

String Field Theory

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In these lectures we are going to talk about string field theory, a quantum field theory of strings. Our specific objective is to understand how homotopy algebras emerge as defining the gauge principles of string theory. We will not assume familiarity with string quantisation of perturbative quantum field theory. Unfortunately these topics are necessary. So we will introduce the basics and that will take up the majority of these lectures. We begin by drawing a table

Definition 1. *Fermat's Principle: The path that light particles choose is the one of least time between given points*

This is also known as the principle of least time. If light travels with constant speed in a vacuum, the shortest path is a straight line. There is an analogue principle called

Definition 2. *The principle of least area: As a string travels in time, it sweeps out a surface. The world-sheet that the string chooses should be the one of minimal area.*

On the other side, light can be understood as a disturbance of the electromagnetic field. This is Maxwell's equations and theory. If we include quantum effects, we get quantum electrodynamics. Other fundamental forces can also be understood as quantum disturbances of a field, such as non-abelian gauge theories. Even gravity can be understood as a gauge theory, but in that case we don't know how to include quantum effects. In these days the framework of quantum field theories has mostly superseded the particle ideas. But it turns out that particle ideas can in fact reproduce a lot of results from QFT. It's not necessarily wrong, just less useful, because in physics we have situations where we are interested in an unbounded number of particles with strong interactions. In such situations, the particle notions start to lose their utility, while

Particles	Strings
Fermat's Principle	Principle of least area
Quantum Electrodynamics (and non-abelian gauge theories)	String field theory

the QFT framework remains effective. On the other hand, most string theorists understand string theory from the principle of least area (Fermat's principle). If we are interested in situations containing an unbounded number of strings, the principle of least area loses utility in the sense that it doesn't lead to useful understanding. A resolution to this problem is to understand string theory as a quantum field theory. If we can do this, then we will understand what string theory is as a complete theory. In practice string field theory is a rather abstract formalism, and in the minds of most physicists it has not yet superseded the principle of least area.

1.1 Gauge symmetry

We will describe our physical system using dynamical variables that redundantly parametrize the set of physically in-equivalent situations. In other words you can have two unequal sets of dynamical variables that describe the same situation. However there is a transformation between physically equivalent dynamical variables which forms a group. This is called a gauge group. We require that physically measurable quantities are invariant with respect to this gauge group. This can provide constraints to the form of the theory.

Remark 1. *In string theory, the gauge symmetry fixes the form of the theory uniquely. This is via homotopy algebras.*

1.2 Particle actions

One thing that Fermat didn't appreciate when he formulated his principle is that light not only follows a trajectory, but also has a polarisation. That is, photons have spin. So unfortunately Fermat's principle doesn't accommodate this. We won't be able to derive Maxwell's equations. Still, we can describe particles with vanishing spin. This means particles of spin 0 and mass m .

Remark 2. *On the other hand strings have internal motion that allow us to parametrize things like spin.*

A particle can follow a trajectory from x_i^μ to x_f^μ , $x^\mu(s)$, $s \in [0, 1]$. The action is given by

$$S[x^\mu(s)] = m \int_0^1 ds \sqrt{\dot{x}^\mu(s) \dot{x}_\mu(s)} \quad (1)$$

We will Wick-rotate to Euclidean signature. The action principle tells you that the path is a straight line, but it does not provide a parameterization s .

1.3 Gauge symmetry

Our parametrized curve redundantly describes the set of equivalent objects, i.e unparametrized paths. There is an action by the diffeomorphism group

Diff([0, 1]). So in particular if $f(s)$ is a diffeomorphism of [0, 1], the action is to re-parametrize the path:

$$f \circ x^\mu(s) = x^\mu(f(s)) \quad (2)$$

And this action leaves the Action invariant:

$$f \circ S[x^\mu(s)] = S[x^\mu(s)] \quad (3)$$

Now we want to include quantum effects. What we should do is take some kind of average between all paths between the two points: “sum over histories”

$$\int [dx^\mu(s)] e^{-S[x^\mu \sigma]} \quad (4)$$

We will introduce a new but equivalent action instead.

$$S[e(s), x^\mu(s)] = \frac{1}{2} \int_0^1 ds \left(\frac{1}{2} e(s)^{-1} \dot{x}^\mu(s) \dot{x}_\mu(s) + m^2 e(s) \right) \quad (5)$$

The einbein $e(s) > 0$ represents the distance on a world-line between s and $s + ds$. We can require that the action is extremal, and then the einbein is in fact determined to be

$$e(s) = \frac{\sqrt{\dot{x}^\mu(s) \dot{x}_\mu(s)}}{m} \quad (6)$$

The advantage of the new formulation is that the action is now a quadratic form in the path of the variable, which we know how to evaluate. This action still admits a gauge invariance under diffeomorphisms of the line segment:

$$\begin{aligned} f \circ x^\mu(s) &= x^\mu(f(s)) \\ f \circ e(s) &= \left(\frac{df(s)}{ds} \right)^{-1} e(f(s)) \\ f \circ S[e(s), x^\mu(s)] &= S[e(s), x^\mu(s)] \end{aligned}$$

Now we can integrate

$$\int [dx^\mu(s)] [de(s)] e^{-S[e(s), x^\mu(s)]} \equiv G(x_i^\mu, x_f^\mu) \quad (7)$$

This is almost the amplitude for a quantum particle to pass from the initial point to our final point. What is amplitude? If one rotates this quantity and takes the square of its magnitude, it should represent a probability. Unfortunately it's not a physically meaningful statement to say that a particle is located x_i^μ at time $s = 0$. If we had this, then what we have would in fact be the amplitude. Thus what we have is something more formal, called the

- “Off-shell amplitude”
- (2-point) Green's function

- propagator

It is a building block that will allow us to compute other quantities. Our task is now to evaluate this path integral.

Problem 1. *We get a divergent factor proportional to the volume of the group of diffeomorphisms of the unit line segment.*

You are supposed to assume that this factor is divided out. How do we remove volume factor?

Observation 1. *For a given configuration $e(s)$, there is a unique diffeomorphism $f(s)$ such that if you transform $f \circ e(s) = \tau$ you get a constant equal to the invariant length of $[0,1]$.*

$$f^{-1}(s) = \frac{1}{\tau} \int_0^s ds' e(s') \quad (8)$$

We can make the einbein constant, but we can not make it equal to one. Thus there is a moduli space given by the length, modulo diffeomorphisms. Thus we can write

$$[de(s)] = d\tau[df(s)](\text{determinant}) \quad (9)$$

The determinant can be accounted for by introducing what are called “ghost fields” $b(s), c(s)$ on the world-line with BRST symmetry. This is the origin of the homotopy algebra structures. In our particle example however this will be trivial, and we can more or less ignore it as it is too degenerate to be useful.

$$G(x_i^\mu, x_f^\mu) = \int_0^\infty d\tau \int [dx^\mu(s)] e^{-S[\tau, x^\mu(s)]} \quad (10)$$

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In our previous lecture we reduced the path integral to the form

$$G(x_i, x_f) = \int_0^\infty d\tau \int [dx^\mu(s)] e^{-\frac{1}{2} \int_0^1 ds (\frac{1}{\tau} \dot{x}^\mu(s) \dot{x}_\mu(s) + m^2 \tau)} \quad (11)$$

Here we integrate over the modulus τ which represents the invariant length of the world-line interval. The trick to evaluate this further is to make a substitution,

$$x^\mu(s) = x_{cl}^\mu(s) + y^\mu(s) \quad (12)$$

Where the first term is the classical path. If you do this and normalize the integral over y^μ , G will evaluate to

$$G = \int \frac{d^D p}{(2\pi)^D} e^{ip_a(x_i^\mu - x_f^\mu)} \int_0^\infty d\tau e^{-\tau(p^\mu p_\mu + m^2)} = \frac{d^D p}{(2\pi)^D} e^{ip_a(x_i^\mu - x_f^\mu)} \frac{1}{p^2 + m^2} \quad (13)$$

which is the propagator of scalar QFT.

2.1 Higher point Green's functions

Suppose we have multiple points, x_1, x_2, x_3, x_4 . And suppose that particles don't just move from one point to another, but may be created or destroyed. We describe this by different diagrams and coupling constants. To compute the 4-point Green's function, we would have to evaluate a lot of terms coming from different diagrams of multi-particle interactions. In principle we could consider an n -point Green's function:

$$G(x_1, x_2, \dots, x_n) = \sum_{\text{sum over graphs}} \Gamma \quad (14)$$

This is generally an asymptotic and divergent series. Graphs with loops can occur, in particular "tadpole graphs". This has one cubic vertex coming with a coupling constant g_{03} . We integrate over the interaction point

$$g_{03} \int d^p y G(y, y) G(y, y) \quad (15)$$

and

$$G(y, y) = \int \frac{d^D p}{(2\pi)^D} \int_0^\infty d\tau e^{-\tau(p^2 + m^2)} = \int_0^\infty d\tau e^{-\tau m^2} \frac{\text{Vol}(S^{D-1})}{(2\pi)^D} \int d|p| |p|^D e^{-\tau|p|^2} \quad (16)$$

yielding

$$G(y, y) = \int_0^\infty \frac{d\tau}{\tau^{D/2}} e^{-\tau m^2} \left(\frac{\text{Vol}(S^{D-1})}{(2\pi)^D} \int_0^\infty d\alpha \alpha^{D-1} e^{-\alpha^2} \right) \quad (17)$$

which is divergent if $D \geq 2$. This is the famous ultraviolet divergence. We want to emphasize that this appears when the length of the loop shrinks down to 0. To deal with this divergence we introduce a "counter-term" which cancels out the divergence, which comes with a coupling constant $g_{1,1}$ representing the fact that it absorbs one term. We may need to introduce counter-terms systematically for each order in perturbation theory for all graphs with some coupling constant g_{3n} . It could happen that the coupling constants are chosen in such a way that you only need a finite number of counter-terms. In that case the theory is called renormalizable. Theories which do not have this property have no predictive power in general.

2.2 Quantum states

We computed Green's functions for particles connecting specific positions. But in the quantum setting a particle might be in a distributed superposition of locations. This is described by a wavefunction

$$\phi(x) : \mathbb{R}^D \rightarrow \mathbb{C} \quad (18)$$

This is also called the quantum state. They are elements of an infinite dimensional vector space \mathcal{H} . In principle we would like this to be a Hilbert space so

that we can take squares. However this space is not physical, and in fact in string theory the space will turn out to not be Hilbert. In the formalism of quantum field theory, $\phi(x)$ is a spacetime field. We know that quantum states evolve according to the Schrödinger equation,

$$H\phi = \frac{\partial}{\partial t}\phi \quad (19)$$

In our case the time parameter $t = s$ is equal to the world-line parameter. The Hamiltonian H can be derived from our action

$$S = \frac{1}{2} \int ds(x^2 + m^2) \quad (20)$$

and it is

$$H = -\partial_\mu \partial^\mu + m^2 \quad (21)$$

It is not meaningful to say that any physical configuration is changing with respect to the world-line parameter s . This motivates a distinction between physical quantum states and unphysical quantum states. The physical quantum states, also called “on-shell” quantum states, satisfy

$$(-\partial^\mu \partial_\mu + m^2)\phi = 0 \quad (22)$$

The space of physical states $\mathcal{H}_{phys} \subset \mathcal{H}$ forms a subset of the full space of quantum states. It might seem that if we want to compute physical quantities, we should compute Green’s functions ...

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So far we have started with Fermat’s principle and tried to derive quantum field theory. Hopefully this will aid in understanding the sketch of string theory which will follow.

1. Fermat \rightarrow Action \rightarrow Quadratic form in position and einbein
2. Evaluate propagator (path integral) for particle starting at x_i and ending at x_f . This involved integrating over a modulus or length τ of the particle world-line.
3. Higher point Green’s functions $G_n(x_1, \dots, x_n)$. This involved several particles at many positions. This involved summing over all the ways the particles could split and join. We discussed loops and ultraviolet divergences and counter-terms.
4. We discussed the notion of a wavefunction (quantum state) $\phi(x) \in \mathcal{H}$. Inside \mathcal{H} there is a subspace of physical quantum states \mathcal{H}_{phys} . These are states which satisfy the Klein-Gordon equation $(\partial^\mu \partial_\mu + m^2)\phi(x) = 0$. These quantum states allow us to deal with superpositions.

We may consider G_n to be a multilinear map:

$$G_n : \mathcal{H}^{\otimes n} \rightarrow \mathbb{C} \quad (23)$$

Where

$$G_n(\phi_1, \dots, \phi_n) = \int dx_1 \dots dx_n G(x_1, \dots, x_n) \phi_1(x) \dots \phi_n(x) \quad (24)$$

If we want to work with something where the probabilities make sense, you might think that we should substitute physical wavefunctions into this equation. This is incorrect. The problem is that physical wavefunctions make a divergence due to not evolving in time. Basically you are integrating $\int_0^\infty \tau d\tau$. What we will do is “amputate” the Green’s function to define “physical amplitudes”. A physical amplitude can be regarded as a linear map

$$\mathcal{A}_n : \mathcal{H}_{phys}^{\otimes n} \rightarrow \mathbb{C} \quad (25)$$

Taking multiple physical states to a number. This is our first physically measurable quantum quantity.

$$G_n(x_1, \dots, x_n) \mapsto (\partial_1^2 + m^2)(\partial_2^2 + m^2) \dots (\partial_n^2 + m^2) G_n(x_1, x_2, \dots, x_n) \quad (26)$$

The above is a naive procedure, because amputating the propagator is not general enough, it only fixes divergences from loops. What needs to be done is to remove each external propagator and replace them with physical states.

Remark 3. *The analogous problem is difficult to deal with in conventional string theory. In that case, the resolution seems to be best understood by applying string field theory, which allows you to compute the quantum corrections to the mass of highly excited strings.*

Now we have

$$\mathcal{A}_n(\phi_1, \dots, \phi_n) = \sum_{\text{moduli space of that graph}} \int \text{amputated graphs} \quad (27)$$

This has a close analogue in string theory.

3.1 Ward identities

Suppose your system has some symmetries. These symmetries yield some identities for \mathcal{A}_n . There are not that many symmetries for particles, but there is translation symmetry. We know that there is not preferred choice of origin in spacetime, so if we translate all ϕ_i , the probability amplitude should be the same.

Example 1.

$$\mathcal{A}_n(\partial_\mu \phi_1, \dots, \phi_n) + \mathcal{A}_n(\phi_1, \partial_\mu \phi_2, \dots, \phi_n) + \dots + \mathcal{A}_n(\phi_1, \phi_2, \dots, \partial_\mu \phi_n) = 0 \quad (28)$$

Such equalities are called Ward identities. It means that $G(x_i, x_f)$ only depends on $x_i^\mu - x_f^\mu$. Ward identities imply identities for vertices where particles interact. We may write a vertex as a multilinear map:

$$\mathcal{V}_{g,n} : \mathcal{H}^{\otimes n} \rightarrow \mathbb{C} \quad (29)$$

where g is the genus.

$$\mathcal{V}_{0,3}(\phi_1, \phi_2, \phi_3) = g_{0,3} \int d^D x \phi_1(x) \phi_2(x) \phi_3(x) \quad (30)$$

Translation identity for \mathcal{A}_n implies

$$\mathcal{V}_{g,n}(\partial_\mu \phi_1, \dots, \phi_n) + \dots + \mathcal{V}_{g,n}(\phi_1, \dots, \partial_\mu \phi_n) = 0 \quad (31)$$

This is essentially how homotopy algebras appear. The Ward identities that appear force the operators to satisfy A_∞ or L_∞ relations.

3.2 Quantum field theory

We promote our quantum states $\phi(x) \in \mathcal{H}$ to a field. Our theory is defined by integrating over $\phi(x)$. It is governed by an action principle

$$S[\phi] = \frac{1}{2!} \int dx \phi(x) (\partial^\mu \partial_\mu + m^2) \phi(x) + \frac{1}{3!} \mathcal{V}_{0,3}(\phi, \phi, \phi) + \frac{1}{4!} \mathcal{V}_{0,4}(\phi, \phi, \phi, \phi) \dots \quad (32)$$

$$+ \frac{1}{1!} \mathcal{V}_{1,1}(\phi) + \frac{1}{2!} \mathcal{V}_{1,2}(\phi, \phi) + \frac{1}{3!} \mathcal{V}_{1,3}(\phi, \phi, \phi) + \dots + \frac{1}{1!} \mathcal{V}_{2,2}(\phi) + \dots \quad (33)$$

Where the latter terms are counter-terms. We get Green's functions

$$G_n(x_1, \dots, x_n) = \int [d\phi(x)] e^{-S[\phi]} \phi(x_1) \dots \phi(x_n) \quad (34)$$

See a textbook for this.

3.3 String theory

1. We start with the principle of least area. \rightarrow quadratic action X^μ with world-sheet metric g_{ab} .
2. Action is invariant under diffeomorphisms of the world-sheet and local scale transformations.
3. The action is defined on a Riemann surface.
4. We may compute the path integral of this action modulo diffeomorphisms and scale transforms. This yields \int moduli space of Riemann surfaces. The Jacobian determinant from equation 9 is now important, and this introduces ghost fields and BRST symmetry.

5. The space of quantum states of the string is a graded vector space once you account for the ghosts, and inside there are physical states that turn out to be invariant under BRST operators. Two physical states can be equivalent if they differ by the BRST difference of another state. This means that all our physical states are equivalent by addition of a BRST exact term, which implies a Ward identity.