Noether's theorem and conserved currents in Covariant Classical and Quantum Mechanics

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Motivation

Last year we celebrated 100 years from publication of the results known as "Noether Theorems". These results was published in the paper

Noether, Emmy (1918). "Invariante Variationsprobleme". Nachr. D. König. Gesellsch. D. Wiss. Zu Göttingen, Math-phys. Klasse. 1918: 235–257.

In my lecture I would like to demonstrate one application of the 2nd Noether's theorem in Covariant Classical and Quantum Mechanics.

For review of possible applications of results of above paper I recommend the monograph

Kosmann-Schwarzbach, Yvette (2010). The Noether theorems: Invariance and conservation laws in the twentieth century. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag. ISBN 978-0-387-87867-6.

Covariant Classical and Quantum Mechanics

Covariant Classical Mechanics (CCM) and **Covariant Quantum Mechanics** (CQM) are geometric approaches to **Classical Mechanics** and **Quantum Mechanics** on a curved spacetime fibred over absolute time and equipped with a riemannian metric on its fibres.

The main features of CCM and CQM are:

- CCM and CQM are *fully covariant*,

- CCM and CQM skip the *spacetime lorentzian metric* and replaces it with a *spacelike riemannian metric*,

- CCM and CQM implement the *principle of General Relativity*,

- CCM and CQM fit *Standard Classical and Quantum Mechanics* in the flat case.

1. General formulation of the 2nd Noether's theorem

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, with $\dim \boldsymbol{B} = m$ and $\dim \boldsymbol{F} = n = m + l$, equipped with a scaled volume form of the base space $\upsilon : \boldsymbol{B} \to \mathbb{U} \otimes \wedge^m T^* \boldsymbol{B}$. We denote the typical fibred charts of $p : \boldsymbol{F} \to \boldsymbol{B}$ by (x^λ, y^i).

Moreover, let us consider a 1st-order lagrangian form

$$\mathsf{L} = \mathfrak{l}\, \upsilon \in \sec(J_1 \boldsymbol{F}, \wedge^m T^* \boldsymbol{B}) , \qquad \text{where} \qquad \mathfrak{l} \in \operatorname{map}(J_1 \boldsymbol{F}, \mathbb{U}^* \otimes \mathbb{R}) ,$$

and the associated Poincaré–Cartan form

$$\Theta = \mathsf{L} + \Pi \in \sec(J_1 \boldsymbol{F}, \wedge^m T^* \boldsymbol{F}) ,$$

with coordinate expression

$$\Theta = \mathfrak{l}\, \upsilon + \partial^\lambda_i \mathfrak{l}\, (d^i - y^i_\mu\, d^\mu) \wedge \upsilon_\lambda , \qquad \text{where} \qquad \upsilon_\lambda := i_{\partial_\lambda} \upsilon .$$

Let us consider a vector field $Y \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ and its holonomic (jet) prolongation

$$Y_1 := r_1 \circ J_1 Y \in \mathrm{pro}_{\boldsymbol{F}}(J_1\boldsymbol{F}, TJ_1\boldsymbol{F}) \,,$$

where $r_1 : J_1 T\boldsymbol{F} \to TJ_1\boldsymbol{F}$ is the natural transformation.

2nd Noether's theorem: If Y is an infinitesimal symmetry of L, i.e. if $L_{Y_1}\mathsf{L} = 0$, then the form, called the **current** associated with Y,

$$\mathfrak{j}[Y] := i_{Y_1}\Theta = i_Y\Theta \in \sec(J_1\boldsymbol{F}, \wedge^{m-1}T^*\boldsymbol{F})$$

turns out to be **conserved** along the sections $s : \boldsymbol{B} \to \boldsymbol{F}$ which are solutions of the Euler–Lagrange equation (“**critical sections**”), hence the **current form** $(j_1 s)^*\mathfrak{j}[Y] \in \sec(\boldsymbol{B}, \wedge^{m-1}T^*\boldsymbol{B})$ is closed, i.e.

$$d\big((j_1 s)^*\mathfrak{j}[Y]\big) = 0 \,.$$

Indeed, in the context of Covariant Classical Mechanics and Covariant Quantum Mechanics, respectively, we get in a covariant way classical and quantum lagrangians and we are able to classify their infinitesimal symmetries (generated by the so called **special phase functions** f). As a consequence we obtain classical and quantum currents conserved on critical sections.

2. Covariant Classical Mechanics

Spaces of scales

Covariance involves **equivariance with respect to coordinates, observers, gauges and scale bases** as well, on the same footing. For this reason, we incorporate the spaces of scales into geometric objects, by a rigorous mathematical procedure.

The basic "**positive spaces**" in the theory are:

$$\begin{aligned} \mathbb{T} &= \text{space of time scales}\,, \\ \mathbb{L} &= \text{space of length scales}\,, \\ \mathbb{M} &= \text{space of mass scales}\,. \end{aligned}$$

In order to account for objects with **scaled dimensions**, these space will be possibly tensorialised with standard tensors, so yielding scaled tensors.

We denote as $\hbar = \hbar_0 \, u^0 \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ the **Planck constant**.

Spacetime

Absolute time is an affine space $\boldsymbol{T}$ of dimension 1 with the associated vector space $\mathbb{R} \otimes \mathbb{T}$.

Spacetime is an oriented manifold of dimension 4, fibred over time

$$t : \boldsymbol{E} \to \boldsymbol{T}\,.$$

The fibred coordinate charts will be $(x^\lambda) = (x^0, x^i)$.

Time 1-form is $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

Motions are section $s : \boldsymbol{T} \to \boldsymbol{E}$.

Classical fields

1) The **metric** is a scaled spacelike riemannian metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}) \,.$$

2) The **gravitational field** is a torsion free linear connection of spacetime (**galilean connection**)

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \,,$$

such that

$$\nabla^\natural dt = 0 \,, \qquad \nabla^\natural g = 0 \,, \qquad R^\natural{}_{i\lambda j\mu} = R^\natural{}_{j\mu i\lambda} \,.$$

3) The **electromagnetic field** is a scaled spacetime 2–form

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*\boldsymbol{E} \,,$$

such that
$$dF = 0 \,.$$

Coordinate expression

The gravitational connection $K^\natural$ is determined by the metric g and an observer dependent and gauge dependent potential $A^\natural : \boldsymbol{E} \to T^*\boldsymbol{E}$

$$K^\natural{}_\lambda{}^0{}_\mu = 0\,,$$
$$K^\natural{}_h{}^i{}_k = K^\natural{}_k{}^i{}_h = -\tfrac{1}{2}\, g^{ij}\,(\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk})\,,$$
$$K^\natural{}_h{}^i{}_0 = K^\natural{}_0{}^i{}_h = -\,\tfrac{1}{2}\; g^{ij}\,(\partial_0 g_{hj} + \partial_h A_j - \partial_j A_h)\,,$$
$$K^\natural{}_0{}^i{}_0 = -g^{ij}\,(\partial_0 A_j - \partial_j A_0)\,,$$

Comment: In fact time-preserving and metric-preserving linear connection is given by a 2-form $\Phi^\natural$. The condition $R^\natural{}_{i\lambda j\mu} = R^\natural{}_{j\mu i\lambda}$ is equivalent with $d\Phi^\natural = 0$, so locally $\Phi^\natural = 2\,dA^\natural$.

Joined spacetime connection

It is possible to incorporate, in a covariant way by a **minimal coupling**, the electromagnetic field into gravitational field $K^\natural$ preserving the properties of $K^\natural$ to be **galileian**

$$K = K^\natural - \tfrac{1}{2}\,\tfrac{q}{m}\,(dt \otimes \hat{F} + \hat{F} \otimes dt)\,, \qquad \text{with} \qquad \hat{F} := g^{\sharp 2}(F)\,,$$

where $m \in \mathbb{M}\,, \quad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ are **mass** and **charge** of a particle.

This **joined spacetime connection** K yields several classical and quantum objects, which split into gravitational and electromagnetic components.

From $dF = 0$ we get the **joined potential** $A = A^\natural + A^\mathfrak{e}$.

From now on, in CCM and CQM, we shall refer to this joined spacetime connection.

Phase space

We consider as **phase space** the space of 1–jets of motions

$$J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}\,.$$

Phase space is **odd dimensional**

$$\dim J_1\boldsymbol{E} = 7$$

with the induced fibred coordinate charts (x^0, x^i, x^i_0).

Joined phase objects

The joined connection K yields in a covariant way the following **objects**, which split into gravitational and electromagnetic components:

1) **phase connection** (affine connection)

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}\,, \qquad \Gamma_{\lambda 0 \mu}^{\;i0} = K_\lambda{}^i{}_\mu\,,$$

2) **dynamical phase connection**

$$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,, \qquad \gamma_{00}^{i} = K_p{}^i{}_q\, x_0^p\, x_0^q + 2\, K_p{}^i{}_0\, x_0^p + K_0{}^i{}_0\,,$$

3) **dynamical phase 2–form** $\Omega : J_1\boldsymbol{E} \to \wedge^2 T^* J_1\boldsymbol{E}$

$$\Omega = \tfrac{m}{\hbar_0}\, g_{ij}\Big(d_0^i - (K_\lambda{}^i{}_p\, x_0^p + K_\lambda{}^i{}_0)\, d^\lambda\Big) \wedge (d^j - x_0^j\, d^0)\,,$$

4) **dynamical phase 2–vector** $\Lambda : J_1\boldsymbol{E} \to \wedge^2 V J_1\boldsymbol{E}$

$$\Lambda = \tfrac{\hbar_0}{m}\, g^{ij}\Big(\partial_i + (K_i{}^h{}_p\, x_0^p + K_i{}^h{}_0)\, \partial_h^0\Big) \wedge \partial_j^0\,.$$

Cosymplectic structure of phase space

Above phase objects satisfy the following identities

$$\begin{gathered} i_\gamma dt = 1\,, \qquad i_\gamma \Omega = 0\,, \qquad i_{dt}\Lambda = 0\,, \\ d\Omega = 0\,, \qquad dt \wedge \Omega \wedge \Omega \wedge \Omega \not\equiv 0\,, \qquad \gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda \not\equiv 0\,, \\ [\gamma, \Lambda] = 0\,, \qquad [\Lambda, \Lambda] = 0\,. \end{gathered}$$

Thus the phase space turns out to be equipped with a scaled **cosymplectic structure** given by the pair of phase forms (dt, Ω). γ is the scaled **Reeb vector field** and Λ is the **Poisson 2-vector field**.

This cosymplectic structure replaces the symplectic structure or the contact structure of other geometric approaches to Classical and Quantum Mechanics. Really, in GR as the phase space is usually defined to be $T^*\boldsymbol{E}$ with the **canonical symplectic 2–form** or the **observer space** (a part of the unit pseudosphere bundle given by timelike future oriented vectors) with the **contact structure**.

Ω is closed and admits **horizontal phase potentials**

$$A^{\uparrow} \in \sec(J_1\boldsymbol{E},\, T^*\boldsymbol{E})\,, \qquad \text{such that} \qquad \Omega = dA^{\uparrow}\,,$$

$$A^{\uparrow} = -(\tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,g_{ij}\,x^i_0\,x^j_0 - A_0)\,d^0 + (\tfrac{m}{\hbar_0}\,g_{ij}\,x^j_0 + A_i)\,d^i\,.$$

For an **observer** $o : \boldsymbol{E} \to J_1\boldsymbol{E}$ the **observed potentials** $o^*A^{\uparrow} = A[o]$ of Ω coincide with the observed potentials of the joined spacetime connection K.

Ω is global, gauge independent, observer independent,
$A^{\uparrow}$ is local, gauge dependent, observer independent,
$A[o]$ is local, gauge dependent, observer dependent.

3. Classical current

The cosymplectic phase 2–form Ω yields the **classical lagrangian**

$$\mathcal{L} : J_1\boldsymbol{E} \to T^*\boldsymbol{T}\,, \quad \mathcal{L} = i_{д} A^{\uparrow} = \mathcal{L}_0\, d^0 = \big(\tfrac{m}{\hbar_0}\,\tfrac{1}{2}\, g_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + A_0\big)\, d^0\,,$$

where д is the **contact mapping** $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$

$$д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)\,.$$

$\mathcal{L}$ is local, gauge dependent, observer independent,

The corresponding **Poincaré–Cartan form** is

$$\Theta \equiv A^{\uparrow} = \mathcal{L}_0\, d^0 + \partial^0_i \mathcal{L}_0\, (d^i - x^i_0\, d^0) = (\mathcal{L}_0 - x^i_0\, \partial^0_i \mathcal{L}_0)\, d^0 + \partial^0_i \mathcal{L}_0\, d^i\,,$$

and the **classical current** associated with a vector field $X = X^\lambda\, \partial_\lambda$ on $\boldsymbol{E}$ is the function

$$\mathfrak{c}[X] = X^0\, (\mathcal{L}_0 - x^i_0\, \partial^0_i \mathcal{L}_0) + X^i\, \partial^0_i \mathcal{L}_0\,.$$

Special phase functions

By the Noether's theorem if we wish to find vector fields X such that the corresponding classical currents are constant on critical sections, we have to find infinitesimal symmetries of $\mathcal{L}$. Such symmetries are generated by **special phase functions** (s.p.f.)

$$f = f^0 \, \tfrac{m}{\hbar_0} \, \tfrac{1}{2} \, g_{ij} \, x_0^i \, x_0^j + f^i \, \tfrac{m}{\hbar_0} \, g_{ij} \, x_0^j + \breve{f} \,,$$

with $f^0, \, f^i, \, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \, \mathbb{R})$.

S.p.f. turn out to be the sources of **classical symmetries**, **quantum symmetries**, **classical** and **quantum currents**.

Theorem: S.p.f. yield a map (**tangent lift**)

$$X : \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f] \,,$$

$$X[f] = f^0 \, \partial_0 - f^i \, \partial_i \,.$$

Holonomic and hamiltonian phase lifts

$$X^{\uparrow}{}_{\mathrm{hol}}[f] := r_1 \circ J_1 X[f] \qquad : J_1 \boldsymbol{E} \to T J_1 \boldsymbol{E} \,,$$
$$X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) : J_1 \boldsymbol{E} \to T J_1 \boldsymbol{E} \,.$$

Here $f'' = f^0 \, \partial_0$ is the "time" component of f. We see that the hamiltonian lift is a **modification of the standard Jacobi lift** where we use only time component of s.p.f. f. In coordinates

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0 \, \partial_0 - f^i \, \partial_i - (\partial_0 f^i + \partial_j f^i \, x^j_0 + \partial_0 f^0 \, x^i_0 + \partial_j f^0 \, x^j_0 \, x^i_0) \, \partial^0_i \,,$$

$$\begin{aligned} X^{\uparrow}{}_{\mathrm{ham}}[f] = {} & f^0 \, \partial_0 - f^i \, \partial_i \\ & + \tfrac{\hbar_0}{m} \, g^{ij} \Big(- f^0 \, (\partial_0 \mathcal{P}_j - \partial_j A_0) + f^h \, (\partial_h \mathcal{P}_j - \partial_j A_h) \\ & \qquad + \partial_j f^0 \, \tfrac{1}{2} \, \tfrac{m}{\hbar_0} \, g_{hk} \, x^h_0 \, x^k_0 + \partial_j f^h \, \tfrac{m}{\hbar_0} \, g_{hk} \, x^k_0 + \partial_j \breve{f} \Big) \, \partial^0_i \,. \end{aligned}$$

Special phase Lie bracket

Theorem: S.p.f. are closed w.r.t. the **special phase Lie bracket**

$$\begin{aligned} [\![f, \acute{f}]\!] &= \{df, d\acute{f}\} + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f \\ &= \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f \,. \end{aligned}$$

We see that the special phase Lie bracket is a modification of the Jacobi bracket used in contact geometry. But here we use only the time components of f, $\acute{f}$ to unscale the Reeb vector field.

So, we have the **Lie algebra of s.p.f.** $(\mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), [\![\ ,\ ]\!])$.

Remark: S.p.f. f can be identified with the pair $(X[f], \breve{f})$. The special phase Lie bracket then corresponds to the **bracket of pairs**

$$\big[(X, \breve{f}), (\acute{X}, \acute{\breve{f}})\big]_{dA} = \big([X, \acute{X}], X.\acute{\breve{f}} - \acute{X}.\breve{f} + dA(X, \acute{X})\big) \,.$$

Theorem: For $f, \acute{f} \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ we have

$$X^{\uparrow}{}_{\mathrm{hol}}\big[\,[\![\, f, \acute{f} \,]\!]\,\big] = \big[X^{\uparrow}{}_{\mathrm{hol}}[f], X^{\uparrow}{}_{\mathrm{hol}}[\acute{f}]\big] \,.$$

For $f, \acute{f} \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ we have

$$X^{\uparrow}{}_{\mathrm{ham}}\big[\,[\![\, f, \acute{f} \,]\!]\,\big] = \big[X^{\uparrow}{}_{\mathrm{ham}}[f], X^{\uparrow}{}_{\mathrm{ham}}[\acute{f}]\big] \,.$$

Distinguished subalgebras in the Lie algebra of s.p.f.

The sheaf of s.p.f. with the special Lie bracket is a Lie algebra with the following subalgebras:

of projectable s.p.f. $:= \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) := \big\{ f \mid \partial_j f^0 = 0 \big\}$.

of time preserving s.p.f. $:= \operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) := \big\{ f \mid \partial_\lambda f^0 = 0 \big\}$,

of affine s.p.f. $:= \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) := \big\{ f \mid f^0 = 0 \big\}$,

of spacetime s.p.f. $:= \operatorname{map}(\boldsymbol{E}, \mathbb{R}) := \big\{ f \mid f^\lambda = 0 \big\}$.

Conserved special phase functions

Conserved phase functions are phase functions such that

$$\gamma . f = 0 .$$

Theorem: The conserved s.p.f. constitute an $\mathbb{R}$–Lie subalgebra $\mathrm{cns}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$ and

$$X^{\uparrow}{}_{\mathrm{ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f] .$$

In coordinates

$$\begin{aligned}
1)& \qquad 0 = \partial_i f^0 , \\
2)& \qquad 0 = \tfrac{m}{\hbar_0} \left(\partial_0 f^0 \, g_{hk} - f^0 \, \partial_0 g_{hk} + f^i \, \partial_i g_{hk} + \partial_h f^i \, g_{ik} + \partial_k f^i \, g_{ih} \right) , \\
3)& \qquad 0 = \partial_h \breve{f} - f^0 \, (\partial_0 A_h - \partial_h A_0) + f^i \, (\partial_i A_h - \partial_h A_i) + \partial_0 f^i \, \tfrac{m}{\hbar_0} \, g_{ih} , \\
4)& \qquad 0 = \partial_0 \breve{f} - f^i \, (\partial_0 A_i - \partial_i A_0) .
\end{aligned}$$

Hence f is projectable and $X[f]$ is an infinitesimal symmetry of $\frac{m}{\hbar}\, g$.

Phase lift of conserved s.p.f.

$$\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})\,.$$

$$X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f]\,.$$

Classical symmetries

Theorem : The $\mathrm{I\!R}$–Lie algebra of infinitesimal symmetries $X^{\uparrow} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ of Ω is constituted by the phase lifts of **conserved s.p.f.**

$$X^{\uparrow}[f] : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,, \qquad \text{with} \qquad f \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})\,.$$

Moreover, the $\mathrm{I\!R}$–Lie algebra of infinitesimal symmetries of the cosymplectic structure (dt, Ω) is constituted by the phase lifts of **conserved time preserving s.p.f.**

Infinitesimal symmetries of the classical lagrangian

Finally, we are interested in infinitesimal symmetries of the classical lagrangian $\mathcal{L}$.

Theorem:

$$L_{X^{\uparrow}_{\mathrm{hol}}} \mathcal{L} = 0 \qquad \text{if and only if} \qquad L_{X^{\uparrow}_{\mathrm{hol}}} A^{\uparrow} = 0\,.$$

So, the Lie algebra of infinitesimal symmetries of the classical lagrangian coinsides with the Lie algebra of infinitesimal symmetries of the potential of the phase 2-form Ω. But any infinitesimal symmetry of $A^{\uparrow}$ is an infinitesimal symmetry of $\Omega = dA^{\uparrow}$. Hence the Lie algebra of infinitesimal symmetries of $A^{\uparrow}$ is a Lie subalgebra in the Lie algebra of infinitesimal symmetries of Ω which is generated by the Lie subalgebra in the Lie algebra of conserved s.p.f..

Theorem: The Lie algebra of infinitesimal symmetries of the classical lagrangian $\mathcal{L}$ is a subalgebra in the Lie algebra of infinitesimal symmetries of Ω and is constituted by the subalgebra in $\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ given by

$$d\breve{f} = -d(i_{X[f]}A)\,. \tag{A}$$

So, infinitesimal symmetries of $\mathcal{L}$ are generated by s.p.f. (given up to a constant) of the type

$$f = f^0\,\tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,g_{ij}\,x_0^i\,x_0^j + f^i\,\tfrac{m}{\hbar_0}\,g_{ij}\,x_0^j - f^0\,A_0 + f^i\,A_i\,,$$

where conditions 1) – 4) are satisfied.

$$\begin{array}{ccccc}
\operatorname{sym}(\mathcal{L}) & \subset & \operatorname{sym}(\Omega) & \subset & \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \\
\updownarrow \text{1–1} & & \updownarrow \text{1–1} & & X^{\uparrow}{}_{\text{hol}} \Uparrow X^{\uparrow}{}_{\text{ham}} \\
\operatorname{cns}\operatorname{spe}_{(A)}(J_1\boldsymbol{E}, \mathbb{R}) & \subset & \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) & \subset & \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})
\end{array}$$

Theorem: For each $f \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ satisfying the condition (A) and each critical motion $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$, the time function (**classical current form**)

$$\mathfrak{c}[f](s) := (j_1 s)^* \mathfrak{c}[X[f]] \in \operatorname{map}(\boldsymbol{T}, \mathrm{I\!R}) ,$$

$$\mathfrak{c}[f](s) = f^0 \, (\tfrac{1}{2} \, G^0_{ij} \, \partial_0 s^i \, \partial_0 s^j - A_0) + f^i \, (G^0_{ij} \, \partial_0 s^j + A_i)$$

turns out to be constant.

Remark: Let us note that the critical sections are solutions of the Euler–Lagrange equations and in the classical case they coincides with sections satisfying

$$\partial_{00} s^i = \gamma^i_{00} \circ s = K_p{}^i{}_q \, \partial_0 s^p \, \partial_0 s^q + 2 \, K_0{}^i{}_p \, \partial_0 s^p + K_0{}^i{}_0 \, .$$

4. Covariant Quantum Mechanics

Quantum bundle is a 1–dimensional complex vector bundle *over spacetime*

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}$$

Upper quantum bundle is the quantum bundle with extended base space

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E} \,.$$

A quantum state is described by a **quantum section**

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q} \,.$$

Quantum objects

1) **Hermitian quantum metric** is a hermitian metric of $\boldsymbol{Q}$ with values into vertical spacelike volume forms

$$\mathfrak{h}_\eta : \boldsymbol{E} \to (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^* \otimes \wedge^3 V^* \boldsymbol{E}) \otimes \mathbb{C}\,.$$

2) **Upper quantum connection** is a linear connection of the vector bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$

$$\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow\,,$$

such that: a) it is **hermitian** $\nabla^\uparrow \mathfrak{h}_\eta = 0\,,$
b) it is **reducible** $(\text{Ч}_i^0 = 0)\,,$
c) its **curvature** is $R^\uparrow = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow\,.$

Coordinate expression

$$\text{Ч}^\uparrow = d^\lambda \otimes \partial_\lambda + d_0^i \otimes \partial_i^0 + \mathfrak{i}\,(A^\uparrow{}_\lambda\, d^\lambda) \otimes \mathbb{I}^\uparrow\,.$$

Dynamical objects

The hermitian quantum metric h_η and the upper quantum connection $Ч^\uparrow$ yield, in a covariant way, all main quantum dynamical objects. The **quantum lagrangian** is

$$\mathsf{L} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \wedge^4 T^* \boldsymbol{E}) \,,$$

$$\mathsf{L}(\Psi) := -\, dt \wedge \big(\mathrm{i}\mathrm{m}\, \mathsf{h}_\eta(\Psi,\, д \lrcorner \nabla^\uparrow \Psi) + \tfrac{1}{2} \,(\tfrac{\hbar}{m}\, \bar{g} \otimes \mathsf{h}_\eta)(\check{\nabla}^\uparrow \Psi,\, \check{\nabla}^\uparrow \Psi)\big) \,,$$

$$\mathsf{L}(\Psi) = \tfrac{1}{2} \,\big(- \tfrac{\hbar_0}{m}\, g^{ij}\, \partial_i \bar{\psi}\, \partial_j \psi + \mathfrak{i}\, A^\lambda_0\, (\bar{\psi}\, \partial_\lambda \psi - \psi\, \partial_\lambda \bar{\psi}) + (2\, A_0 - A_i\, A^i_0)\big)\, \upsilon^0 \,.$$

The **quantum Poincaré–Cartan form** is

$$\Theta[\mathsf{L}] := \mathsf{L} + \vartheta \mathbin{\bar{\wedge}} V_{\boldsymbol{Q}} \mathsf{L} : J_1 \boldsymbol{Q} \to \wedge^4 T^* \boldsymbol{Q} \,,$$

$$\begin{aligned} \Theta[\mathsf{L}] = {} & \tfrac{1}{2}\, \mathfrak{i}\, (\bar{z}\, dz - z\, d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2} \,\big(\tfrac{\hbar_0}{m}\, g^{ij}\, (\bar{z}_i\, dz + z_i\, d\bar{z}) \\ & + \mathfrak{i}\, A^i_0\, (\bar{z}\, dz - z\, d\bar{z})\big) \otimes \upsilon^0_j + \big(\tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, g^{ij}\, \bar{z}_i\, z_j + (A_0 - \tfrac{1}{2}\, A_i\, A^i_0) \bar{z}\, z\big)\, \upsilon^0 \,, \end{aligned}$$

where $\upsilon^0_\lambda := i_{\partial_\lambda} \upsilon^0$.

Further, we get the **Schrödinger operator**

$$\mathsf{S}(\Psi) := \tfrac{1}{2}\Big(\text{д} \lrcorner \nabla^{\uparrow}\Psi + \delta_{\text{ч}^{\uparrow}}\big(\mathsf{Q}(\Psi)\big)\Big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})\,,$$

$$\begin{aligned}
\mathsf{S}_0(\psi) =& \nabla_0\psi + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi - \mathfrak{i}\,\tfrac{1}{2}\,\Delta_0\psi \\
=& \partial_0\psi - \tfrac{1}{2}\,\mathfrak{i}\,\tfrac{\hbar_0}{m}\,g^{ij}\,\partial_{ij}\psi - (A_0^j + \tfrac{1}{2}\,\mathfrak{i}\,\tfrac{\hbar_0}{m}\,\frac{\partial_i(g^{ij}\,\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi \\
& + \tfrac{1}{2}\,\Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\,\sqrt{|g|})}{\sqrt{|g|}} - \mathfrak{i}\,\,(2\,A_0 - A_i\,A_0^i)\Big)\,\psi\,.
\end{aligned}$$

Remark: Let us note that solutions of the S.e. $\mathsf{S}(\Psi) = 0$ coincide with solutions of the E.-L.e. $\mathcal{E}[\mathsf{L}](\Psi) = 0$ (**(quantum) critical sections**).

Quantum and upper quantum lifts of s.p.f.

We assume the pullback $o^* Ч^\uparrow = Ч[o]$ and obtain the hermitian connection on $\boldsymbol{Q}$. Then we obtain injective morphisms of Lie algebras

$$Y_\eta : \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{Q}, T\boldsymbol{Q}) \,.$$

$$\begin{aligned} Y_\eta[f] &= Ч[o]\big(X[f]\big) + (\mathfrak{i}\, \breve{f} - \tfrac{1}{2} \operatorname{div}_\eta X[f])\, \mathbb{I} \\ &= f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\, (\breve{f} + f^0\, A_0 - f^i\, A_i) - \tfrac{1}{2} \operatorname{div}_\eta X[f]\big)\, \mathbb{I} \,. \end{aligned}$$

$$Y^\uparrow{}_\eta : \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \,.$$

$$\begin{aligned} Y^\uparrow{}_\eta[f] &= Ч^\uparrow\big(X^\uparrow{}_{\mathrm{hol}}[f]\big) + \big(\mathfrak{i}\, \breve{f} - \tfrac{1}{2} \operatorname{div}_\eta X[f]\big)\, \mathbb{I}^\uparrow \\ &= f^0\, \partial_0 - f^i\, \partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\, \partial^0_i \\ &\quad + \big(\mathfrak{i}\, (\breve{f} + f^0\, A_0 - f^i\, A_i) - \tfrac{1}{2} \operatorname{div}_\eta X[f]\big)\, \mathbb{I}^\uparrow \,. \end{aligned}$$

Quantum symmetries

Theorem: The $\mathbb{R}$–Lie algebra of infinitesimal symmetries $Y^\uparrow : \boldsymbol{Q}^\uparrow \to T\boldsymbol{Q}^\uparrow$ of

$$\mathfrak{h}_\eta : \boldsymbol{E} \to (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*) \otimes \mathbb{C}\,, \qquad \text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$$

is constituted by the upper quantum lifts of conserved s.p.f.

$$Y^\uparrow{}_\eta[f] : \boldsymbol{Q}^\uparrow \to T\boldsymbol{Q}^\uparrow\,, \qquad \text{with} \qquad f \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})\,.$$

The map

$$\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{sym}(\mathfrak{h}_\eta, \text{Ч}^\uparrow) : f \mapsto Y^\uparrow{}_\eta[f]$$

is an isomorphism of $\mathbb{R}$–Lie algebras.

Remark: Quantum symmetries have the same generators as symmetries of the classical cosymplectic 2-form Ω.

Symmetries of the quantum lagrangian

Theorem: Infinitesimal symmetries of L are quantum vector fields $Y = Y_\eta[f]$, with $f \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

In other words, they are the quantum vector fields of the type

$$Y = Y_\eta[f] = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\breve{f} + A_0\,f^0 - A_i\,f^i)\,\mathbb{I}\,,$$

where the functions $f^0\,,\,f^i\,,\,\breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R})$ fulfill conditions

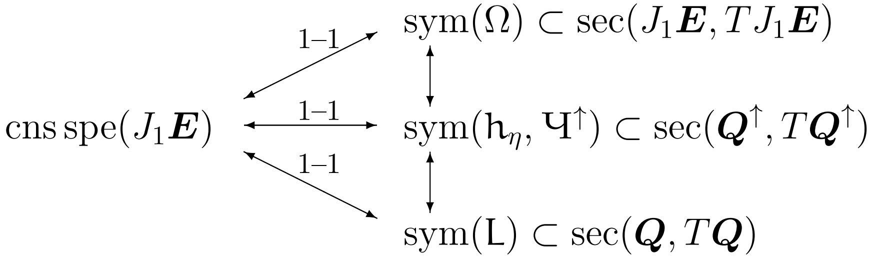
1) $0 = \partial_i f^0\,,$

2) $0 = \frac{m}{\hbar_0}\,\big(\partial_0 f^0\,g_{hk} - f^0\,\partial_0 g_{hk} + f^i\,\partial_i g_{hk} + \partial_h f^i\,g_{ik} + \partial_k f^i\,g_{ih}\big)\,,$

3) $0 = \partial_h \breve{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\,\frac{m}{\hbar_0}\,g_{ih}\,,$

4) $0 = \partial_0 \breve{f} - f^i\,(\partial_0 A_i - \partial_i A_0)\,.$

Remark: So infinitesimal symmetries of L have the same generators as infinitesimal symmetries of the quantum structure and the classical cosymplectic 2-form Ω.



sym(Ω) ⊂ sec(J1 E, T J1 E)
cns spe(J1 E)
1–1
1–1
1–1
sym(hη, Ч↑) ⊂ sec(Q↑, T Q↑)
sym(L) ⊂ sec(Q, T Q)

5. Quantum currents

Quantum current associated with $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$

$$\mathfrak{j}_\eta[f] := -\, i_{Y_{\eta\,1}[f]}\, \Theta[\mathsf{L}] \in \sec(J_1\boldsymbol{Q}, \wedge^3 T^*\boldsymbol{Q})\,.$$

The special phase Lie bracket yields a **Lie bracket of quantum currents**

$$\big[\mathfrak{j}_\eta[f]\,,\,\mathfrak{j}_\eta[\acute{f}]\big] := \mathfrak{j}_\eta\big[\,[\![\, f\,,\,\acute{f}\,]\!]\,\big]\,.$$

Quantum current forms

We define the **quantum current form** associated with $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ to be the spacetime 3–form

$$\mathfrak{j}_\eta[f](\Psi) := (j_1\Psi)^*\mathfrak{j}_\eta[f] \in \sec(\boldsymbol{E}, \wedge^3 T^*\boldsymbol{E})\,.$$

As a consequence of the 2nd Noether's theorem we get

Theorem: For each $f \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ which is a solution of the Schrödinger equation, the associated quantum current form

$$\mathfrak{j}_\eta[f](\Psi) \in \sec(\boldsymbol{E}, \wedge^3 T^* \boldsymbol{E})$$

turns out to be closed, i.e.

$$d\big(\mathfrak{j}_\eta[f](\Psi)\big) = 0\,.$$

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Thank you for your attention.