 & e^{\phi_1-2\phi_2}\mathrm{d}\bar{z} & -*\mathrm{d}\phi_2 \end{pmatrix}$$

The flatness condition is now equivalent to Liovuille/Toda field theory EoM's.

Particle coupled to the gravity

The usual way the particle is coupled to a gravitational field is

$$S[\gamma; g] = m \int_{\gamma} \mathrm{d}s\,,$$

which gives rise to usual geodetic equation. However, this is insufficient for particle with spin where dynamics is governed by Mathisson-Papapetrou-Dixon equation and further degrees of freedom are needed.

Wilson line as a path ordered exponential

In Chern-Simons theory the only observables are *Wilson lines*.

$$W_{\mathcal{R}}(\gamma) = \mathrm{Tr}_{\mathcal{R}}\left[\mathcal{P}\exp\int_{\gamma} A\right]$$

the wilson line is characterized by a representation we are taking the trace in.

Wilson line as an auxiliary 1dim QM system

One can define an auxiliary one dimensional system and represent a Wilson line by a path integral.

$$\int \mathcal{D}U\mathcal{D}P \mathrm{e}^{\int_\gamma PU^{-1}DU+F(P)}$$

The 'F-term' encodes the information about the underlying representation. For pure gravity

$$\lambda\left(\mathrm{Tr}P^2 - c_2\right)$$

In higher spin case, higher Casimirs are included.

Wilson line as an auxiliary 1dim QM system

As an example we can take $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Two quadratic casimirs have to be specified.

$$h + \bar{h} = m \qquad h - \bar{h} = s$$

Backreaction to the gravity

The presence of the Wilson line introduce sources for the connection.

$$\frac{k}{4\pi}F_{mn} = \frac{\mathrm{i}}{2}\epsilon_{mnr}\int \mathrm{d}x^r P\delta^3(x - x(s))$$

When toda gauge is invoked

$$\Delta\phi + \mathrm{e}^{-2\phi} = 4\pi G m\delta(z - z_0)$$

Holography

Due to the existence of conserved currents of the Toda system, one can in principle reconstruct the whole bulk solution from the knowledge of the CFT currents.

$$T_{\mathrm{HW}}(x_+) = -1 + \mathrm{e}^{2\mathrm{i}x_+} T_{\mathrm{Toda}}(\mathrm{e}^{\mathrm{i}x_+})$$
$$W_{\mathrm{HW}}(x_+) = \mathrm{e}^{3\mathrm{i}x_+} W_{\mathrm{Toda}}(\mathrm{e}^{\mathrm{i}x_+})$$

Conclusions

- We have discussed generalization of three dimensional gravity.
- Wilson lines as probes of our system
- Two gauges have been considered. HW gauge and 'Toda' gauge.
- For applications in multi-centered solutions see O. Vasilakis' talk.

Thank you!