Hodge theory, associated bundles and C^*-modules

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Motivation

Geometry - cohomology of sheaves of sections Banach bundles (Illusie, Rohrl, Lempert, Kim)

Determining of cohomology by kernel of Laplacians (harmonic elements)

Non-locality of Quantum Theory – EPR-paradox in Copenhagen interpretation (at least)

Partial Inversions (related e.g. to Lippmann-Schwinger equation in QM-scattering theory)
$|\psi\rangle = |\phi\rangle + (\Delta - E \pm \imath\epsilon \,\mathrm{Id})^{-1} |\psi\rangle$

1. Hodge theory in additive categories

2. Hodge Theory for pre-Hilbert spaces

3. Hodge theory for complexes of C^*-modules

Classical Hodge theory

Suppose

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3) complex of differential operators (not necessary de Rham complex) $d_i : \mathcal{C}^\infty(F^i) \to \mathcal{C}^\infty(F^{i+1})$. Complex is *elliptic* means - symbol of Δ_i is isomorphism out of the zero section of T^*M, where $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$ (**Laplacians**)

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1) $\mathcal{C}^\infty(F^i) \simeq \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^* \oplus \operatorname{Ker} \Delta_i$ (='Hodge theory holds')
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Cohomology groups are Hausdorff

Dagger categories

Dagger category = category $\mathfrak{C}$ with a *dagger functor* $\dagger$ which is a contravariant and idempotent endofunctor on $\mathfrak{C}$ which is identity on objects (preserves objects, reverse direction of morphisms and applied twice it is identity)

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TQFT is a functor from a bord-n category to the monoidal tensor category over a fixed topological vector space

Pseudoinverses on additive dagger categories

Let $\mathfrak{C}$ be an *additive category* (finite two-sided products of objects exist, each set of morphism is an abelian group and compositions are bilinear with respect to the abelian structure $+$: e.g., $h \circ (f + g) = h \circ f + h \circ g$) with *dagger* $\dagger$

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We say that a complex $(E^i, d_i)_i$ in $\mathfrak{C}$ is *Green complex* if for $\Delta_i = d_i^\dagger d_i + d_{i-1} d_{i-1}^\dagger$, there are morphisms g_i and p_i in $\mathfrak{C}$ such that $Id_{E^i} = g_i \Delta_i + p_i = \Delta_i g_i + p_i$, $d_i p_i = 0$ and $d_{i-1}^\dagger p_i = 0$.

Theorem (S. Krysl): If $(E^i, d_i)_i$ is a Green complex in an additive dagger category, then p_i and the pseudoinverses g_i are morphisms of the complex, i.e., $g_{i+1} d_i = d_i g_i$.

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Δ_i are also morphism of complexes

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Theorem (S. Krysl [SK2]): Let $E^\bullet = (E^i, d_i)_i$ be a complex in the category of **Hilbert spaces** and adjointable maps. Then $E^\bullet$ is a Green complex with self-adjoint maps p_i **if and only if** the Hodge theory holds for it.

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Hilbert C^*-modules

Generalization of Hilbert spaces

1) $(R, || \, ||, *)$ a C^*-algebra, i.e., R is associative algebra over complex numbers, $|| \, ||$ is a norm satisfying submultiplicativity and $*$ is an linear idempotent map such that $||aa^*|| = ||a||^2$ and $(R, || \, ||)$ is complete

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3) **Hilbert R-module** $= (H, (,))$ is complex vector space which is a right R-module and $(,) : H \times H \to R$ is R-sesquilinear, hermitian ($(u, v) = (v, u)^*$), positive definite ($(v, v) \in R^+$ and $(v, v) = 0$ implies $v = 0$) and $(H, | \, |)$ is complete, where $|v| = \sqrt{||(v, v)||} \in \mathbb{R}$

Closed images of Laplacians implies Hausdorff cohomologies

Hilbert C^*-bundles = Banach bundles with fibers a fixed Hilbert C^*-module H, and transition maps map into $\mathrm{Aut}_{C^*}(H)$ - the C^*-automorphism group H (linear bijection with $T(hr) = [T(h)]r$)

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Elliptic complexes of Hilbert C^*-bundles

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Assumption on **closed images is 'hard' to verify** in specific cases.

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If H is a Hilbert space, and $R = K(H)$ is its C^*-algebra of compact operators, than Laplacians' images are closed.

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Theorem (S. Krysl, J. Geom. Phys.): Let $(q_i : F^i \to M)_i$ be finitely generated projective $K(H)$-bundles over compact manifold M, and $E^\bullet = (\mathcal{C}^\infty(F^i), d_i)_i$ be an elliptic complex of $K(H)$-invariant pseudodiff. operators. Then the Hodge theory holds for the complex and its cohomology groups are finitely generated projective Hilbert C^*-modules and $H^i(E^\bullet)$ are finitely generated and projective over $K(H)$.

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Underlying result Bakić, Guljaš [BG] ('rigidity' of $K(H)$-modules), and a transfer theorem from Hilbert to pre-Hilbert spaces [SK3].

Example from Spin symplectic geometry

(M^{2n}, ω) symplectic manifold, $Mp(2n, \mathbb{R})$ double cover of the symplectic group (*metaplectic group*), S the complex Segal–Shale–Weil representation ([Shale], [Weil]) on the Hilbert space $H = L^2(\mathbb{C}^n)$, $\mathcal{S}$ the *Shale–Weil bundle* induced by S to the principal bundle of symplectic frames on M.

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Covariant derivatives d_i^{Fed} induced to $\mathcal{S}^i$ by a symplectic (Fedosov) connection ∇^{Fed} need not be pseudodiff. $K(H)$-operators.

Cohomology of symplectic spinors

Theorem (S. Krysl [SK–CMP]): If M is compact than for $d_\bullet^\nabla$ the Hodge theory holds, i.e., $\mathcal{C}^\infty(\mathcal{S}^i) \simeq \operatorname{Im} d_{i-1}^\nabla \oplus \operatorname{Im} d_i^{\nabla *} \oplus \operatorname{Ker} \Delta_i^\nabla$. Further, the cohomological groups are $K(H)$-isomorphic to $H^k_{deRham}(M) \otimes H$, and the images of d_i^∇ and of the Laplacians Δ_i^∇ are closed spaces.

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