# Webs of $\mathcal{W}$-algebras

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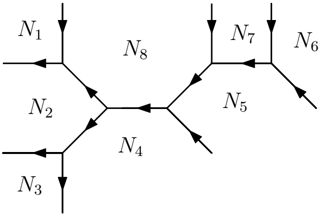
# Plan

- $\mathcal{W}_3$, construction of $\mathcal{W}$-algebras
- $\mathcal{W}_\infty$ as Yangian of affine $\mathfrak{u}(1)$
- Gaiotto-Rapčák gauge theory construction
- Gluing with examples



# W algebras

$\mathcal{W}$-algebas: extensions of the Virasoro algebra by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV, KP)
- (old) matrix models
- instanton partition functions and AGT
- algebras of BPS states in 4d gauge theories
- holographic dual description of 3d higher spin gravities
- quantum Hall effect
- topological strings

Recent: Gaiotto&Rapčák: trivalent junction of D5-NS5-(1,1) bc in Kapustin-Witten twist of 4d $\mathcal{N} = 4$ SYM

# Zamolodchikov $\mathcal{W}_3$ algebra

As in illustration, the $\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(z)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

We need to find the OPE of $W$ with itself such that the resulting chiral algebra is associative.

The result:

$$
\begin{aligned}
W(z)W(w) \sim\ & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
& + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) \\
& + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right)
\end{aligned}
$$

$\Lambda$ is a quasiprimary 'composite' (spin 4) field,

$$
\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).
$$

The algebra is non-linear, not a Lie algebra in the usual sense (in fact linearity should not be expected for spins $\geq 3$).

# Construction of $\mathcal{W}$-algebras

Many different ways of constructing $\mathcal{W}$-algebras:

- solving the associativity conditions for a given spin content; e.g. Zamolodchikov $\mathcal{W}_3$, Gaberdiel-Gopakumar $\mathcal{W}_\infty$
- affine Lie algebra $\rightsquigarrow$ Casimir subalgebra $\widehat{sl(N)_1}/sl(N)$ or more generally GKO coset
- Hamiltonian reduction via BRST: Drinfeld-Sokolov reduction; e.g. $\mathcal{W}_4^{(2)}$ from $\widehat{\mathfrak{su}(4)}$

$$\left(\begin{array}{ccc|c} * & 1 & 0 & 0 \\ * & * & 1 & * \\ * & * & * & * \\ \hline * & * & 0 & * \end{array}\right)$$

- free field constructions - Miura transformation

# $\mathcal{W}_\infty$

**Construction #1:** (Gaberdiel-Gopakumar) solving the associativity conditions (crossing relations)

- assuming the spin content $2, 3, 4, \ldots$ a two-parametric family of algebras is found: $\mathcal{W}_\infty[\lambda, c]$
- specializing $\lambda \to N \rightsquigarrow$ truncation to $\mathcal{W}_N[c]$
- in this sense $\mathcal{W}_\infty$ interpolates between all $\mathcal{W}_N$
- triality symmetry: for each value of $c$ three solutions $\lambda_j$ giving the same algebra $\mathcal{W}_\infty[\lambda_1, c] \simeq \mathcal{W}_\infty[\lambda_2, c] \simeq \mathcal{W}_\infty[\lambda_3, c]$,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \qquad (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) = c$$

- the vacuum character is the MacMahon function (counting three-dimensional partitions)

**Construction #2:** (Maulik-Okounkov) R-matrix, spin chain

- Miura transform (free field representation, oper...) for $\mathcal{W}_N$

$$(\alpha_0\partial + J_1(z))\cdots(\alpha_0\partial + J_N(z)) = \sum_{k=0}^{N} U_k(z)(\alpha_0\partial)^{N-k}$$

where $J_j(z)$ are free bosons $J_j(z)J_k(w) \sim \delta_{jk}(z-w)^{-2}$

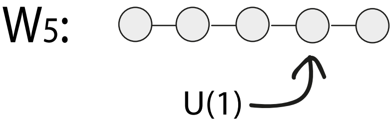
- intertwiner between different embeddings

$$R_{12}(\alpha_0\partial+J_1(z))(\alpha_0\partial+J_2(z)) = (\alpha_0\partial+J_2(z))(\alpha_0\partial+J_1(z))R_{12}$$

satisfies the Yang-Baxter equation

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)$$

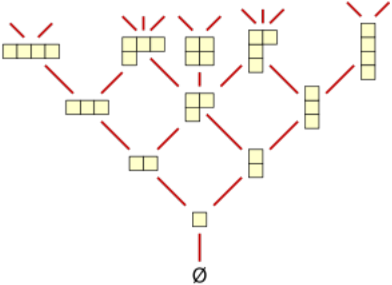
$\rightsquigarrow$ algebraic Bethe ansatz



**Construction #3 (Yangian):** (Schiffmann-Vasserot, Tsymbaliuk) an associative algebra with generators $\psi_j, e_j, f_j$, $j = 0, 1, \ldots$ and relations of the form

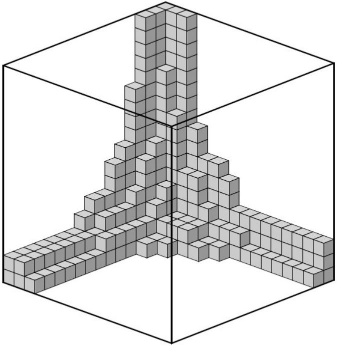
$$
\begin{aligned}
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k] \\
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad + \sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= \mathrm{Sym}_{(j_1,j_2,j_3)}\,[e_{j_1}, [e_{j_2}, e_{j_3+1}]] \\
&\ldots
\end{aligned}
$$

- integrability - an infinite set of commuting charges $\psi_j$
- representations on plane partitions: $e_j/f_j$ are box addition/removal operators
- non-locally related to previous constructions



- characters of maximally degenerate $\mathcal{W}_{1+\infty}$ reps are counting plane partitions with given Young diagram asymptotics

- $L_0$ level $\leftrightarrow$ number of boxes
- this is exactly what the topological vertex of topological strings is counting
- $\rightsquigarrow$ characters, conformal dimensions, ... can be determined combinatorially



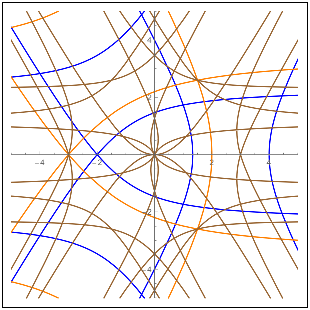
- truncation to integer $\lambda$ by restricting the height

# Truncations

- for special values of parameters the algebra can be truncated
- truncations labeled by triple of non-negative integers

$$\frac{N_1}{\lambda_1}+\frac{N_2}{\lambda_2}+\frac{N_3}{\lambda_3}=1$$

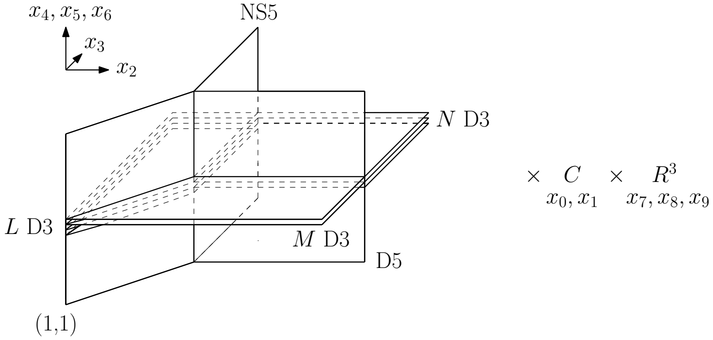
- the first singular vector corresponds to a box at $(N_1+1, N_2+1, N_3+1)$




- all the $\mathcal{W}_N$ minimal models correspond to intersections of two curves of this type, i.e. Ising c=1/2: $(0,0,2)\cap(2,1,0)$ Lee-Yang c = -22/5: $(2,0,0)\cap(0,3,0)$...

# Brane construction (Gaiotto-Rapčák)

trivalent junction of codim 1 interfaces in the Kapustin-Witten twist of $\mathcal{N} = 4$ SYM $\rightsquigarrow$ vertex operator algebra at the corner

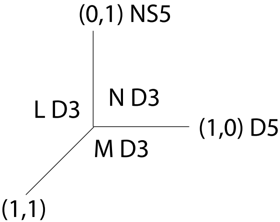



- Mikhaylov-Witten: degress of freedom associated to $(k, 1)$ interface: $U(N|L)_{\Psi+k}$ Chern-Simons theory
- Chern-Simons theory at 2d boundary produces vertex operator algebra

- 5-branes meeting at 2 dimensional subspace $\rightsquigarrow$ gluing of $U(N|L)_\Psi \times U(M|L)_{\Psi+1}$ VOA; bc: Gaiotto-Witten

- Gaiotto-Rapčák proposal (for $N > M$):

$$Y_{LMN}[\Psi] = \frac{W^{DS}_{N-M}[U(N|L)_\Psi]}{U(M|L)_{\Psi-1}}$$



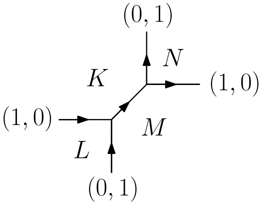

1. Start with $U(N|L)_\Psi \times U(M|L)_{1-\Psi}$
2. Drinfeld-Sokolov reduction in $U(N-M) \subset U(N)$ (BRST)
3. Perform BRST coset, gluing $U(M|L) \subset U(N|L)$ with $U(M|L)$

- comparing vacuum characters, special choices of $N$ as well as large $N$ limits, one concludes that these $Y_{LMN}$ can be identified with the truncations of $\mathcal{W}_{1+\infty}$ discussed above

# Gluing

We can now consider more complicated interfaces and their associated VOAs:

- vertices: degrees of freedom at junction described by $Y_{LMN}$ algebras
- internal edges: degrees of freedom associated to line operators along the interfaces - bi-modules



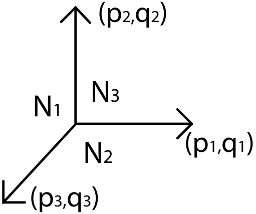

- the resulting VOA is obtained by conformally extending $Y_{LMN}$ algebras (vertices) by bi-modules associated to internal edges $\rightsquigarrow$ the total central charge is a sum of individual central charges (no contribution from the line ops) and can be read off directly from the diagram

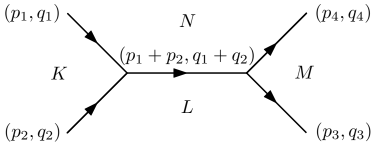
# Vertex and edge

- five-brane charge conservation at vertex $\sum_j (p_j, q_j) = 0$
- parameters of the vertex algebra are given by $(\Psi \equiv -\epsilon_2/\epsilon_1)$

$$\lambda_j = \frac{\sum_k (N_k p_k \epsilon_1 + N_k q_k \epsilon_2)}{p_j \epsilon_1 + q_j \epsilon_2}$$

- the conformal dimension of $\square$ is (half-)integral (independent of $\Psi$) and the statistics of the gluing matter (boson/fermion) depends on the relative orientation of the two vertices



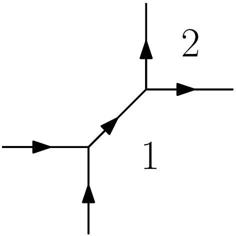
# Example - $\mathcal{N} = 2$ superconformal algebra

An extension of Virasoro algebra by spin 1 current $J$ and two charge $\pm 1$ fermionic spin $\frac{3}{2}$ supercurrents $G^{\pm}$

$$\begin{aligned}
J(z)J(w) &\sim \frac{c}{3(z-w)^2} \\
G^+(z)G^-(w) &\sim \frac{2c}{3(z-3)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+\partial J(w)}{z-w} \\
G^{\pm}(z)G^{\pm}(w) &\sim reg.
\end{aligned}$$

The vacuum character (generic $c$)

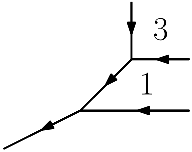
$$\chi = \prod_{n=0}^{\infty} \frac{\left(1 - zq^{\frac{3}{2}+n}\right)\left(1 - z^{-1}q^{\frac{3}{2}+n}\right)}{(1-q^{1+n})(1-q^{2+n})}$$




is up to an $U(1)$ factor exactly what one gets from the gluing

# Example - Bershadsky-Polyakov $\mathcal{W}_3^{(2)}$

- an extension of Virasoro algebra by spin 1 current $J$ and two charge $\pm 1$ *bosonic* spin $\frac{3}{2}$ currents $G^{\pm}$
- the same spin content as $\mathcal{N} = 2$ SCA but different statistics of the gluing fields
- can be obtained from $U(3)$ Kac-Moody by DS reduction using the non-principal embedding $3 = 2 + 1$



# $\mathcal{N} = 2$ $\mathcal{W}_\infty$

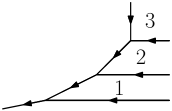
- $\mathcal{N} = 2$ $\mathcal{W}_\infty$ symmetry algebra of Kazama-Suzuki coset models is glued from two copies of bosonic $\mathcal{W}_\infty$
- the central charge can be decomposed as

$$c = \frac{c(\mu+1)(c+6\mu-3)}{3(c+3\mu)^2} - \frac{(c-3\mu)(c(\mu-2)-3\mu)}{3(c+3\mu^2)} + 1$$

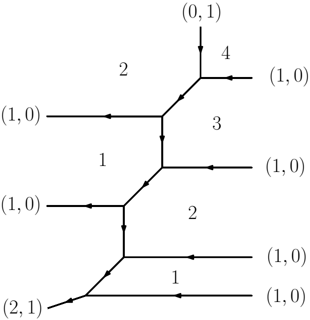
corresponding to central charges of the two $\mathcal{W}_\infty$ vertices

- the symmetry of the diagram is $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is also the duality symmetry of the algebra
- the four basic minimal representations correspond to four external legs

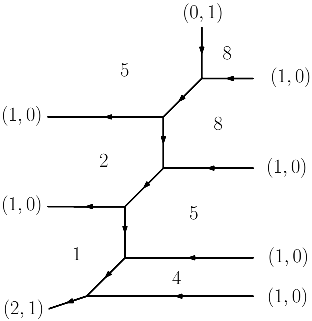
# Other examples and questions




$\widehat{U(3)}$




$\widehat{U(4|2)}$




more general DS reduction

# Further progress

- free field representations of $Y_{N_1 N_2 N_3}$ by modification of Miura transformation $\rightsquigarrow$ free field representations of a very large class of algebras by gluing
- even spin $\mathcal{W}_\infty$ (orthosymplectic) a subalgebra of $\mathcal{W}_{1+\infty}$
- W-algebras associated to $A_n, B_n, C_n, D_n, E_6, E_7, E_8, G_2$ subalgebras of $\widehat{\mathfrak{u}(1)} \times \mathcal{W}_{n-1}$
- $W[F_4]$? Grassmannian cosets? gluing of $N = 3$ and $N = 4$ SCA? higher spin square?
- Yangian description of even $\mathcal{W}_\infty$?

Thank you!