Parabolic geometries and geometric compactifications lecture 1

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- This first lecture starts with an introduction to the general concept of a Cartan geometry associated to a homogeneous space.
- In particular, I will outline how Riemannian geometry can be encoded in that way.
- The example of conformal structures shows how Cartan geometries can be used to encode "higher order information" leading to unusual geometric objects.
- The homogeneous model for conformal structures is of rather special type (a generalized flag manifold) and taking more general homogeneous spaces of this type leads to parabolic geometries.

Contents

- In the spirit of F. Klein's Erlangen program, a classical geometry is specified by a homogeneous space G/P.
- If G is a Lie group, there is a definition of an associated geometric structure due to E. Cartan based on the following.

- $p : G \to G/P$ is an P-principal bundle that carries the left Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ with $\mathfrak{g} = \text{Lie}(G)$.
- The left actions of elements of G are exactly the diffeomorphisms of G/P that admit a P-equivariant lift $\Phi : G \to G$ such that $\Phi^*\omega = \omega$.
- Observe that $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ by the Maurer-Cartan equation.

The definition of a Cartan geometry is obtained by replacing G/P by a manifold M of the same dimension and requiring exactly those properties of the Maurer Cartan form that make sense in the general setting.

Definition

(1) A *Cartan geometry* of type (G, P) on a smooth manifold M is given by a principal P-bundle $p : \mathcal{G} \to M$ and a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.

- each $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism
- $(r^g)^*\omega = \mathrm{Ad}(g)^{-1} \circ \omega$ for all $g \in P$ (equivariancy)
- $\omega(\zeta_X) = X$ for all $X \in \mathfrak{p} \subset \mathfrak{g}$ (fundamental fields)

(2) The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the geometry $(\mathcal{G}, \omega)$ is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.

- Such geometries exist only for $\dim(M) = \dim(G/P)$.
- There is an obvious notion of morphisms, and morphisms induce local diffeomorphisms between the base spaces.
- The curvature of a Cartan geometry vanishes identically if and only if it is locally isomorphic to its *homogeneous model* G/P.

Example

The nature of the concept of Cartan geometries is illustrated nicely by the example related to Euclidean geometry. Put $G = \mathrm{Euc}(n)$ and $P = O(n)$, so G/P is Euclidean space $\mathbb{E}^n$. Consider an n-manifold M and a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type G/P.

- $\mathfrak{g} = \mathfrak{o}(n) \oplus \mathbb{R}^n$ (semi-direct sum) and splitting $\omega = \gamma \oplus \theta$ accordingly, both components are $O(n)$-equivariant
- θ is equivalent to making $\mathcal{G}$ the orthonormal frame bundle of a Riemannian metric g on M
- γ defines a metric linear connection ∇ on TM
- The curvature K encodes curvature and torsion of ∇.

Existence and uniqueness of the Levi-Civita connection $\iff$ n-dimensional Riemannian manifolds are (categorically) equivalent to Cartan geometries of type (G, P) for which K has values in $\mathfrak{p} \subset \mathfrak{g}$. This similarly works for $G = O(n+1)$ and $G = O(n,1)$.

General features of Cartan geometries

- K defines a fundamental and complete invariant
- representations of P induce natural vector bundles
- For the representation on $\mathfrak{g}/\mathfrak{p}$ induced by Ad, one obtains $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM$, so all tensor bundles are associated.
- Starting from distinguished curves in G/P, one obtains general notions of distinguished curves in Cartan geometries.
- Natural notion of infinitesimal automorphisms of a Cartan geometry in $\mathfrak{X}(\mathcal{G})$. Automorphisms of $(\mathcal{G}, \omega)$ form a Lie group of dimension $\leq \dim(G)$ with Lie algebra formed by complete infinitesimal automorphisms.
- Several constructions relating geometries of different type (Correspondence spaces, Fefferman constructions, extension functors).

The conformal sphere

Put $G := SO_0(n+1,1)$ for a Lorentzian inner product on $\mathbb{R}^{n+2}$. Then G acts transitively on S^n, viewed as a space of isotropic rays. Hence $S^n = G/P$, where $P \subset G$ is the stabilizer of one such ray. Elementary arguments show that the action ℓ_g of $g \in G$ on S^n sends the round metric of S^n to a conformally related metric.

- Denoting by $o \in S^n$ the point fixed by P, the map $g \mapsto T_o\ell_g$ defines a surjective homomorphism $P \to G_0 := CO(n)$.
- The kernel of this homomorphism is normal subgroup $P_+ \subset P$ isomorphic to $\mathbb{R}^{n*}$ and $P = G_0 \ltimes P_+$.
- For any $g \in P_+$, ℓ_g coincides with id_{S^n} to first order in o, but for $g \neq e$, they are different on any open neighborhood of o. In particular, there is no G-invariant linear connection on TS^n.

This "higher order issue" will be crucial in what follows.

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Factoring by the action of $P_+ \subset P$, we obtain $\mathcal{G}_0 := \mathcal{G}/P_+$ and $p_0 : \mathcal{G}_0 \to M$ is a principal bundle with structure group $P/P_+ \cong CO(n)$. Projecting the values of ω to $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$, the result descends to a strictly horizontal form $\theta \in \Omega^1(\mathcal{G}_0, \mathbb{R}^n)^{G_0}$. Hence we obtain an underlying conformal structure on M (i.e. an inner product up to scale on each tangent space).

Theorem (E. Cartan)

Any conformal structure arises in this way. Imposing a normalization condition on the curvature K makes the inducing Cartan geometry unique up to isomorphism and one obtains an equivalence of categories.

There are two approaches to proving this, which are very different in spirit. Since each of them has interesting advantages, we'll sketch both of them, starting with the classical approach.

Sketch of classical proof

- Starting from a conformal structure $(\mathcal{G}_0, \theta)$, one first observes that there are torsion-free principal connections γ on $\mathcal{G}_0$.
- For each $u_0 \in \mathcal{G}_0$, the values $\gamma(u_0)$ form an n-dimensional affine space. Attaching this to u_0 one constructs a bundle $\mathcal{G} \to M$ and extending the action of G_0 on $\mathcal{G}_0$ defines a principal right action of P on $\mathcal{G}$.
- Using the connection forms of the γ, one defines a natural form $\omega_0 \in \Omega^1(\mathcal{G}, \mathfrak{g}_0)$. For each $u \in \mathcal{G}$ over u_0, $\theta(u_0) \oplus \omega_0(u)$ defines a linear isomorphism $T_{u_0}\mathcal{G}_0 \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.
- The possible lifts to a linear isomorphism $T_u\mathcal{G} \to \mathfrak{g}$ that is compatible with fundamental fields form an affine space and the corresponding curvature K always has values in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.
- One then shows that there is a unique such lift for which the $\mathfrak{g}_0$-component of K has vanishing Ricci-type contraction.

Sketch of "abstract" proof

- Starting from $(\mathcal{G}_0, \theta)$, define $\mathcal{G} := \mathcal{G}_0 \times_{G_0} P$, so $\mathcal{G}/P_+ \cong \mathcal{G}_0$.
- Choose a principal principal connection on $\mathcal{G}$ and use it and θ to define a Cartan connection $\hat{\omega}$ on $\mathcal{G}$. Then $(\mathcal{G}, \hat{\omega})$ has underlying structure $(\mathcal{G}_0, \theta)$.
- Cartan connections on $\mathcal{G}$ inducing θ form an affine space and there is a concept of homogeneity, which also applies to curvature. The change of curvature in lowest homogeneity is tensorial and induced by a Lie algebra cohomology differential.
- Finding a normalization condition becomes a purely algebraic problem. Having done this, one can normalize $\hat{\omega}$ homogeneity by homogeneity to obtain a normal Cartan connection ω on $\mathcal{G}$.
- Using information on $H^1(\mathbb{R}^n, \mathfrak{g})$ one shows that two normal Cartan connections on $\mathcal{G}$ that induce θ are related by an automorphism covering the identity on $(\mathcal{G}_0, \theta)$.

tractor bundles

We know that for the representation of P on $\mathfrak{g}/\mathfrak{p}$ induced by Ad, we get $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM$. This representation factors through $P \to P/P_+ \cong G_0$, so to recover higher order information, other constructions are needed:

Via equivariant extension, the Cartan connection ω induces a principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$. Taking a representation $\mathbb{V}$ of G and restricting to P, we obtain $\mathcal{V}M := \mathcal{G} \times_P \mathbb{V} = \tilde{\mathcal{G}} \times_G \mathbb{V}$, so this inherits a canonical linear connection. ("tractor bundles and tractor connections")

- Choosing g in the conformal class, its Levi-Civita connection ∇ defines a section $\mathcal{G}_0 \to \mathcal{G}$.
- Using this, one identifies $\mathcal{V}M$ with a bundle associated to $\mathcal{G}_0$ and describes the canonical connection in terms of ∇.
- It can be made explicit how all this changes when rescaling g.

The abstract proof is robust and in particular applies to all pairs (G, P) where G is semisimple and $P \subset G$ is a parabolic subgroup. Here the relevant information on Lie algebra cohomology is provided by Kostant's theorem. Interpretations in the spirit of the classical proof can then be recovered via so-called *Weyl structures*.

Parabolic subgroups are characterized by the fact that there is a Lie algebra grading $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ such that $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. Putting $\mathfrak{g}^i := \oplus_{j \geq i} \mathfrak{g}_j$ makes $\mathfrak{g}$ into a filtered Lie algebra. Since $\mathfrak{p} = \mathfrak{g}^0$, the filtration is P-invariant and there are natural subgroups $G_0, P_+ \subset P$ corresponding to $\mathfrak{g}_0$ and $\mathfrak{p}_+ := \mathfrak{g}^1$.

Filtrations and associated graded objects are crucial for the theory. Recall that for a filtration by smooth subbundles $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ such that $[\Gamma(T^iM), \Gamma(T^jM)] \subset T^{i+j}M$ the Lie bracket induces a tensorial bracket on $\mathrm{gr}(T_xM) = \oplus_i (T_x^iM / T_x^{i+1}M)$ ("symbol algebra at x").

The underlying structure for parabolic geometries

Given a type (G, P) corresponding to $\mathfrak{g} = \oplus_i \mathfrak{g}_i$, the underlying structure consists of

1. A filtration $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ such that $\mathrm{gr}(TM)$ becomes a locally trivial bundle of Lie algebras modeled on $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i$.
2. This then has a natural frame bundle with structure group $\mathrm{Aut}_{gr}(\mathfrak{g}_-)$ that contains G_0 as a subgroup and the second ingredient is a reduction to that structure group.

A standard example arises from $G = SU(n+1, 1)$ with P the stabilizer of an isotropic complex line. Here $\mathfrak{g}_-$ is a Heisenberg algebra, so 1 is a contact structure $H \subset TM$. G_0 consists of those automorphisms that are complex linear on $\mathfrak{g}_{-1} \cong \mathbb{C}^n$, so 2 is an almost complex structure on H.

Conformal structures are among the examples in which (1) is vacuous, and one obtains just a G_0-structure ("AHS structures"). There are examples for which (2) is vacuous since $G_0 = \mathrm{Aut}_{gr}(\mathfrak{g}_-)$, e.g. various generic distributions.

Projective structures are one of two examples in which the Cartan geometry is not determined by the underlying structure. Here $G = SL(n+1, \mathbb{R})$ and P is the stabilizer of a ray in $\mathbb{R}^{n+1}$, so $G_0 = GL^+(n, \mathbb{R})$. Then $\mathcal{G}_0 \to M$ is the full oriented frame bundle of M. Any G_0-equivariant section $\mathcal{G}_0 \to \mathcal{G}$ pulls back the $\mathfrak{g}_0$-component of ω to a principal connection on $\mathcal{G}_0$.

Hence there is a class of distinguished connections on TM. It turns out that they all are torsion-free and have the same geodesics up to parametrization. This leads to a "projective equivalence class" of torsion-free connections, which is equivalently encoded by the Cartan geometry.

Parabolic geometries and geometric compactifications lecture 2

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Recap / program

- In the first lecture, we have discussed the description of conformal structures as Cartan geometries and the generalization to parabolic geometries.
- Today's lecture will start with a fundamental example of geometric compactifications. Starting from the example of hyperbolic space, I will introduce the concept of conformally compact metrics and of Poincaré-Einstein metrics, which are of interest in a broad variety of slightly different settings.
- We then show an efficient description of such metrics via the standard tractor bundle associated to the conformal Cartan geometry. This relates Poincaré-Einstein metrics to parallel tractors and hence to reductions of conformal holonomy.
- I'll briefly outline work of R. Gover and A. Waldron on a resulting boundary calculus and generalizations of the Willmore energy.

Contents

The model example for a geometric compactification is adding the sphere S^n as a boundary at infinity to hyperbolic space $\mathcal{H}^{n+1}$. Let $\overline{M} \subset \mathbb{R}^{n+1}$ be the closed unit ball, $\mathcal{H}^{n+1}$ its interior endowed with the *hyperbolic metric* $g := \frac{4}{(1-r^2)^2} g_{Euc}$ and S^n its boundary.

The function $\rho := 1 - r^2$ is an example of a *defining function* for the boundary $S^n \subset \overline{M}$. This means that $\rho : \overline{M} \to \mathbb{R}$ is smooth with zero set S^n and $d\rho|_{S^n}$ is nowhere vanishing. Any other defining function is of the form $f\rho$, where $f : \overline{M} \to \mathbb{R}$ is smooth and nowhere vanishing (locally around S^n).

Turning things around, g has the property that $\rho^2 g$ admits a smooth extension to all of $\overline{M}$ with the boundary values defining a Riemannian metric on S^n (the round one). This then holds for any defining function, but one obtains a metric on S^n conformal to the round one. Then $Isom(\mathcal{H}^{n+1}) \cong Conf(S^n)$. Observe that $\rho^\alpha g$ does not extend for $\alpha < 2$, while for $\alpha > 2$ it extends, but the boundary values are zero.

There is a general concept of local defining functions (and sections of line bundles) for arbitrary hypersurfaces $\Sigma \subset M$. In particular, this applies to the boundary in any manifold with boundary. The crucial feature of those is that any smooth function f such that $f|_\Sigma = 0$ can be written as ρh for a smooth function h. This leads to a notion of *order of vanishing* on Σ and of *growth* towards Σ.

Definition

Let $\overline{M}$ be a smooth manifold with boundary ∂M and interior M. A Riemannian metric g on M is called *conformally compact* if for any local defining function ρ for ∂M, the metric $\rho^2 g$ admits a smooth extension to all of $\overline{M}$, whose restrict to $T\partial M$ is non-degenerate. If g in addition is Einstein with negative scalar curvature, then it is called a *Poincaré-Einstein metric* (PE metric).

This leads to a well defined conformal structure $[\rho^2 g|_{T\partial M}]$ on ∂M, the *conformal infinity* of g. One is led to a variety of interesting problems in different settings:

- Starting from (M, g), one studies asymptotic aspects of Riemannian geometry, in particular in the PE case.
- Looking for asymptotic invariants of metrics that are asymptotic to the hyperbolic metric leads to a hyperbolic version of mass. (Here the PE case is trivial.)
- Given a conformal structure on ∂M, one can try to "fill in" a PE metric on M. This is interesting both on a formal level (Fefferman-Graham) and on an analytical level.
- The picture is the setup for the AdS/CFT correspondence and various versions of holography in physics.
- This is the model for compactifications of symmetric spaces. In general, the boundary structure is much more involved and it is difficult to endow boundary components with reasonable geometric structures.

We will next describe the setup from the point of view of conformal geometry.

densities

- A metric g on N defines a volume density $\mathrm{vol}_g = \sqrt{\det(g_{ij})}$.
- Forming powers and duals of the resulting line bundle, one obtains a family $\mathcal{E}[w] \to N$ of line bundles for $w \in \mathbb{R}$. The standard convention is that $\mathrm{vol}_g \in \Gamma(\mathcal{E}[-n])$.
- $\mathcal{E}[w]$ is associated to $\mathcal{G}_0$ via a representation of $Z(CO(n))$.

For a choice of metric g, $(\mathrm{vol}_g)^{-w/n}$ is a nowhere vanishing section of $\mathcal{E}[w]$, thus identifying $\Gamma(\mathcal{E}[w])$ with $C^\infty(N, \mathbb{R})$. Changing from g to $\hat{g} = f^2 g$, this identification changes as $\hat{\sigma} = f^{-w}\sigma$, which explains the convention.

Conversely, for $w \neq 0$, any nowhere vanishing $\sigma \in \Gamma(\mathcal{E}[w])$ determines a unique metric g in the class such that σ is parallel for the connection induced by the Levi-Civita connection ∇^g.

We will use abstract indices, so $\mathcal{E}^a = TN$, $\mathcal{E}_a = T^*N$ and so on. Adding $[w]$ indicates a tensor product with $\mathcal{E}[w]$.

The conformal class spans a line subbundle of $\mathcal{E}_{(ab)}$ isomorphic to $\mathcal{E}[-2]$. This defines a tautological section $\mathbf{g}_{ab} \in \Gamma(\mathcal{E}_{(ab)}[2])$ ("conformal metric"). This has an inverse $\mathbf{g}^{ab} \in \Gamma(\mathcal{E}^{(ab)}[-2])$. Hence we may raise and lower indices at the expense of a weight.

We next describe the *standard tractor bundle* $\mathcal{E}^A$ which is an equivalent encoding of the Cartan geometry associated to a conformal structure. Recall that this has type (G, P), where $G = SO_0(n+1, 1)$. Restricting the standard representation gives a representation of P on $\mathbb{V} := \mathbb{R}^{n+2}$ and $\mathcal{E}^A = \mathcal{G} \times_P \mathbb{V}$ and we get:

- A Lorentzian bundle metric h_{AB} with inverse h^{AB}.
- A line subbundle $\cong \mathcal{E}[-1]$ (isotropic for h) whose inclusion defines $X^A \in \Gamma(\mathcal{E}^A[1])$.
- A surjection $\mathcal{E}^A \to \mathcal{E}[1]$ given by $X_A = h_{AB}X^B$.

The natural line subbundle in $\mathcal{E}^A$ is isotropic, thus contained in its orthocomplement and defining a filtration. We write this as a composition series $\mathcal{E}^A = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. Choosing a metric in the conformal class defines a splitting of the filtration, thus identifying $\mathcal{E}^A$ with a direct sum, which we denote by vectors.

Changing from g to $\hat{g} = f^2 g$, we put $\Upsilon_a = f^{-1}df$ and the splitting changes as $\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \\ \rho - \mathbf{g}^{ab}(\Upsilon_a \mu_b + \frac{1}{2}\Upsilon_a \Upsilon_b \sigma) \end{pmatrix}$. $\mathcal{E}^A$ carries the canonical tractor connection. In the splitting for g this is given in terms of $\nabla = \nabla^g$ and the Schouten tensor P_{ab} of g as $\nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab}\rho + \mathsf{P}_{ab}\sigma \\ \nabla_a \rho - \mathbf{g}^{ij}\mathsf{P}_{ai}\mu_j \end{pmatrix}$. Next, $D^A \tau := \begin{pmatrix} w(n+2w-2)\tau \\ (n+2w-2)\nabla_a \tau \\ -\mathbf{g}^{ij}(\nabla_i \nabla_j + \mathsf{P}_{ij})\tau \end{pmatrix}$ defines a natural operator $D^A : \Gamma(\mathcal{E}[w]) \to \Gamma(\mathcal{E}^A[w-1])$.

We will mainly use this on $\mathcal{E}[1]$ and put $I^A := \frac{1}{n} D^A \sigma$, which has σ as its top component ("BGG splitting operator"). Of course, we can then form $|I|^2 := h_{AB} I^A I^B$, which is a smooth function.

To interpret $|I|^2$, we first look at $U := \{x : \sigma(x) \neq 0\}$. The metric $g_{ab} := (1/\sigma^2)\mathbf{g}_{ab}$ on U satisfies $\nabla_a \sigma = 0$, and in this scale, it is evident that $|I|^2$ is a negative multiple of $Scal(g)$. For $x \notin U$, $\nabla_a \sigma(x)$ is independent of the choice of metric and outside of U, we get $|I|^2 = \mathbf{g}^{ij}(\nabla_i \sigma)(\nabla_j \sigma)$.

Parallel sections of $\mathcal{E}^A$ are closely related to Einstein metrics in the conformal class:

- Any parallel section is of the form I^A as above. (Determined by the top component.)
- For U and g as above, $\nabla_a I^A|_U = 0$ is equivalent to the Schouten tensor P_{ab} of g being proportional to g_{ab} and hence to g being Einstein.
- If I^A is parallel, then $|I|^2$ is constant and on U is a negative multiple of the Einstein constant of g.

Consider a conformal manifold $\overline{M} = M \cup \partial M$ with boundary, let g be a metric in the class on M and take $\sigma := (\text{vol}_g)^{-1/n} \in \Gamma(\mathcal{E}[1])$.

Then g is conformally compact iff σ extends by zero to a defining density for ∂M.

Proof: For a local defining function ρ for ∂M put $\hat{g} := \rho^2 g$. If g is conformally compact, $\hat{g}$ is a metric in our class defined on all of $\overline{M}$. Thus $\hat{\sigma}$ is nowhere vanishing. But $\text{vol}_{\hat{g}} = \rho^n \text{vol}_g$ and hence $\sigma = \rho\hat{\sigma}$ on M, which shows that σ extends as required. Conversely, if σ extends to a defining density, then $\rho^{-1}\sigma$ smoothly extends to $\overline{M}$ and the metric it determines coincides with $\rho^2 g$ on M.

Theorem

For $\overline{M} = M \cup \partial M$ let g be a negative Einstein metric on M such that the conformal class $[g]$ smoothly extends to $\overline{M}$, but g itself does not admit a smooth extension to any neighborhood of a boundary point (e.g. because g is complete).
Then g is conformally compact and hence Poincaré-Einstein.

- $\mathcal{E}^A$ and the tractor connection are defined on $\overline{M}$.
- The tractor I^A determined by g is parallel over M hence can be smoothly extended to a parallel tractor on $\overline{M}$.
- Projecting I^A to $\Gamma(\mathcal{E}[1])$ provides a (unique) smooth extension of σ to all of $\overline{M}$.
- If $\sigma(x) \neq 0$ for some $x \in \partial M$, one obtains a smooth extension of g to a neighborhood of x, so all boundary values are zero.
- Since $|I|^2$ is constant on $\overline{M}$ and nonzero on M, σ is a defining density.

The setup described here is the starting point for a detailed analysis in several articles of R. Gover and A. Waldron: Given $\overline{M} = M \cup \partial M$ and a conformally compact metric g on M, take the corresponding conformal structure on $\overline{M}$, the defining density $\sigma \in \Gamma(\mathcal{E}[1])$ for ∂M selected by g and put $I^A := \frac{1}{n} D^A \sigma$. We assume that $|I|^2$ is nowhere vanishing.

Consider $I \cdot D : \Gamma(\mathcal{E}[w]) \to \Gamma(\mathcal{E}[w-1])$, $\tau \mapsto h_{AB} I^A D^B \tau$. This naturally extends to sections of weighted tractor bundles.

- On M, this is a Yamabe type operator associated to g.
- If $|I|^2 \equiv 1$ close to ∂M, then it restricts to the conformally invariant Robin operator on a neighborhood of ∂M.
- Together with multiplication by σ and a weight operator, $I \cdot D$ forms an $\mathfrak{sl}_2$-triple. This allows for very efficient computations (analysis of eigenfunctions, problems of harmonic extension, operators acting tangentially, etc.)

There are very interesting applications to the study of (oriented) hypersurfaces Σ in a conformal manifold $(N,[g])$. The natural question here is whether one can find a defining density $\sigma\in\Gamma(\mathcal{E}[1])$ for $\Sigma\subset N$ such that the corresponding tractor $I^A=\frac{1}{n}D^A\sigma$ satisfies $|I|^2\equiv 1$ (singular Yamabe problem).

Starting from any defining density σ_0 for Σ the problem can be studied formally along Σ:

- If $n=\dim(N)$, there exists σ (unique up to $\mathcal{O}(\sigma^{n+1})$) such that $|I|^2=1+\mathcal{O}(\sigma^n)$.
- For this σ, $\sigma^{-n}(|I|^2-1)$ is a smooth section of $\mathcal{E}[-n]$ defined locally around Σ, whose restriction to Σ is an invariant of $(N,[g],\Sigma)$.
- for $n=3$, this produces the Willmore energy, so one obtains a natural family of higher order Willmore energies and invariants.

Parabolic geometries and geometric compactifications lecture 3

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Recap / program

- In the second lecture we have seen that Poincaré-Einstein metrics can be described in terms of conformal structures that admit a parallel section of the standard tractor bundle. This provides a relation to holonomy reductions of Cartan connections which will be discussed in the first part of today's talk.
- In the second part of the talk, I will discuss how the holonomy perspective leads to the concepts of projective compactness that has been developed in several recent articles of R. Gover and myself. I will also make some remarks on c-projective compactness.
- In the last part of the talk, I will sketch applications of the holonomy theory to the study of compactifications of symmetric spaces and more general homogeneous spaces.

Contents

- Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Since ω trivializes $T\mathcal{G}$, it has no holonomy in a naive sense.
- As we have seen, there is an induced principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$, which has a holonomy group $\subset G$.

For conformal Cartan connections, this was studied since the early 2000's under the name "conformal holonomy". There were some early results, including classification results on possible holonomy groups (S. Armstrong), but some basic geometric features remained unnoticed for a longer time.

The fact that a parallel standard tractor is equivalent to an Einstein metric on a dense open subset was fundamental for the classification results. But the relation to Poincaré-Einstein was first noticed in a 2009 article of R. Gover, and the general holonomy theory appeared in a 2014 article of R. Gover, M. Hammerl and myself:

Let $\mathcal{O}$ be a homogeneous space of G. For a geometry $(\mathcal{G},\omega)$ of type (G,P), one gets $\mathcal{G}\times_P\mathcal{O}\cong\tilde{\mathcal{G}}\times_G\mathcal{O}$, so this inherits an Ehresmann connection. One defines a *holonomy reduction of type* $\mathcal{O}$ to be a parallel section of that bundle. This corresponds to a P-equivariant smooth function $f:\mathcal{G}\to\mathcal{O}$.

Let $\mathcal{O}=\sqcup\mathcal{O}_i$ be the decomposition of $\mathcal{O}$ into P-orbits. Then for each $x\in M$, $f(\mathcal{G}_x)\subset\mathcal{O}$ is one of these orbits and if this is $\mathcal{O}_i$, we say that $i\in\mathcal{O}/P$ is the *P-type of x*. Correspondingly, we get a decomposition $M=\sqcup M_i$ according to P-types.

For G/P, one verifies that there is a unique holonomy reduction of type $\mathcal{O}$ for each $\alpha\in\mathcal{O}$, corresponding to the P-equivariant function $G\to\mathcal{O}$ given by $f(g)=g^{-1}\cdot\alpha$. Putting $H:=G_\alpha$ (so we can identify $\mathcal{O}$ with G/H) we conclude that the decomposition of G/P according to P-types coincides with the decomposition into H-orbits. Thus P-types are indexed by $H\backslash G/P$ in general.

It is a general result that H-orbits in G/P are initial submanifolds, and of course they are homogeneous spaces of H. In the curved case, one uses a version of normal coordinates to prove:

Let $x \in M$ be a point of P-type i, fix a holonomy reduction of G/P corresponding to H for which eP has P-type i. Let L_i be the stabilizer of eP in H. Then

1. There are open neighborhoods U of x and V of eP and a diffeomorphism $\varphi : U \to V$ that is compatible with the decomposition into P-types. In particular, $M_i \subset M$ is an initial submanifold.
2. The initial submanifold M_i inherits a Cartan geometry of type (H, L_i) from the holonomy reduction.

This motivates the terminology "curved orbit decomposition". The curved orbits $M_i \subset M$ cannot look worse than the H-orbits in G/P.

For the case relevant to Poincaré-Einstein, $G = SO_0(n+1,1)$, P is the stabilizer of a null ray and $H \cong SO(n,1)$ is the stabilizer of a positive vector in $\mathbb{R}^{n+1,1}$. This gives

- $M = M_- \cup M_0 \cup M_+$, with $M_\pm$ open and M_0 (if non-empty) a separating hypersurface according to the inner product of the vector with the ray.
- $L_\pm \cong SO(n)$, so $H/L_\pm$ is hyperbolic space and on $M_\pm$ obtains induced Riemannian metrics. Einstein follows from normality of the conformal Cartan connection.
- L_0 is the stabilizer of a null line in $\mathbb{R}^{n,1}$, H/L_0 is the conformal $(n-1)$-sphere and on M_0 one obtains the normal Cartan geometry associated to a conformal structure.

So this not only provides the picture for Poincaré-Einstein we had, but it also leads to the canonical Cartan geometries associated to all the geometric structures on the curved orbits.

Projective structures are one of two examples of parabolic geometries that are not determined by the underlying structure discussed in lecture 1. Here $G = SL(n+1,\mathbb{R})$ and P is the stabilizer of a ray in $\mathbb{R}^{n+1}$, so $G_0 = GL^+(n,\mathbb{R})$.

For a Cartan geometry of type (G,P), the underlying bundle $\mathcal{G}_0 \to N$ is the full oriented frame bundle of N. But one can use G_0-equivariant sections $\mathcal{G}_0 \to \mathcal{G}$ to pull back the $\mathfrak{g}_0$-component of ω to a principal connection on $\mathcal{G}_0$.

This defines a family of linear connections ∇ on TN, that turn out to be torsion free and have the same geodesics up to parametrization. Thus they form a "projective equivalence class" and it turns out that the Cartan geometry equivalently encodes this class respectively the family of geodesic paths.
There are again density bundles $\mathcal{E}(w)$ and non-vanishing densities select connections in the class.

There are two relevant holonomy reductions here: First, we can take the stabilizer H_1 of a linear functional α on $\mathbb{R}^{n+1}$. On the homogeneous model, this decomposes S^n into two open hemispheres which inherit flat connections and the totally geodesic equator.

On a curved geometry, such a reduction is given by a parallel sections of $\mathcal{G} \times_P \mathbb{R}^{(n+1)*}$ and provides

- $M = M_- \cup M_0 \cup M_+$ with $M_\pm$ open and M_0 a totally geodesic separating hypersurface.
- Ricci flat connections in the projective class on $M_\pm$.
- The normal Cartan geometry associated to a projective structure on M_0.

If the connections on $M_\pm$ are Levi-Civita for some metric, there is an induced holonomy reduction for the Cartan geometry on M_0.

For the stabilizer H_2 of a non-degenerate bilinear form of signature $(n,1)$ on $\mathbb{R}^{n+1}$, the orbit structure on G/P is more complicated:

- Two copies of Riemannian hyperbolic space (negative rays)
- two copies of a Riemannian conformal sphere (null rays)
- One copy of Lorentzian de Sitter space (positive rays)

On curved geometries, such holonomy reductions are equivalent to *Klein-Einstein metrics*. The setup looks similar to the PE case: Open curved orbits inherit Einstein metrics and there are separating hypersurfaces that inherit the normal Cartan geometries associated to a conformal structure.

From a geometric point of view, there are several remarkable differences to the Poincaré-Einstein case:

- Here the metrics on the open orbits have different signature (Riemannian and Lorentzian). Crossing a boundary, the signature changes.
- What one actually gets on the open orbits are is a connection ∇ with non-degenerate, parallel Schouten-tensor (which then is an Einstein metric).
- The induced conformal structure on M_0 can be described in terms of the projective second fundamental form (which is well defined up to scale).
- The conformal tractor bundle on M_0 is the projective standard tractor bundle endowed with the the parallel bundle metric defining the holonomy reduction.

Trying to weaken the two types of holonomy reductions to obtain an analog of conformal compactness, it turns out that they both fit into a broader picture:

A projective modification of ∇ is described by a one-form Υ via $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. One defines projective compactness by the requirement that certain modifications admit a smooth extension, but this allows for an additional parameter.

Fix $\alpha \in (0, 2]$. For $\overline{M} = M \cup \partial M$ a connection ∇ on M is called *projectively compact of order* α if for a local defining function ρ for ∂M, the projective modification of ∇ determined by $\Upsilon := \frac{d\rho}{\alpha\rho}$ admits a smooth extension to $\overline{M}$.

- α is related to the rate of volume growth, $\alpha \in (0, 2]$ is needed for the boundary to be at infinity.
- The holonomy reductions from before are special cases for $\alpha = 1$ (Ricci flat) and $\alpha = 2$ (Einstein), respectively.
- If ∇ preserves a volume form, projective compactness of order α is equivalent to a defining density in $\mathcal{E}(\alpha)$.

Selected results on projective compactness

- Asymptotic forms for metrics that are sufficient for projective compactness of any order α, thus many local examples.
- For $\alpha = 2$, the asymptotic form is equivalent to projective compactness.
- If the projective structure $[\nabla]$ determined by a metric g on M admits a smooth extension to $\overline{M}$, then $Scal(g)$ extends to $\overline{M}$. Locally around boundary points in which this extension is non-zero, g is projectively compact of order 2.
- Suppose that g is Einstein on M, $[\nabla]$ extends to $\overline{M}$ but ∇ does not extend to any open subset of ∂M. Then g is projectively compact of order 1 if $Ric(g) = 0$ and of order 2 if $Ric(g) \neq 0$ (different rates of volume growth are forced).
- Explicit description of the conformal boundary tractor bundle and connection from the projective data in the interior.

There is an almost complex analog of projective compactness of order 2, which involves c-projective geometry ("c-projective compactness"). In its metric version, this involves (quasi-)Kähler metrics in the interior and (almost) CR structures on the boundary.

Results

- The complete Kähler metrics on smoothly bounded domains constructed from defining functions are c-projectively compact.
- Equivalent characterization via an asymptotic form.
- Holonomy reductions to $U(p,q) \subset SL(n,\mathbb{C})$ are contained as a special case. These lead to Kähler-Einstein metrics in the interior.

Suppose that G/P is a generalized flag manifold and $H \subset G$ is a subgroup that acts with finitely many orbits on G/P. Then there are open H-orbits H/L which inherit a (locally flat) parabolic geometry of type (G, P) and the closure in G/P defines a compactification of H/L. This is not a rare situation:

Theorem (J. Wolf, 1976)

Let G be a real simple Lie group and θ an involutive automorphism of G with fixed point group $H \subset G$. Then for each parabolic subgroup $P \subset G$, H acts on G/P with finitely many orbits.

Example: For $p \leq q$ put $n = p + q$, take $G := SL(n, \mathbb{R})$ and P such that $G/P = Gr(p, \mathbb{R}^n)$. Let $H := SO(p, q) \subset G$ defined by a bilinear form b on $\mathbb{R}^n$. H-orbits in G/P are determined by rank and signature of the restriction of b, and for positive definite restriction, one gets the Riemannian symmetric space H/K, $K = S(O(p) \times O(q))$.

Hence we get a compactification of H/K, in which all positive semi-definite subspaces are added as boundary components. The space H/K carries an invariant Grassmannian structure (decomposition of TM as a tensor product), which admits a smooth extension across the boundary.

Results

- b can be used to construct parallel sections of tractor bundles that project to solutions of invariant operators.
- Description of orbit-closures as zero sets of such solutions.
- Construction of defining sections for orbits, which lead to slice theorems that describe the neighborhood of each orbit in the compactification.
- Local models turn out to be symmetric matrices decomposed according to rank.