Invariant differential operators for Hermitian symmetric spaces

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The most important slide

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Homogeneous vector bundles

- homogeneous space G/P
- P-representation $\sigma : P \to GL(\mathbb{V})$

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homogeneous bundle $\mathcal{V} = G \times_\sigma \mathbb{V} \to G/P$

$$G \times_\sigma \mathbb{V} = G \times \mathbb{V}/[(g, v) \simeq (gp, \sigma(p^{-1})v)]$$

with space of sections

$$\Gamma^\infty(G/P, \mathcal{V}) \simeq \mathcal{C}^\infty(G, \mathbb{V})^P = \{f \colon G \to \mathbb{V} \mid f(gp) = \sigma(p^{-1})f(g)\}$$

and *induced action*

$$(\widetilde{\sigma}(g)s)(x) = s(g^{-1}x)$$

Invariant differential operators

invariant differential operators between sections of two such bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ must respect the induced actions of G:

$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}_1) \to \Gamma^\infty(G/P, \mathcal{V}_2)$$
$$\mathcal{D} \circ \widetilde{\sigma_1} = \widetilde{\sigma_2} \circ \mathcal{D}.$$

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linear differential operator $\mathcal{D}$ of order k is given by a linear bundle map from the k-th jet prolongation

$$D\colon \Gamma^\infty(G/P, \mathcal{J}^k\mathcal{V}_1) \to \Gamma^\infty(G/P, \mathcal{V}_2)$$

Jet prolongations

The bundle $\mathcal{J}^k\mathcal{V}$ is homogeneous, i.e. there exists P-representation $j^k\sigma$ on $J^k\mathbb{V}$ such that

$$\mathcal{J}^k\mathcal{V} = G \times_{j^k\sigma} J^k\mathbb{V}$$

Invariant differential operator corresponds to equivariant bundle map

$$G \times_{j^k\sigma_1} J^k\mathbb{V}_1 \to G \times_{\sigma_2} \mathbb{V}_2$$

and thus is given by a P-equivariant morphism

$$\varphi\colon J^k\mathbb{V}_1 \to \mathbb{V}_2.$$

Induced representations

Dualizing

$$\varphi\colon J^k\mathbb{V}_1 \to \mathbb{V}_2$$

we get

$$\varphi^*\colon \mathbb{V}_2^* \to \left(J^k\mathbb{V}_1\right)^* \simeq \mathfrak{U}(\mathfrak{g})^k \otimes_{\mathfrak{p}} \mathbb{V}_1^*$$

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Taking direct limit $k \to \infty$ we obtain

$$\mathrm{Hom}_P\left(\mathbb{V}_2^*, \mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{V}_1^*\right) \simeq \mathrm{Hom}_{\mathfrak{g},P}\left(\mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{V}_2^*, \mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{V}_1^*\right)$$

The case of parabolic subgroup

If P is a parabolic subgroup of a complex simple Lie group G then

$$\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u} \qquad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$$

and

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V} \simeq_{\mathfrak{l}} \mathfrak{U}(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} \mathbb{V} \simeq_{\mathbb{C}} S(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} \mathbb{V}.$$

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If $\mathbb{V}$ is a highest weight $\mathfrak{l}$-module $\mathbb{F}_\lambda$ with highest weight λ, then

$$M_\lambda = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{F}_\lambda$$

is also highest weight module and

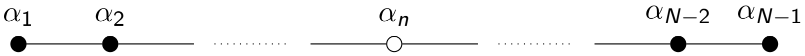
$$\mathrm{Hom}_{\mathfrak{g}}(M_{\lambda_1}, M_{\lambda_2}) = \mathrm{Hom}_{\mathfrak{l}}(\mathbb{F}_{\lambda_1}, M_{\lambda_2})^{\mathfrak{u}}$$

Special linear algebra and its maximal parabolics

$$\mathfrak{g} = \mathfrak{sl}(n+m, \mathbb{C}) \quad \mathfrak{l} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(m, \mathbb{C}) \oplus \mathbb{C}$$

$$\mathfrak{u} = M_{n,m}(\mathbb{C}) \simeq \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \quad \bar{\mathfrak{u}} = M_{m,n}(\mathbb{C})$$

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A \in \mathfrak{gl}(n, \mathbb{C}), D \in \mathfrak{gl}(m, \mathbb{C}), \mathrm{Tr}(A) + \mathrm{Tr}(B) = 0 \right\}$$



$$N = n + m$$

Explicit basis

$$f_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix} \qquad e_{ij} = \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}$$

$$h_c = \begin{pmatrix} \frac{1}{n} I_n & 0 \\ 0 & -\frac{1}{m} I_m \end{pmatrix}$$

$$h_{A,B} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\operatorname{Tr} A = \operatorname{Tr} B = 0$$

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Scalar case:

$$M_\lambda = \mathbb{C}[\overline{\mathfrak{u}}^*] \otimes \mathbb{C}_{\lambda_\alpha}, \alpha \in \mathbb{C}$$

y_{ij} ... linear coordinate functions on $\mathfrak{u}$

Explicit action for $M_{\lambda-\rho_{\mathfrak{u}}}$

$$E = \sum_{k=1}^{m}\sum_{\ell=1}^{n} y_{k\ell}\partial_{y_{k\ell}}$$

$$R_A = -\sum_{i,j=1}^{n}\sum_{k=1}^{m} a_{ij}y_{kj}\partial_{y_{ki}}, \qquad L_B = \sum_{i,j=1}^{m}\sum_{k=1}^{n} b_{ij}y_{ik}\partial_{y_{jk}}$$

$$\begin{aligned}
\hat{\pi}_\lambda(f_{ij}) &= -y_{ij}\\
\hat{\pi}_\lambda(h_c) &= -\left(\frac{1}{n}+\frac{1}{m}\right)E + \left(\lambda(h_c) - \frac{n+m}{2}\right)\\
\hat{\pi}_\lambda(h_{A,B}) &= R_A + L_B\\
\hat{\pi}_\lambda(e_{ij}) &= \sum_{k=1}^{m}\sum_{\ell=1}^{n} y_{k\ell}\partial_{y_{ki}}\partial_{y_{j\ell}} - \left(\lambda(h_c) - \frac{n+m}{2}\right)\partial_{y_{ji}}
\end{aligned}$$

Explicit action – continued

$$L_{E_{st}} = \sum_{i,j=1}^{m} \sum_{k=1}^{n} (\delta_{is}\delta_{tj}) y_{ik} \partial_{y_{jk}}$$
$$= \sum_{k=1}^{n} y_{sk} \partial_{y_{tk}}$$

"Replacing t^{th} row with s^{th} row."

$$R_{E_{st}} = -\sum_{k=1}^{n} y_{kt} \partial_{y_{ks}}$$

"Replacing s^{th} column with t^{th} column."

$$\hat{\pi}_{\lambda_\alpha}(e_{ij}) = \sum_{k=1}^{m} \partial_{y_{ki}} L_{kj} - \left(\alpha - \frac{n-m}{2}\right) \partial_{y_{ji}}$$

Multiplicity-free decomposition

Let $\alpha \in \mathbb{C}$ and let $r = \min\{n, m\}$ and define

$$q_k = \det \begin{pmatrix} y_{1,n-k+1} & y_{1,n-k+2} & \cdots & y_{1,n} \\ y_{2,n-k+1} & y_{2,n-k+2} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k,n-k+1} & y_{k,n-k+2} & \cdots & y_{k,n} \end{pmatrix}$$

for $k = 1, 2, \ldots, r$.

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for $k = 1, 2, \ldots, r$.
The vectors

$$v_\mu = q_1^{a_1} q_2^{a_2} \ldots q_r^{a_r} \in \mathbb{C}[\mathfrak{u}] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho}$$

are highest weight vectors of $\mathfrak{l}$.

$$\mu = \sum_{i=1}^{r} a_i \omega_{n-i} + \sum_{i=1}^{r} a_i \omega_{n+i} + \left(\alpha - \frac{n+m}{2} - 2\sum_{i=1}^{r} a_i\right)\omega_n$$

Multiplicity-free decomposition – continued

We have

$$\mathbb{C}[\mathfrak{u}] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho} \simeq_{\mathfrak{l}} \bigoplus_{\mu \in \Lambda^+_\alpha(\mathfrak{p})} \mathbb{F}_\mu$$

as $\mathfrak{l}$-modules, where $\mathbb{F}_\mu$ is the simple $\mathfrak{l}$-module with highest weight $\mu \in \mathfrak{h}^*$ and

$$\Lambda^+_\alpha(\mathfrak{p}) = \bigg\{ \sum_{i=1}^r a_i \omega_{n-i} + \sum_{i=1}^r a_i \omega_{n+i} + \bigg(\alpha - \frac{n+m}{2} - 2\sum_{i=1}^r a_i\bigg)\omega_n \;\bigg|\; a_1, a_2, \dots, a_r \in \mathbb{N}_0 \bigg\}.$$

Proof

$$v_\mu = q_1^{a_1} q_2^{a_2} \dots q_r^{a_r}$$

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- s largest such that $a_s \neq 0$

$$\hat{\pi}_\lambda(e_{n-s+1,s}) v_\mu = \left(a_s - s - \alpha + \frac{n+m}{2} \right) \partial_{y_{s,n-s+1}} v_\mu$$

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- t such that $a_t \neq 0$ and $a_l = 0$ for $l = t+1, \dots, s-1$

$$\partial_{y_{t,n-t+1}} \hat{\pi}_\lambda(e_{n-t+1,t}) v_\mu = a_t(a_s + a_t + s - t)(\partial_{y_{t,n-t+1}} q_s)(\partial_{y_{t,n-t+1}} q_t)$$

Proof

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- verify that

$$\hat{\pi}_\lambda(e_{i,j}) q_s^{a_s} = 0, \quad i = 1, \ldots, n, j = 1, \ldots, m$$

Result

We have

$$M_\lambda^{\mathfrak{u}} \xrightarrow{\simeq} \begin{cases} \mathbb{F}_{\mu_{\alpha,r}} & \text{if } \alpha + r \notin \mathbb{N}, \\ \mathbb{F}_{\mu_{\alpha,r}} \oplus \bigoplus_{k=0}^{\min\{r-1,\alpha+r-1\}} \mathbb{F}_{\mu_{\alpha,k}} & \text{if } \alpha + r \in \mathbb{N}, \end{cases}$$

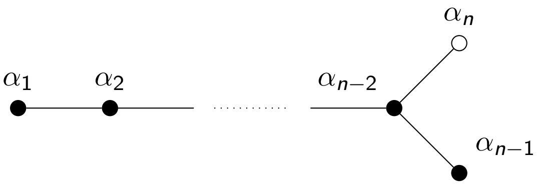
where

$\mu_{\alpha,k} = (\alpha + r - k)\omega_{n-r+k} + (\alpha + r - k)\omega_{n+r-k} - (\alpha + 2r - 2k)\omega_n$
for $k = 0, 1, \ldots, r$.

Other classical Hermitian Symmetric Spaces

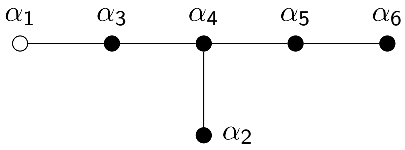
$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & -A^{\mathrm{T}} \end{pmatrix} \middle| A, B \in M_{n,n}(\mathbb{C}),\ B^{\mathrm{T}} = B \right\}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & -A^{\mathrm{T}} \end{pmatrix} \middle| A, B, C \in M_{n,n}(\mathbb{C}),\ B^{\mathrm{T}} = -B,\ C^{\mathrm{T}} = -C \right\}$$



Exceptional case E_6

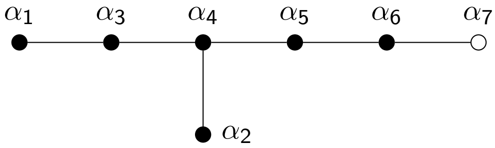
$E_6^{-14}/U(1)Spin(10)$



$\dim \mathfrak{u} = 16 \quad \deg q_1 = 1, \deg q_2 = 2$

Exceptional case E_7

$E_7^{-25}/U(1)E_6^{\mathrm{cpt}}$



$\dim \mathfrak{u} = 27 \quad \deg q_1 = 1, \deg q_2 = 2, \deg q_3 = 3$

$\mathfrak{u} = \mathrm{Herm}(3, \mathbb{O}) \quad q_3 = \det$