Generalized Killing spinors on 3-Sasakian manifolds

Petr Zima
(joint with P. Somberg and I. Agricola)

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

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(M, g) a (pseudo-) Riemannian spin manifold,
∇^g the Levi-Civita and the induced spin connection

Killing spinors

$$\nabla^g_X \Psi = a\, X \cdot \Psi \tag{1}$$

- *Killing number* $a \in \mathbb{C} \sim$ eigenvalue of the Dirac operator
- 1st integrability condition: $\mathcal{R}_{X,Y}\, \Psi = -a^2\, [X\cdot, Y\cdot]\, \Psi$

  $\Rightarrow (M, g)$ is **Einstein** with $\mathrm{Scal} = 4\, a^2\, n(n-1)$

Generalized Killing spinors

$$\nabla^g_X \Psi = S(X) \cdot \Psi \tag{2}$$

- $S(X)$ is a section of *symmetric endomorphisms* of TM
- **Not invariant!**
  (unless we consider $S(X)$ as a part of the solution)

V a vector bundle over M, ∇^V a linear connection in V, extend ∇^V by ∇^g also to V-valued differential forms

Let $\Phi \in \Omega^p(M, V)$ and $\Xi \in \Omega^{p+1}(M, V)$

Killing(-Yano) forms

$$\nabla^V_X \Phi = X \lrcorner \Xi \tag{3}$$

$*$-Killing forms

$$\nabla^V_X \Xi = X^\flat \wedge \Phi \tag{4}$$

Special Killing forms

$$\nabla^V_X \Phi = X \lrcorner \Xi, \qquad \nabla^V_X \Xi = -c\, X^\flat \wedge \Phi \tag{5}$$

- Does **not** imply Einstein, but we have $\mathrm{Scal} = c\, n(n-1)$
- *Cone construction:* spec. Killing $\Leftrightarrow$ parallel on the cone $\Rightarrow$ **Holonomy classification** via Berger's list

Ordinary special Killing forms

All the examples where M is compact are:

- *Round spheres:* a solution for arbitrary (Φ_0, Ξ_0)
- *Sasakian:* $\Phi^{(k)} = \eta \wedge (\mathrm{d}\eta)^k$, η the contact form
- Exceptional: *nearly Kähler* in dim $= 6$, G_2 in dim $= 7$

Spinor-valued special Killing forms

$V = \Sigma$ is the spinor bundle and ∇^V the *Killing spinor connection*

$$\nabla^V_X \Psi = \nabla^g_X \Psi - a\, X \cdot \Psi, \qquad a \in \mathbb{C} \tag{6}$$

- The cone construction works only when $c = 4a^2$.
- *Round spheres:* again a solution for arbitrary (Φ_0, Ξ_0)
- Q: Can $c = 4a^2 = \frac{1}{n(n-1)} \operatorname{Scal}$ be deduced in general from the integrability conditions?

“2^{nd} order Killing spinors”

≡ spinor-valued spec. K. 0-forms; combining the equations ⇔

$$
\begin{aligned}
(\nabla^g)^2_{X,Y}\Psi = -a^2\, X\cdot Y\cdot \Psi\, + \\
+\, a\,(Y\cdot(\nabla^g_X\Psi) + X\cdot(\nabla^g_Y\Psi)) - c\, g(X,Y)\,\Psi
\end{aligned}
\tag{7}
$$

- 1^{st} integrability condition *the same as for Killing spinors,* so again ⇒ (M, g) is **Einstein**.
- Includes Killing spinors with Killing number $a' = -a$. ⇒ **Invariant** generalization of Killing spinors.
- Future research: Higher order equations derived from rank ≥ 2 *symmetric Killing tensor-spinors.*

Holonomy classification:

- *3-Sasakian:* Admit an additional solution!
- *Sasakian*, G_2*:* No additional solutions possible.
- *Nearly Kähler:* Remains to be checked.

Sasakian manifolds

$(M, g, \varphi, \xi, \eta)$, $\dim M = 2m+1$, such that:

- *almost contact:* $\varphi^2 = -\operatorname{Id}_{TM} + \eta \otimes \xi$, $\eta(\xi) = 1$
- *normal:* Nijenhuis torsion $N_\varphi = [\varphi, \varphi] + \mathrm{d}\eta \otimes \xi = 0$
- *compatible metric:* $g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$
- *contact:* $\mathrm{d}\eta = 2\Phi$ where $\Phi(X, Y) = g(X, \varphi(Y))$

$\Leftrightarrow$ $|\xi| = 1$, $\eta = \xi^\flat$ is a **special Killing 1-form** with $c = 1$

$\Rightarrow$ reduction of the structure group to $\mathrm{U}(m)$

3-Sasakian manifolds

$(M, g, \varphi_i, \xi_i, \eta_i)$, $\dim M = 4m+3$, $i = 1, 2, 3$, such that each $(\varphi_i, \xi_i, \eta_i)$ is a Sasakian structure compatible with g and

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j,$$
$$\xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \quad \eta_k = \eta_i\phi_j = -\eta_j\varphi_i.$$

$\Rightarrow$ reduction of the structure group to $\mathrm{Sp}(m)$; always **Einstein!**

3-(α, δ)-Sasakian manifolds

Split $TM = \mathcal{V} \oplus \mathcal{H}$, the *vertical* and *horizontal distribution*,

$$\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle, \qquad \mathcal{H} = \ker \eta_1 \cap \ker \eta_2 \cap \ker \eta_3 .$$

Rescale g on $\mathcal{V}$ and $\mathcal{H}$ $\rightsquigarrow$ 2-parameter family of metrics $\rightsquigarrow$ $\rightsquigarrow$ *3-(α, δ)-Sasakian manifolds*

Proposition

(M, g) is Einstein iff $\delta = \alpha$ *or* $\delta = (2m + 3)\,\alpha$.

Dimension 7

$\delta = \alpha = 1$	$g = g_1$	the original *3-Sasakian structure*
$\delta = 5\alpha$	$\widetilde{g} = g_{1:5}$	canonical *cocalibrated* G_2*-structure*

- The cocalibrated G_2-structure with metric $\widetilde{g}$ possesses the so called **canonical spinor** satisfying $\nabla^c \Psi_0 = 0$.
- With respect to the original 3-Sasakian metric g the spinor field Ψ_0 becomes a *generalized Killing spinor.*

Canonical spinor

- Ψ_0 is a **generalized Killing spinor**.

$$\nabla^g_\xi \Psi_0 = \frac{1}{2}\,\xi \cdot \Psi_0, \quad \nabla^g_Y \Psi_0 = -\frac{3}{2}\,Y \cdot \Psi_0, \quad \xi \in \mathcal{V},\ Y \in \mathcal{H} \tag{8}$$

- Reeb vector fields ξ_i are **special Killing** with $c = 1$.
- $\Psi_i = \xi_i \cdot \Psi_0$ are **Killing spinors** with $a = \frac{1}{2}$.

$$\nabla^g_X \Psi_i = \frac{1}{2}\,X \cdot \Psi_i, \qquad X \in TM;\ i = 1, 2, 3 \tag{9}$$

Proposition

*Ψ_0 is also a **2nd order Killing spinor** with $a = -\frac{1}{2}$ and $c = 1$ which is not a Killing spinor.*

- Invariant description of the canonical spinor Ψ_0.
- WIP: Describe Ψ_0 in general for $\dim M = 4m + 3$ without the detour to the G_2-structure.

References

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THANK YOU FOR YOUR ATTENTION!