Compatibility complexes for the Killing equation

arXiv:1805.03751

Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences, Prague

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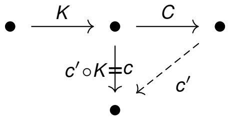
Problem to solve

- Consider a (pseudo-)Riemannian manifold (M, g).
- ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a.
- $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- The Killing equation $K[v]_{ab} = 0$ is an over-determined equation of finite type.
- Given g, what is the full compatibility complex of $K[v]_{ab} = 0$?

$$T^*M \xrightarrow{K} S^2T^*M \xrightarrow{?} \cdots \xrightarrow{?} \cdots$$

- **Def:** C is a compatibility operator for K if $c \circ K = 0 \implies c = c' \circ C$.

$c \circ K = 0$



$C \circ K = 0$

- Complete answer known previously only for constant curvature (Calabi, 1961) and locally symmetric (Gasqui-Goldschmidt, 1983) cases.

Solution strategy and new results

- Possible solutions:
  - General Spencer-Goldschmidt theory. (too cumbersome)
  - Computer algebra. (not enlightening, breaks special structure)
  - Reduction to **canonical form** of a flat connection. (✓)
- Updated list of known cases:
  - constant curvature $\tilde{g}$ (dim $M \geq 2$): $\tilde{R} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$
  - locally symmetric $\nabla_a R_{bcde} = 0$ (not explored after Gasqui-Goldschmidt)
  - homogeneous and isotropic **FLRW** cosmology (dim $M \geq 2$): [arXiv:1805.03751]

$$g = -dt^2 + f(t)^2 \tilde{g}$$

  - spherical **Schwarzschild-Tangherlini** black hole (dim $M \geq 4$): [arXiv:1805.03751]

$$g = -f\,dt^2 + f^{-1}dr^2 + r^2\tilde{g}, \quad f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}$$

  - rotating **Kerr** black hole (dim $M = 4$): work in progress [arXiv:1803.05341]

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Canonical form for PDEs of finite type

- Our solution strategy:
  - reduce the Killing equation $K[v] = 0$ to **canonical form**;
  - in canonical form, the full compatibility complex is known;
  - convert results back to the original Killing equation.
- The Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$ is of **finite type**. There exists an $N < \infty$ such that $v(x), \partial v(x), \ldots, \partial^N v(x)$ determines the solution v uniquely on a neighborhood of x. It is **regular** when $\dim(\ker K)$ is locally constant.
- The **canonical form** for any regular PDE of finite type is $\mathbb{D}u = 0$, where $\mathbb{D}$ is a **flat connection** on a (possibly new) set of fields u:

$$\mathbb{D}_a u^\alpha = \partial_a u^\alpha + \Gamma_{a\beta}^{\alpha} u^\beta = 0, \quad \text{where} \quad [\mathbb{D}_a, \mathbb{D}_b] = 0.$$

- Starting with $\mathbb{D}_0 := \mathbb{D}$, define $\mathbb{D}_p w = \mathbb{D} \wedge w^\alpha$ for any **vector valued** p**-form** w^α. Then $\mathbb{D}_p \circ \mathbb{D}_{p-1} = 0$ is the **de Rham complex twisted by** $\mathbb{D}$; it is a full compatibility complex.
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Remark

The **canonical form** need not always be a **flat connection**. It need only be a PDE with a **known** compatibility complex.

But the **twisted de Rham** complex $\mathbb{D}_p$ is a particularly simple construction.

Reduction to canonical form

Equations $K[v] = 0$ and $\mathbb{D}u = 0$ are **equivalent** only when there are local formulas $v \leftrightarrow u$ that are **bijective on solutions**:

$$K[V_0[u]] \propto \mathbb{D}u = V_1[\mathbb{D}u].$$

The basic idea is that $V_0[u](x) = \sum_\alpha u^\alpha(x)\mathbf{v}_\alpha(x)$ for a complete set of independent solutions $K[\mathbf{v}_\alpha] = 0$. Also,

$$\begin{aligned} \mathbb{D}U_0[v] &\propto K[v] = U_1[K[v]], \\ v - V_0[U_0[v]] &\propto K[v] = H_0[K[v]], \\ u - U_0[V_0[u]] &\propto \mathbb{D}u = H_0'[\mathbb{D}u]. \end{aligned}$$

The relationships between these differential operators are **visually summarized** in the following diagram:

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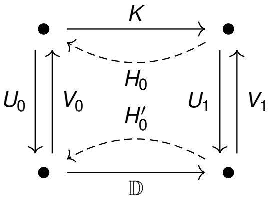
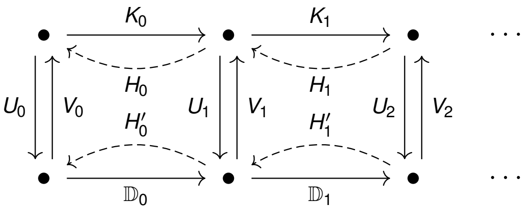


Sufficient conditions for completeness

Two complexes of differential operators, K_i and $\mathbb{D}_i$, are **equivalent up to homotopy** when the diagram



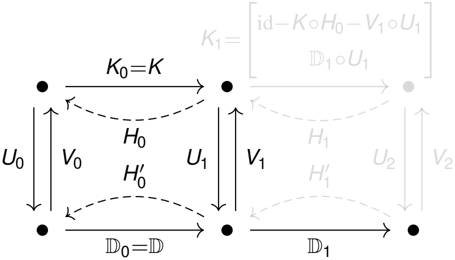
$$\mathrm{id} - V_i \circ U_i = H_i \circ K_i + K_{i-1} \circ H_{i-1},$$
$$\mathrm{id} - U_i \circ V_i = H'_i \circ \mathbb{D}_i + \mathbb{D}_{i-1} \circ H_{i-1}$$

exists, where the solid arrows commute and the dashed arrows are homotopy corrections.

Lemma (homotopy equivalence as witness)

Consider an equivalence up to homotopy between complexes K_i and $\mathbb{D}_i$, $i \geq 0$. Then, if $\mathbb{D}_i$ is a full compatibility complex, then so is K_i.

Lifting compatibility operators



Lemma

*After reduction to **canonical form**:*

(a) $\mathbb{D}_1 \circ U_1[K[v]] = 0$

(b) $(\mathrm{id} - K \circ H_0 - V_1 \circ U_1)[K[v]] = 0$

(d) $\exists$ *compatible* U_2, V_2, H_1, H_1'

(c) *the set (a), (b) is **complete***

Lemma

*When K_1 and K_1' can be **factored through each other**, $K_1' = U' \circ K_1$, $K_1 = U \circ K_1'$, if one is complete then both are complete.*

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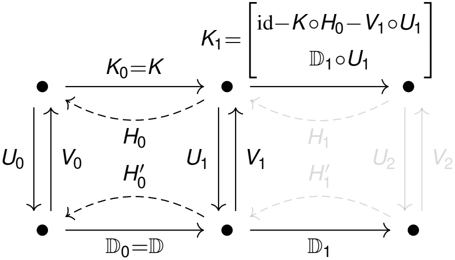

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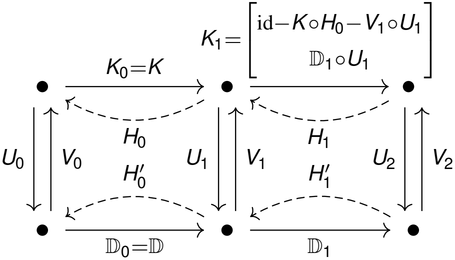

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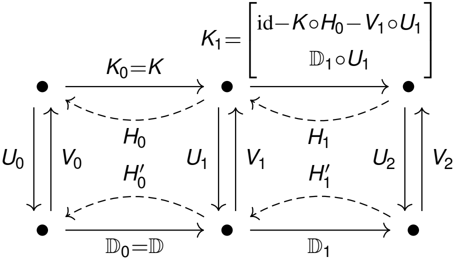

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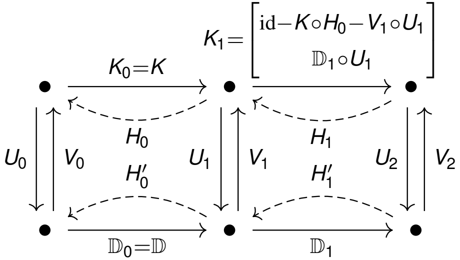

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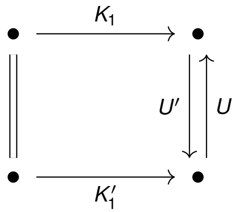

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- Hence, we can **iterate** the argument (simplifying at each step) to get a **full compatibility complex** for K_0, with an explicit **equivalence up to homotopy** with the de Rham complex twisted by $\mathbb{D}$.
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Example: (A)dS

- The simplest example is of a **constant curvature** space, identified by

$$C[g] := R_{abcd}[g] - \alpha(g_{ac}g_{bd} - g_{ad}g_{bc}) = 0.$$

- The Calabi complex (reviewed in [arXiv:1409.7212])

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Example: FLRW cosmology

- For an $(n-1)$-dimensional **constant curvature** Riemannian metric $\tilde{g}$ with $\tilde{R}[\tilde{g}]_{abcd} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$, let

$$g = -(dt)^2 + f^2(t)\tilde{g} \quad (f'(t) \neq 0).$$

- **Parametrize** $v_a = -\boldsymbol{A}\, f(dt)_a + f^2 \tilde{\boldsymbol{X}}_a$ and $h_{ab} = \boldsymbol{p}\,(dt)^2_{ab} + 2f^2(dt)_{(a}\tilde{\boldsymbol{Y}}_{b)} + f^2\tilde{\boldsymbol{Z}}_{ab}$.
- The **Killing operator** $h = K[v]$ becomes

$$\begin{bmatrix} p \\ \hline \tilde{Y} \\ \tilde{Z} \end{bmatrix} = K \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix} = \begin{bmatrix} -2(Af)' \\ \hline \tilde{X}' - f^{-1}\tilde{\nabla} A \\ \tilde{K}[\tilde{X}] + 2Af'\tilde{g} \end{bmatrix} = \left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g} & \tilde{K} \end{array}\right] \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix},$$

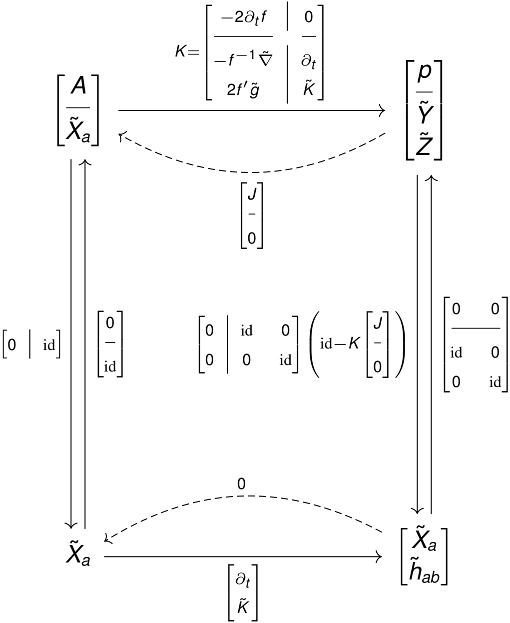
where $(-)' = \partial_t(-)$, while $\tilde{\nabla}$ and $\tilde{K}$ come from $\tilde{g}$.

FLRW: canonical form

- Any **solution** of $K[v] = 0$ has $A = 0$, $\tilde{K}[\tilde{X}] = 0$ and $\partial_t \tilde{X} = 0$.
- There **exists** an operator J such that $J[K[v]] = A$.
- Since $\mathcal{R}' = \partial_t R_{ab}{}^{ab}[g] \neq 0$, we can take

$$J[h] = \frac{1}{f\mathcal{R}'}\dot{\mathcal{R}}[h].$$

- Since we know the constant curvature full compatibility complex for $\tilde{K}$, there is **no need to fully reduce** it to a flat connection.



Discussion

- A **full compatibility complex** K_i can now be constructed for K on any spacetime of sub-maximal (but uniform) symmetry.
- The core of the calculation consists of explicitly identifying integrability conditions of the **Killing equation** $K[v] = 0$ and putting it into the **canonical form** of a **flat connection**.
- The construction outputs a **homotopy equivalence** of K_i with a twisted de Rham complex, which **witnesses** the completeness of each K_i.
- Examples of new method: **FLRW** and **Schwarzschild** (including higher dimensions); **Kerr** (together with Aksteiner, Andersson, Bäckdahl and Whiting).
- **TODO:** extend to other geometries, Myers-Perry, Kerr-Newman, ...
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Thank you for your attention!