Linear infinitesimal braidings for abelian 2-groups

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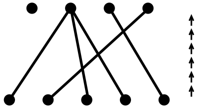
Joint work with Pavol Ševera

Recall: The category of *finite sets* FinSet:

objects: finite sets $\{\bullet \cdots \bullet\}$

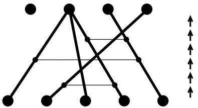
morphisms:



the category of *chorded sets* CSet:

the same objects $\{\bullet \cdots \bullet\}$

morphisms:



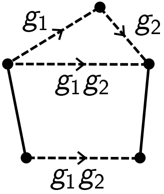
$$[t_{ij}, t_{kl}] = 0, \quad i, j, k, l \text{ distinct},$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0, \quad i, j, k \text{ distinct},$$

and a bit more.

Motivation: functors CSet $\to C$ can be quantized using a Drinfeld associator.

in Pavol's lecture: we've seen the *nerve functor* for a group G:

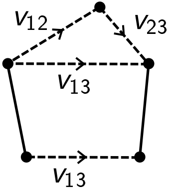
$$F : \text{FinSet} \to \text{Vect}$$
$$\{n \text{ elements}\} \mapsto C^\infty(G^{\times(n-1)})$$



extending to $\tilde{F}$: CSet $\to$ Vect $\Leftrightarrow$ Poisson-Lie structure on G.

We consider a different functor: For a vector space V,

$$F_2 : \text{FinSet} \to \text{Vect}$$
$$\{n \text{ elements }\} \mapsto C^\infty(V^{\times\binom{n}{2}})$$

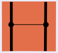


This is the nerve of the 2-group (V, V, 1_V, $v \mapsto 1_V$)

The problem: What data is needed to extend this functor to $\tilde{F}_2 : \mathrm{CSet} \to \mathrm{Vect}$?

Proposition

To extend F_2, it is enough to give the following morphisms:

$A =$  $: F_2(2) \to F_2(2) \quad \textit{or} \quad C^\infty(V) \to C^\infty(V)$

$B =$  $: C^\infty(V) \otimes C^\infty(V) \to C^\infty(V^{\times 6})\,.$

Moreover, both are derivations, e.g.

$$B(f, g) = B^{\alpha\beta}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) \frac{\mathrm{d}f(v_{12})}{\mathrm{d}x_{12}^{\alpha}} \frac{\mathrm{d}g(v_{34})}{\mathrm{d}x_{34}^{\beta}}$$

Denote

$$B^{\alpha\beta}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) = B^{\alpha\beta}(1234)$$

However, there are more conditions, coming from $[t_{ij}, t_{kl}] = 0$ etc.

For example:

$$B^{\delta\gamma}(ikli)\frac{\partial B^{\beta\alpha}(kiji)}{\partial x^{\delta}_{ik}} + B^{\delta\gamma}(jkli)\frac{\partial B^{\beta\alpha}(kiji)}{\partial x^{\delta}_{jk}}$$
$$-B^{\beta\delta}(ikli)\frac{\partial B^{\gamma\alpha}(liji)}{\partial x^{\delta}_{li}} - B^{\beta\delta}(iklj)\frac{\partial B^{\gamma\alpha}(liji)}{\partial x^{\delta}_{lj}} = 0$$

Simplifying ansatz: We assume that A and B are linear: it is enough to specify linear maps

$$\mathbf{b}\colon V^* \otimes V^* \to V^* \quad \text{and} \quad \mathbf{a}\colon V^* \to V^* .$$

Then

$$B^{\alpha\beta}(1234) = \sum_{\gamma} \mathbf{b}^{\alpha\beta}_{\gamma}(v^{\gamma}_{12} - v^{\gamma}_{13} - v^{\gamma}_{23}) + \mathbf{b}^{\beta\alpha}_{\gamma}(v^{\gamma}_{23} + v^{\gamma}_{24} - v^{\gamma}_{34}) ,$$
$$A^{\alpha}(v_{12}) = \sum_{\beta} \mathbf{a}^{\alpha}_{\beta} v^{\beta}_{12} .$$

Recapitulation: we are looking for $\mathbf{b}\colon V^* \otimes V^* \to V^*$ and $\mathbf{a}\colon V^* \to V^*$, which extend the functor F_2 to a functor $\tilde{F}_2 : \mathrm{CSet} \to \mathrm{Vect}$.

For the nerve of the group $(V, +)$, this would be a Lie bracket on V^*.

Proposition

The necessary and sufficient condition for $\mathbf{b}$ *and* $\mathbf{a}$ *to give a functor* $\mathrm{CSet} \to \mathrm{Vect}$ *are the following equations:*

$$\mathbf{b}(\mathbf{b}(x,y),z) = \mathbf{b}(\mathbf{b}(x,z),y)\,, \qquad \mathbf{b}(x,\mathbf{b}(y,z)) = \mathbf{b}(x,\mathbf{b}(z,y))$$

$$\mathbf{a}(\mathbf{b}(x,y)) = \mathbf{b}(\mathbf{a}(x),y) = \mathbf{b}(x,\mathbf{a}(y)) = 0\,,$$

where $x, y, z \in V^*$.

The conditions can be rewritten, using $\mathbf{b}(a,b) \equiv a \cdot b + [a,b]$, as:

Theorem

A functor CSet $\to$ Vect *extending the functor F_2 with linear A and B is equivalent to a Lie bracket $[-,-]$ on V^* and a commutative, nonassociative product $\cdot$ on V^* such that*

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = -2[b,[a,c]] + [b \cdot c, a] - [b \cdot a, c]\,,$$
$$a \cdot [b,c] + [a,[b,c]] = 0\,,$$

and $\mathbf{a} \colon V^* \to V^*$ *s.t.* $\mathbf{a}(a \cdot b) = \mathbf{a}([a,b]) = [\mathbf{a}(a), b] = \mathbf{a}(a) \cdot b = 0.$

Further questions:

- Is this structure already known?
- We can quantize by BrSet $\xrightarrow{\Phi}$ CSet $\xrightarrow{\tilde{F}_2}$ Vect.
  What is the quantum structure? For $(V,+)$ with a Lie bracket, it is $U(V^*)$.
- Is it possible to take non-linear B and A?