Perturbative Quantum Field Theory
with Homotopy Algebras
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Abstract

The BV-formalism associates to any Lagrangian field theory an $L_\infty$-algebra. Any $L_\infty$-algebra comes with minimal models, and the minimal models of $L_\infty$-algebras originating from Lagrangian field theories capture the scattering amplitudes of these. The minimal models are easily computed using the homological perturbation lemma, which leads to recursive formulas helpful for their studies.

Pointers to literature (containing references to the original papers):

[1]: Review, conventions, technical details, classical field theory
[2]: shorter version of above
[3]: tree level amplitudes, scalar and Yang–Mills theory, Berends–Giele recursion
[4]: loop level amplitudes, homological perturbation lemma, non-planar diagrams

1. Lecture I

1.1. Motivation: Scattering amplitudes

Particle Accelerator (LHC at CERN):

- Beams of protons at very high velocity (0.99999999c) circulate in ring with diameter 27km.
- Collision in a small area, particles deflected and decay into new particles ("scatter")
- Detector measures these particles and their properties
- Initial and final particles are asymptotic and "free," i.e. non-interacting, not feeling potential

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Transition from asymptotic incoming to asymptotic outgoing particle configurations via S-matrix ("scattering matrix"):

\[ |\text{asymptotic out} \rangle = S |\text{asymptotic in} \rangle , \quad (1.1) \]

Note:
- \( |\ldots \rangle \) denote asymptotic (i.e. \( t = \pm \infty \)) configurations.
- \( |\ldots \rangle \) are vectors/rays in a Hilbert space \( \mathcal{H} \)
- \( \mathcal{H} \) is infinite dimensional (labeled e.g. by momentum of particle, spin, \ldots )
- \( S \) is a unitary operator

Scattering amplitude (probability amplitude) and probability:

\[ A := \langle \text{asymptotic out} | S |\text{asymptotic in} \rangle \quad \text{and} \quad |A|^2 . \quad (1.2) \]

Scattering amplitudes \( A \)
- are crucial to understand nature
- have interesting and surprising structures.
- computed via heuristics dubbed Quantum Field Theory

Various prescriptions for computing \( A \), but usually involving much machinery
- Classical action
- Quantization of fields
- Wick’s theorem
- (amputated) Feynman diagrams
- Dyson series
- LSZ reduction
- ...

This is complicated, hard to explain to mathematicians. Better:
- field theory actions \( \xleftarrow{\text{BV formalism}} L_\infty \)-algebras
- \( L_\infty \)-algebras have minimal models (from homological perturbation lemma)
- Minimal models encode scattering amplitudes

Also: exposes many structural results on scattering amplitudes
1.2. Homotopy algebras

(Name: generalizations of the classical algebras: associative, Lie, Leibniz, commutative, etc., in which the identities hold only “up to homotopy”)

Strict $L_{\infty}$-algebras: Differential graded Lie algebra $(\mathfrak{g}, [-, -], d)$:

- graded vector space: $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$
- $a \in \mathfrak{g}_i, b \in \mathfrak{g}_j$: $da \in \mathfrak{g}_{i+1}, [a, b] \in \mathfrak{g}_{i+j}$
- $[a, b] = (-1)^{ab}[b, a]$ $[a, [b, c]] = [[a, b], c] + (-1)^{ab}[b, [a, c]]$
- $d^2 = 0, d[a, b] = [d, a, b] + (-1)^a[a, db]$
- Maurer–Cartan: $da + \frac{1}{2}[a, a]$ for $a \in \mathfrak{g}_1$

Metric/quadratic extension: (“cyclic”)

- graded symmetric bilinear form $\langle - , - \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$
- compatibility: $\langle da, b \rangle + (-1)^a \langle db, a \rangle = 0 \quad \langle [a, b], c \rangle + (-1)^{ab} \langle b, [a, c] \rangle = 0$
- Maurer–Cartan action: $S = \frac{1}{2}\langle a, da \rangle + \frac{1}{3!}\langle a, [a, a] \rangle$.

Alternative description I: Codifferential on cocommutative coalgebra:

- $d$ of degree 1, $[-, -]$ of degree 2
- grade-shift: $\mathfrak{g}[1]$, defined via $(\mathfrak{g}[i])_j := \mathfrak{g}_{i+j}$, all have grade 1
- combine into codifferential $D = d + [-, -]$ on $\odot(\mathfrak{g}[1])$, $D^2 = 0$

Alternative description II, Dually: Differential graded commutative algebra:

- basis $\tau_\alpha$ of $\mathfrak{g}$, coordinate functions $\xi^\alpha \in \mathfrak{g}[1]$, $\xi^\alpha : \mathfrak{g}[1] \rightarrow \mathbb{R}$
- $Q = D^*$ is differential on $\odot(\mathfrak{g}[1]^*)$, $Q^2 = 0$
- actually: vector field on $\mathfrak{g}$: $Q = d^\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha} + \frac{1}{2} f^{\alpha\beta\gamma} \xi^\alpha \xi^\beta \frac{\partial}{\partial \xi^\gamma}$
- cyclic structure is a symplectic form $\omega$ on $\mathfrak{g}[1]$.
- “Chevalley–Eilenberg algebra $CE(\mathfrak{g})$ of $\mathfrak{g}$”

Extensions:

- general $L_{\infty}$-algebra: allow for arbitrary polynomial coefficients in $Q$ get totally antisymmetric linear “products” $\mu_i : \mathfrak{g}^\wedge i \rightarrow \mathfrak{g}$
- $D = \mu_1 + \mu_2 + \mu_3 + \ldots \quad D^2 = 0 \quad \Leftrightarrow \quad \mu_1^2 = 0, \ldots, \mu_1 \mu_3 = \mu_2 \mu_2$
- $L_{\infty}$-algebroids: allow for graded vector bundle over manifold
- $A_{\infty}$-algebra: replace $\odot$ by $\otimes$ everywhere
  Analogue of matrix algebras for matrix Lie algebras, Antisymmetrization: $L_{\infty}$
Special cases:
- trivial: $\ast$, $g_i = \ast$.
- strict: $\mu_i = 0$ for $i > 2$: differential graded Lie algebras
- skeletal: $\mu_1 = 0$
  example: $g = (R[1] \xrightarrow{0} su(n))$, $\mu_1 = 0$, $\mu_2 = [-, -]$, $\mu_3 = \langle -, [-, -] \rangle$ ("string Lie 2-algebra")
- linearly contractible: $\mu_i = 0$ for $i \geq 2$, $H^*_{\mu_1}(g)$ is trivial
  example: $R[q] \xrightarrow{id} R[q - 1]$

Relation of $L_\infty$-algebras to other concepts:
- differential graded commutative algebras
- codifferential graded cocommutative coalgebras
- $L_\infty$-algebras are useful models of $\infty$-categorified Lie algebras:
  $g = g_{-k} \oplus \cdots \oplus g_0$ is a "Lie $(k + 1)$-algebra.
- Differentiation $\infty$-groups/-oids yields $L_\infty$-algebras/-oids.
- Strict $L_\infty$-algebras crossed modules, hypercrossed modules of Lie algebras
- Relation to operads.

1.3. Quasi-isomorphisms

$g$, $\tilde{g}$: $L_\infty$-algebras. Note: $\mu_1$, $\tilde{\mu}_1$ are differentials!

Appropriate notion of morphisms:
- strict morphism: $\phi : g \to \tilde{g}$ such that $\tilde{\mu}_i(\phi(a_1), \ldots, \phi(a_i)) = \phi(\mu_i(a_1, \ldots, a_i))$
  corresponds to morphism $\phi^* : CE(\tilde{g}) \to CE(g)$ linear in the generators $\xi^a$.
- general morphism: $\phi^* : CE(\tilde{g}) \to CE(g)$ without linear restriction
  corresponds to morphism $\phi$ encoded in maps $\phi_i : g^{\wedge i} \to g$ of degree $1 - i$

Quasi-isomorphism (appropriate notion of isomorphism for $L_\infty$-algebras):
- (General) morphism of $L_\infty$-algebras, inducing isomorphism on cohomologies:
  $\phi_* : H^*_{\mu_1}(g) \cong H^*_{\mu_1}(\tilde{g})$

Structural theorems: Any $L_\infty/A_\infty$-algebra is
- isomorphic to linearly contractible$\oplus$skeletal
- quasi-isomorphic to a skeletal one ("minimal model")
- quasi-isomorphic to a strict one
Examples:

- $\mathbb{R}[q] \xrightarrow{id} \mathbb{R}[q-1]$ is quasi-isomorphic to $\ast$ ("trivial pairs" in BV gauge fixing)
- $\mathbb{R}[1] \to g$ is quasi-isomorphic to $\hat{L}_0g[1] \to P_0g$

1.4. Higher Chern–Simons theory

Ingredients:

- compact $d \geq 3$-dimensional manifold $M$
- higher gauge algebra $g = g_{d-3} \oplus \cdots \oplus g_0, \langle -,- \rangle$
- Note: dgca $\otimes L_\infty$-algebra carries $L_\infty$-structure:
  - $\hat{\mu}_1 = d \otimes 1 + 1 \otimes \mu_1$
  - $\hat{\mu}_i = 1 \otimes \mu_i$
  - $\langle \alpha \otimes a, \beta \otimes b \rangle = (\int_M \alpha \wedge \beta) \langle a, b \rangle$
- Maurer–Cartan action:
  - action: $S = \sum_{k \geq 1} \frac{1}{(k+1)!} \mu_k(a, \ldots, a), a \in g_1$ (not S-matrix!!)
  - Critical points: $\frac{\delta S}{\delta a} = \sum_{k \geq 1} \frac{1}{k!} \mu_k(a, \ldots, a) =: f$
  - Invariance of action $S$ under $a \to a + \delta a$
    with $\delta a = \sum_{k \geq 0} \frac{1}{k!} \mu_{k+1}(a, \ldots, a, c_0), c_0 \in g_0$
- This defines higher Chern–Simons theory in any dimension!

Example: $d = 3$

- $g = g_0$ is Lie (1-)algebra, $\mu_2 = [-,-]$
- $a = A \in \Omega^1(M, g)$
- $f = F = dA + \frac{1}{2}[A, A]$
- $S = \int_M \frac{1}{2} \langle A, dA \rangle_g + \frac{1}{3!} \langle [A, [A, A]] \rangle_g =: \int_M \text{cs}(A)$
- $d \text{cs}(A) = \langle F, F \rangle$

Example: $d = 4$.

- $g = g_{-1} \oplus g_0$
- $a \in \Omega^4(M, g): a = A + B \in \Omega^1(M, g_0) \oplus \Omega^2(M, g_{-1})$
- $f = dA + \frac{1}{2}[A, A] + \mu_1(B) + dB + \mu_2(A, B) + \frac{1}{3!} \mu_3(A, A, A)$
- $S = \int_M \langle B, dA + \frac{1}{2} \mu_2(A, A) + \frac{1}{2} \mu_1(B) \rangle_g + \frac{1}{3!} \langle \mu_3(A, A, A), A \rangle_g =: \int_M \text{cs}(A, B)$
- $d \text{cs}(A, B) = \langle F, H \rangle$
2. Lecture II

2.1. Field Theory

A classical field theory is given by:

- space-time $M$, e.g. $M = \mathbb{R}^{1,3}$, better $M$: compact.
- set of "fields" $\mathfrak{F}$ (connections on some principal fiber bundle or sections of associated vector bundles)
- action functional $S : \mathfrak{F} \to \mathbb{R}$ defined via a Lagrangian $L : \mathfrak{F} \to \Omega^{\text{top}}(M)$:
  \[ S = \int_M L. \quad (2.1) \]
- Classical equations of motion: $\frac{\delta S}{\delta \phi} = 0$ for all $\phi \in \mathfrak{F}$.
- Example:
  - Scalar field $\varphi : \mathbb{R}^{1,3} \to \mathbb{R}$
  - Action:
    \[ S = \int_{\mathbb{R}^{1,3}} d^4x \left( \begin{array}{c}
      \frac{1}{2} \varphi(-\Box - m^2) \varphi \\
      \frac{-\kappa}{3!} \varphi^3 - \frac{\lambda}{4!} \varphi^4
    \end{array} \right), \quad (2.2) \]
    \[ \kappa, \lambda \in \mathbb{R}, \lambda > 0. \]
- Connections:
  - encoded in local, Lie-algebra valued 1-forms $A \in \Omega^1(M, g)$
  - redundancy: gauge equivalence. Infinitesimal gauge transformations: $A \to A + \delta A$ with $\delta A = dc + [A, c], c \in \Omega^0(M, g)$
  - Yang–Mills Action: $S = \int_{\mathbb{R}^{1,3}} \langle F, \ast F \rangle_g$, "curvature" $F := dA + \frac{1}{2}[A, A]$
- Observables: Functions on $\mathfrak{F}$ which satisfy the equations of motion and which are invariant under gauge transformations.

Quantum Field Theory:

- want to compute expressions like ("path integrals"):
  \[ \int \mathcal{D}\Phi \ f(\Phi) e^{-\frac{i}{\hbar} S[\Phi]} \quad (2.3) \]
  - Not rigorously possible, heuristics from finite Gaussian integrals, stationary phase formula, perturbation theory.
  - Very roughly: sum over all Feynman diagrams: graphs constructed from vertices given by the monomials in the interaction term.
2.2. The Batalin–Vilkovisky formalism – classical part

For QFT, we first need to describe observables. Thus:

- Divide out redundancy / gauge equivalence
- Impose classical equations of motion

Gauge equivalence via Chevalley–Eilenberg resolution:

- Bad idea: divide field space by gauge transformations
- Note: Fields+gauge transformations: action Lie groupoid
- Infinitesimal: Lie algebroid

\[
\delta_{\text{BRST}} = \delta_{-1}^{\text{BRST}} \oplus \delta_{0}^{\text{BRST}}
\]

(2.4)

- Dually:
  - coordinate functions \( c \) and \( A \) (degrees 1 and 0)
  - differential:

\[
Q_{\text{BRST}} A = \delta A = dc + [A, c] \quad \text{and} \quad Q_{\text{BRST}} c = -\frac{1}{2}[c, c] \quad (2.5)
\]

- complex:

\[
0 \rightarrow C^\infty(\mathfrak{g}^{\text{BRST}}) \xrightarrow{Q_{\text{BRST}}} C^\infty(\mathfrak{g}^{\text{BRST}}) \xrightarrow{Q_{\text{BRST}}} \cdots \quad (2.6)
\]

- Chevalley–Eilenberg resolution: \( C^\infty(\mathfrak{g}/\text{gauge}) \cong H^0(\mathfrak{g}^{\text{BRST}}) \)

Equations of motion via Koszul–Tate resolution:

- Add to each field \( \Phi^A \) an “anti-field” \( \Phi^+_A \)
- symplectic form \( \omega = d\Phi^A \wedge d\Phi^+_A \rightarrow \) Poisson bracket (“anti-bracket”)
- \( Q = \{ S, \text{ original action} \} + \{ \cdots, \text{ at least linear in } \Phi^+, - \} \)

Both put together: BV-complex \( \mathfrak{g}^{\text{BV}}, \omega^{\text{BV}}, S^{\text{BV}}, Q^{\text{BV}} := \{ S^{\text{BV}}, - \} \)
2.3. Homotopy algebras from field theories

The BV-complex is a dgca. There is a dual $L_\infty$-algebra:

<table>
<thead>
<tr>
<th>$\cdots$</th>
<th>$g_{-1}$</th>
<th>$g_0$</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots$</td>
<td>gauge-of-gauge trans.</td>
<td>gauge trans.</td>
<td>physical fields</td>
<td>equations of motion</td>
<td>Noether identities</td>
<td>higher Noether</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

More direct computation:
- Guess $n$-ary products to recover action as homotopy Maurer–Cartan
- Example:

$$S = \int_{\mathbb{R}^{1,3}} d^4x \left( \frac{1}{2} \varphi (-\Box - m^2) \varphi - \frac{\kappa}{3!} \varphi^3 - \frac{\lambda}{4!} \varphi^4 \right),$$

has $L_\infty$-algebra

$$\begin{array}{cccccc}
\ast & \to & C^\infty(\mathbb{R}^{1,3}) & \to & C^\infty(\mathbb{R}^{1,3}) & \to & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_0 & g_1 & g_2 & g_3 & \cdots & \cdots & \cdots \\
\end{array}$$

with higher products

$$\mu_1(\varphi_1) := (-\Box - m^2) \varphi_1, \quad \mu_2(\varphi_1, \varphi_2) := -\kappa \varphi_1 \varphi_2,$$

$$\mu_3(\varphi_1, \varphi_2, \varphi_3) := -\lambda \varphi_1 \varphi_2 \varphi_3$$

Yang–Mills theory: often more useful to work with $A_\infty$-algebras:
- Action: (gauge algebra: $\mathfrak{gl}(N, \mathbb{C})$

$$S = \int_M (F, \ast F) = \int_M \text{tr}(F \wedge \ast F)$$

- Higher products (some examples):

$$m_1(A) = d^! dA + \ldots,$$

$$m_2(A_1, A_2) = d^!(A_1 \wedge A_2) + \ast (A_1 \wedge dA_2) + \ldots,$$

$$m_3(A_1, A_2, A_3) = \ast (A_1 + \ast (A_2 \wedge A_3)) + \ldots$$

- Full gauge fixed, suitable for $A_\infty$:

$$S_{YM,gf} := \int \text{tr} \left\{ \frac{1}{2} F \wedge \ast F - (A^+ + d\bar{c}) \wedge \ast \nabla c - \frac{e}{2} c^+ \wedge \ast [c, c]
\right.\
- b \wedge \ast (\bar{c}^+ + d^! A - \frac{\xi}{2} b) \right\},$$
2.4. First results

- Equivalent Field Theories \(\leftrightarrow\) Quasi-isomorphic \(L_\infty\)-algebras
  Example: 1st/2nd order Yang–Mills theory. Easier than integrating out etc.
- Strictification theorem \(\Rightarrow\) Any field theory is equivalent to a field theory with only cubic vertices.
  Example: 1st/2nd order Yang–Mills theory

3. Lecture III

3.1. Homological Perturbation Lemma

Scattering amplitudes: encoded in minimal models of theory:
\[
\mathcal{A}(\phi_0, \ldots, \phi_k) = \langle \phi_0, \mu_k^1(\phi_1, \ldots, \phi_k) \rangle^0
\]

How to compute? Homological Perturbation Lemma:

- Start from \(A_\infty\)-algebra \(a\).
- Focus: underlying complex \((a, m_1)\).
- Link to minimal model via contractible homotopy

\[
\begin{align*}
1 &= m_1 \circ h + h \circ m_1 + e \circ p, \quad p \circ e = 1, \\
p \circ h &= h \circ e = h \circ h = p \circ m_1 = m_1 \circ e = 0.
\end{align*}
\]

- Lift to codifferential coassociative coalgebra \(T(a) := \otimes^*(a[1]), D_0 = m_1:\)

\[
\begin{align*}
H_0 \circ (T(a), D_0) &\xrightarrow{P_0} (T(a^o), 0), \\
P_0|_{T^k(a)} &= p^\otimes^k, \quad E_0|_{T^k(a^o)} := e^\otimes^k, \quad H_0|_{T^k(a)} := \sum_{i+j=k-1} 1^\otimes^i \otimes h \otimes (e \circ p)^\otimes^j.
\end{align*}
\]

- \(D = \mu_1 + \mu_2 + \mu_3 + \cdots = D_0 + D_{int}\), regard \(D_{int}\) as perturbation
- Homological perturbation lemma: \(D_{int}\) yields deformations
  \[
  P = P_0 \circ (1 + D_{int} \circ H_0)^{-1}, \quad H = H_0 \circ (1 + D_{int} \circ H_0)^{-1}, \\
  E = (1 + H_0 \circ D_{int})^{-1} \circ E_0, \quad D^o = P \circ D_{int} \circ E_0.
  \]

Proof: Computation. Note: existence of \((1 + D_{int} \circ H_0)^{-1}\) for small \(D_{int}\).

- Outlook:

\[
D^o = P_0 \circ D_{int} \circ E \quad E = E_0 - H_0 \circ D_{int} \circ E
\]

\(\Rightarrow\) Recursion relations!
3.2. Field theory and Feynman diagrams

Recall:

- Field theory in $A_\infty$-algebra $\mathfrak{a}$
- $\mathfrak{a}^\circ$: kernel of $m_1$, “free fields,” $\mathfrak{a}^\circ = \ker_c(m_1)$
- $\mathfrak{a}$: all fields, some choice, e.g. $\mathfrak{a} = \ker_c(m_1) + S(\mathbb{R}^{1,3})$
- contracting homotopy: inverse of $m_1$: “propagator”
- Claim: Amplitudes
  $$A(\phi_0, \ldots, \phi_k) = \langle \phi_0, \mu_k^\circ(\phi_1, \ldots, \phi_k) \rangle^0 = \sum_{\sigma \in S_k} \langle \phi_0, m_k^\circ(\phi_{\sigma(1)}, \ldots, \phi_{\sigma(k)}) \rangle$$  \hspace{1cm} (3.5)

- Feynman diagrams from recursion relation:
  $$D^\circ = P_0 \circ D_{\text{int}} \circ E \quad E = E_0 - H_0 \circ D_{\text{int}} \circ E$$ \hspace{1cm} (3.6)

  Construct “current” $m_k^\circ(\phi_1, \ldots, \phi_k)$ (by chopping off one leg of amplitude).

Example: Scalar fields, $D_{\text{int}} = m_2 + m_3$

- 4-point tree-level amplitude:

  $$\begin{align*}
  \varphi_0 &\quad \varphi_1 \\
  \varphi_2 &\quad \varphi_3 \\
  \varphi_0 &\quad \varphi_1 \\
  \varphi_2 &\quad \varphi_3 \\
  \varphi_0 &\quad \varphi_1
  \end{align*}$$

- $D^\circ = P_0 \circ D_{\text{int}} \circ (E_0 - H_0 \circ D_{\text{int}} \circ E_0 + \ldots) = \cdots + m_3^\circ + \ldots$

  - sketch

Note:

- Indeed amplitudes, “amputated diagrams”
- Recursion relation $E = E_0 - H_0 \circ D_{\text{int}} \circ E$ was observed for Yang–Mills in 1988, Nucl. Phys. B, same journal as birthplace of $L_\infty$-algebras!
- Berends–Giele recursion lead to Parke–Taylor formula, hugely important, inspired many things, etc. twistor strings.

3.3. Full quantum amplitudes

This all was “tree-level,” i.e. our diagrams did not contain any loops. To get these:
• Recall in BV: classical master equation to quantum master equation:

\[ Q_{\text{BV}} := \{S_{\text{BV}}, -\}, \quad Q_{\text{BV}}^2 = 0 \quad \rightarrow \quad \hbar \Delta + \{S_{\text{BV}}, -\}, \quad 2\hbar \Delta S_{\text{BV}} + \{S_{\text{BV}}, S_{\text{BV}}\} = 0 \quad (3.8) \]

\[ \Delta = \omega^{AB} \frac{\delta}{\delta \Phi^A} \frac{\delta}{\delta \Phi^B} = \frac{\delta}{\delta \Phi} \frac{\delta}{\delta \Phi^*}, \quad \text{“takes away” field-antifield pairs.} \]

• In homological perturbation lemma, replace \( D_{\text{int}} \) by \( D_{\text{int}} + \hbar \Delta^* \)

\( \Delta^* \) inserts field-antifield pairs in all possible ways.

• Structures in HPL get distorted:
  
  ◦ \( P, E \) no longer algebra morphisms
  
  ◦ minimal model as “quantum \( L_\infty \)-algebra”
  
  ◦ homotopy Jacobi identities distorted.

• Still: recursion relation

\[ D^0 = P_0 \circ D_{\text{int}} \circ E \quad E = E_0 - H_0 \circ (D_{\text{int}} + \hbar \Delta^*) \circ E \quad (3.9) \]

Example: Scalar field theory with \( D_{\text{int}} = \mu_2 + \mu_3 \):

• Restrict \( E, D \) to \( E^{i,j}, D^{i,j} \) with \( i, j \) out/inputs.

• \( l \): \# of loops, \( v \): \# of vertices

• Recursion relation:

\[ E_{l,v}^{i,j} = \delta_{l}^{0} \delta_{v}^{0} \delta_{i}^{i} E_{0}^{i,i} - H_0 \circ \sum_{k=2}^{i+2} D_{\text{int}}^{i,k} \circ E_{l,v-1}^{k,j} + i\hbar H_0 \circ \Delta^* \circ E_{l-1,v}^{i-2,j} \quad (3.10) \]

— sketch : 1-loop correction to propagator —

Note: not properly amputated. Can be implemented, but irrelevant for combinatorics

Thus: We get Berends–Giele for any Lagrangian field theory and at loop level!

Applications

• Planar vs non-planar diagrams —sketch—

\( \ell \)-loop, \( n \)-point have maximally \( t = \max\{\ell, n\} \) traces

Thus: come with a factor of \( N^\ell - t + 1 \)

• Relation between planar and non-planar diagrams at 1-loop, knowing planar is sufficient!

• Hopefully: double copy

• Rewrite QFT books using homotopy algebras.
References


Perturbative QFT & Homology Algebras.

Ref: arXiv 1803.03899, 1503.02887, 1503.05713, 1812.06695

Robinson
LHC

\[ d = 27\text{km} \]
\[ v = 0.9 \times 3 \times 10^8 \text{ m/s} \]

\[ A < \text{out} \rangle \text{amp} > \]

Scattering amplitudes:
- are crucial for understanding nature
- relating & mysterious properties
- compute: heuristics "Quantum Field Theory"

Problem: Explain QFT: problematic & complicated

1) Field theory action \( \frac{1}{2f} \) \( \text{Lo}-\)algebra
2) \( \text{Lo}-\)algebras have "minimal models"
3) Minimal models are given by scattering amplitudes.
Homotopy Algebras

dg Lie algebra: shift Lie algebra
  graded vector space \( g = \bigoplus_{i \in \mathbb{Z}} g_i \)
  \( a, b, c \in g \)
  \( [a, b] \in g_{i+j} \)
  \( [a, b] = \pm [b, a] \)
  \( [a, [b, c]] = \pm ([a, b], c) + [c, [a, b]] \)
  \( d^2 = 0 \)
  \( d[a, b] = [da, b] + [a, db] \)
  Maurer-Cartan equation: \( a, b \in g \), \( da + \frac{1}{2} [a, a] = 0 \)

Metric / quadratic extension ("cycles")
  graded sym. bilinear form \( \langle - , - \rangle : g \times g \to \mathbb{R} \)
  compatibility: \( \langle da, b \rangle = \pm \langle a, db \rangle \) etc.
  Maurer-Cartan action: \( S = \frac{1}{2} \langle a, da \rangle + \frac{i}{3} \langle a, [a, a] \rangle \)

Alternative description I: Codifferential comonoid, coalgebra.
  \( \partial : \mathbb{C}[-,-] \) have "degrees" \( , 0 \)
  grade shift: \( g[1] \)
  \( (V[1])^q = V_{p+q} \)
  combine \( \partial \) and \( \mathbb{C}[-,-] \) codifferential \( \mathcal{D} = \partial + \mathbb{C}[-,-] \)

on \( \mathcal{O}(g[1]) \)
  \( \mathcal{D}^2 = 0 \) of degree 1

Alternative II: dual dg differential algebra
  basis \( e \) of \( g[1] \) coordinate functions \( \mathcal{E}: g[1] \to \mathbb{R} \)
  \( Q = \mathcal{D}^* \) differential dual \( \mathcal{O}(g[1]^*) \)
  \( Q^2 = 0 \)
  e.g. Lie algebra \( g \), dual functions \( \mathcal{E} \) of degree 1
  \( A = \frac{1}{2} \mathcal{E} \frac{\partial}{\partial \mathcal{E}} \)
  \( Q^2 = 0 \) \( \Rightarrow \) Jacobi identity
Extensions:
- Allow coefficients in $Q$ of arbitrary degree.

$\Rightarrow$ $L_\infty$-algebras with products: $\mu_i : g^i \rightarrow g$

$\mu_i$ are of degree $2 - i$

$D = \mu_1 + \mu_2 + \mu_3 + \ldots$ \hspace{1cm} $D^2 = 0$ \hspace{1cm} $\mu_i^2 = 0$

- $L_\infty$-algebras: graded vector bundles
- $A_\infty$-algebras: $\otimes \rightarrow \otimes$ \hspace{1cm} $\mu_i : g^i \otimes g \rightarrow g$

Examples:
- Trivial: $g = *$
- Strict: $\mu_i = 0$ for $i \geq 3$
- Skeletal: $\mu_i = 0$
- Non-skeletal: $\mu_i = 0$ for $i > 2$ \hspace{1cm} $H^*_\mu(g) = 0$

Quasi-isomorphisms
- Morphism: $L_\infty$-algebras have underlying complex $\int$ strict chain map $\phi : g \rightarrow g$
- Morphism $\phi$ is a quasi-isomorphism of $L_\infty$-algebras

Quasi iso: $\phi : g \rightarrow g$

(general) morphism of $L_\infty$-algebras

$\phi : (T^*_\mu(g) \xrightarrow{\sim} H^*_\mu(g))$

Structural theorems:
- Any $L_\infty$ algebra
- is iso to skeletal $+$ lia. contrab.
- is quasi iso to skeletal
- is quasi iso to a strict one
E.g. $R[G] \rightarrow R[\mathfrak{g}]$ also to *

1.4. Higher Chern-Simons theory

Ingredients:
- $d \geq 3$-dim manifold $M$, compact
- $\mathfrak{g}$-algebra:
  \[ \mathfrak{g} = \mathfrak{g}_3 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_0 \]

Note: $\text{dgca} \otimes \mathfrak{h}_\text{algebra has natural $\mathfrak{g}$-algebra structure}$

- $\hat{\lambda}^i = d \otimes 1 + 1 \otimes \mu^i$
- $\hat{\mu}^i = 1 \otimes \mu^i$

Moreover, $\langle \alpha \otimes \lambda, \lambda \otimes \beta \rangle = \langle \chi, \lambda \otimes \beta \rangle$,

Magnetic action

\[ S = \sum_{\mathfrak{g} \cong \mathfrak{g}_1} \frac{1}{\mathfrak{g}_1} \langle \chi, \hat{\mu}^\chi (\chi, \ldots, \chi) \rangle \]

$\rightarrow$ higher Chern-Simons form:

\[ \omega^\mathfrak{g}(M) \otimes \mathfrak{g} \]

$d = 3$:

\[ \mathfrak{g} = \mathfrak{g}_0 \]

\[ \hat{\mathfrak{g}} = \mathfrak{g}(M) \otimes \mathfrak{g}_0 \]

\[ a = A \in \mathfrak{g}(M, \mathfrak{g}) \]

MC eqn:

\[ F = dA + \frac{1}{2} C_A = 0 \]

\[ S = \sum \left( dA + \frac{1}{2} [A, [A]] \right) = \sum \text{cs}(A) \]

\[ \chi \text{cs}(A) = \langle \chi, F \rangle \]

$d = 4$:

\[ \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0 \]

\[ a = \mathfrak{g}(M) \otimes \mathfrak{g} \quad a = A + B \in \mathfrak{g}(M, \mathfrak{g}_0) \oplus \mathfrak{g}(M, \mathfrak{g}_0) \]

\[ S = \sum \left( dA + \frac{1}{2} F_2 + \frac{1}{3} F_3 + \frac{1}{4} F_4 \right) + \frac{1}{4} [A, [A, [A, A]]] \]

\[ = \sum \text{cs}(A, B) \]

\[ \chi \text{cs}(A, B) = \langle \chi, F \rangle \]
Recap:

Field theory actions \( \xrightarrow{\text{BV}} \) \( L_\infty \)-algebras

\[
\text{Scattering amplitudes} \xrightarrow{\text{yield}} \text{minimal models}
\]

\( L_\infty \)-algebras

Graded vector space \( g = \bigoplus_{i \in \mathbb{Z}} g_i \)

Coalgebra pic: \( \bigcirc g[i] \)

Dga-pic: \( \bigcirc g[i]^* \)

\( M_i : g^* \rightarrow i, \quad \mu_1 = 2 - i \rightarrow D = \mu_1 + \mu_2 + \cdots \)

1-form = \( Q = D^* \)

Differential: \( D^2 = 0 \)

\( A_\infty \)-algebras: replace 0 \( \rightarrow \) \( \bigoplus \)

\[
a = \bigoplus_{i \in \mathbb{Z}} a_i, \quad m_i : a^\otimes i \rightarrow a, \text{ etc.}
\]

Homotopy Maurer-Cartan:

\[
S = \sum_{k \geq 1} \frac{1}{(k+1)!} \langle a_1, \ldots, a_k \rangle, \quad a \in g
\]

\[
S = \sum_{k \geq 1} \frac{1}{(k+1)!} \langle a_1, \ldots, a_k \rangle, \quad a \in g
\]
Field Theory

Classically:
1) spacetime \( \mathcal{M} \), \( \mathcal{M} = \mathbb{R}^4 \), (\( \mathcal{M} \) compact)
2) set of fields \( \mathcal{F} \) (connections on principal bundles, sections of associated vector bundles)
3) action functional \( S : \mathcal{F} \to \mathbb{R} \) from Lagrangian \( L : \mathcal{F} \to \mathcal{E}^{\text{field}}(\mathcal{M}) \)

\[ S = \int_{\mathcal{M}} L \]

Classical eqm: \[ \frac{\delta S}{\delta \Phi} = 0 \]
\( \Phi : \text{field} \)

Example: Scalar fields \( \phi : \mathbb{R}^4 \to \mathbb{R} \)

\[
S = \int_{\mathbb{R}^4} d^4x \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right)
\]

\( V, \lambda, \kappa > 0 \)

Connections:
- gauge Lie algebra
- \( \mathfrak{g} \) = gauge Lie algebra valued 1-forms \( A \in \mathfrak{g}(\mathcal{M}) \)
- gauge equivalence \( C \) bundle isomorphisms
- infinitesimally \( C \in \mathfrak{g}(\mathcal{M}) \)

\[
A \to A + \delta A = d c + [A, c]
\]

Actions:

\[
S = \int d^4x \left( \mathcal{L}_A + \frac{i}{2} [\mathcal{L}_A, \mathcal{L}_A] \right)
\]

Observables:
- functions on \( \mathcal{F} \): for charg satisfying eqn which are invariant under gauge transform

Quantum Field Theory:

Compute expressions like

\[
\int D\Phi \ e^{-\frac{i}{\hbar} S[\Phi]}
\]

measure on \( \mathcal{F} \)

Not rigorous, heuristics from Gaussian integrals

\( \Rightarrow \) perturbation theory

1) Feynman diagrams. \( \phi^3 \) graphs

\[ x \to x^4 \]
BR Formalism — classical part
need observables:
• factor out gauge symmetry ✓
• impose equations of motion

Gauge symmetry: Chevalley—Eilenberg resolution
• dividing spaces by group actions: bad idea
  instead: action groupoid \( A \hookrightarrow A \)
  action Lie algebroid

\[
\mathfrak{F}_{\text{BRS}} = \mathfrak{F}_{\mathcal{A}^-} \oplus \mathfrak{F}_0
\]

Gauge fields

Dually: cocon. fiber
\( A : \mathfrak{F}_0^{\text{BRS}} \rightarrow \mathbb{R} \)
\( c : \mathfrak{F}_{\mathcal{A}^-} \rightarrow \mathbb{R} \) slow

BRS differential
\( Q_{\text{BRS}} A = \delta A = \delta c + [A, c] \)
\( Q_{\text{BRS}} c = -\frac{1}{2} [c, c] \)

complex
\( 0 \rightarrow C^0(\mathfrak{F}_{\text{BRS}}) \rightarrow C^0(\mathfrak{F}_{\text{BRS}}) \rightarrow \cdots \)

Resolution: \( C^0(\mathfrak{F}_{\text{BRS}}) \rightarrow H^0(\mathfrak{F}_{\text{BRS}}) \)

Koszul–Tate resolution:
Add to each field/ghost etc. antifield \( \bar{\phi} \)
Symp. form \( \omega = d\bar{\phi} \wedge d\phi^+ \)
\( A = \text{trans on all types of fields} \)
\( Q = \{ \underbrace{S_{\text{BV}}}_\text{original action (ghost)} \} \)

\( \{ S_{\text{BV}}, S_{\text{BV}}^+ \} = 0 \)

\( \Rightarrow \) BV complex \( \mathfrak{F}_{\text{BV}}, Q_{\text{BV}}, S_{\text{BV}}, Q_{\text{BV}} = \{ S_{\text{BV}}, \cdots \} \)

\( Q_{\text{BV}}^2 = 0 \)

\( \Rightarrow \) dgca. \( S_{\text{BV}}, \bar{\bar{\mathbb{E}}}^+ S = 0 \)
Homology algebra & Field theory

... \( g_{-1} \rightarrow g_0 \rightarrow g_1 \rightarrow g_2 \rightarrow g_3 \rightarrow g_4 \rightarrow \cdots \)

... ghosts fields anti-fields Noether higher gauge theory com anti-fields highest ghost

More directly: \( \text{"guess n-ary product such that action is bMC action of an co-algebra"} \)

E.g. \( S = \int_{\mathbb{R}^{13}} d^4x \left( \frac{1}{2} \partial \cdot \partial \phi - \frac{1}{2} \phi^2 - \frac{1}{4!} \phi^4 \right) \)

\[
S = \sum_{a \in \mathbb{Z}_0} \frac{1}{a!} \left\langle c_1, \chi \partial \cdot \partial \phi, \cdots, \phi \right\rangle
\]

\[
= \sum_{a \in \mathbb{Z}_0} \frac{1}{a!} \left\langle c_1, \chi \partial \cdot \partial \phi, \cdots, \phi \right\rangle
\]

\[
\Rightarrow \quad \gamma_1 \rightarrow C^\infty(\mathbb{R}^{13}) \quad \gamma_2 \rightarrow C^\infty(\mathbb{R}^{13}) \quad \gamma_3 \rightarrow \gamma_2 \phi_1 \phi_2 \phi_3
\]

all form higher product trivial.

Yang-Mills (gauge: \( A \), \( c \))

Action:

\[
S = \int_{\mathbb{R}^{13}} d^4x \left( F \cdot F \right) = \sum_{a \in \mathbb{Z}_0} \frac{1}{a!} \left\langle c_1, \chi \partial \cdot \partial \phi, \cdots, \phi \right\rangle
\]

\[
a = A \quad m_1(A) = d^+dA \quad d^+ = \star d^+
\]

\[
m_2(A_1, A_2) = d^+(A_1 \wedge A_2) + \star (A_1 \wedge dA_2)
\]

\[
m_3(A_1, A_2, A_3) = \star (A_1 \wedge A_2 \wedge A_3)
\]

\[
j_c(H, \phi) \rightarrow j_c(H, \phi)
\]

incorporate ghosts, gauge fixing ... \( S_{BF} \) ...

Equivalence of Field Dories <-> quasi-iso of \( \mathcal{C} \)-algebras

Equivalence theorem: any locally q.i. to a strict one. \( q_i = 0 \quad i = 3 \)
Recap:

- Field theory actions $\xrightarrow{\text{BV}}$ $L_\infty$-algebras
- Scattering amplitudes $\xrightarrow{\text{yield}}$ minimal models

Scalar field theory: $S = \int d^4x \left( \frac{1}{2} \phi (-\Box - m^2) \phi - \frac{\kappa}{3!} \phi^3 - \frac{\lambda}{4!} \phi^4 \right)

$g_0 \rightarrow g_1 \rightarrow g_2 \rightarrow g_3$

$x \rightarrow C_0g_1(13) \rightarrow C_0g_2(13) \rightarrow x$

$\phi^+ \rightarrow C\phi^+$

$A_i(\phi) = (-i\omega - m^2)\phi$

$\mu_2(\phi_1, \phi_2) = -\kappa \phi_1 \phi_2$

$\mu_3(\phi_1, \phi_2, \phi_3) = -\lambda \phi_1 \phi_2 \phi_3$

Equivalence of $\mathcal{FT} \xrightarrow{\text{qiso}} L_\infty$-algebras

$\xrightarrow{\text{quick way of proving equivalence}}$

Strictification theorem $\Rightarrow$ For any $\mathcal{FT}$, $\mathcal{I}$ equivalent $\mathcal{FT}$ with only cubic vertices $\Lambda$.

E.g. 1st/2nd order YM theory

E.g. Yang-Mills:

\[
\begin{align*}
S_0 &\rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \\
\Sigma^0(M, g(M)) &\xrightarrow{d} \Sigma^1(M, g(M)) \xrightarrow{d} \Sigma^2(M, g(M)) \xrightarrow{d} \Sigma^3(M, g(M)) \\
&\xrightarrow{\Lambda} \Lambda^+ \xrightarrow{\text{c}} \Lambda^+
\end{align*}
\]
Homological Perturbation lemma:

$A_\infty$-algebra $a$, focus on underlying complex $\ldots \xrightarrow{a_{n+1}} a_n \xrightarrow{a_n} \ldots$

$\exists C(\pi, m_i) \xrightarrow{P} (a_0 = H^\bullet(a), 0)$

$\text{p.o.e. } = \text{id}_{a_0}$

$\text{id}_a = \text{p.o.e.} + m_0 h_0 + h_0 m_0$

Hodge Decomposition

$poh = h_0 = h_0 h = h_0 m_0 = m_0 e = 0$

Lift to coadjoint coalgebra picture $D_0 = m_1$

$H_0 C(\otimes a, D_0) \xrightarrow{P_0} (\otimes a_0, 0)$

$P_0 \text{ acts } = \text{p.o.e.}$

$E_0 \text{ same}$

$H_0 \otimes a = \sum_{i+j = k} \otimes (\text{p.o.e.})^{(ij)}$

$I = \cdots + m_2 + m_4 + \cdots = \text{p.o.e.} + \text{Dilatation}$

$H P L : P = P_0 (1 + \text{Dilatation})$

$E = (1 + H_0 \circ \text{Dilatation})^{-1} E_0$

$D_0 = P_0 \circ \text{Dilatation} \circ E_0$

Useful:

$D_0 = P_0 \circ \text{Dilatation} \circ E_0$

$E = E_0 - (H_0 \circ \text{Dilatation} \circ E_0)$

Field theory & Feynman diagrams

- $\text{FT} \in A_\infty$-algebra $a$
  - $a_0 = \text{free}(m_1)$
  - $a = \text{free}(m_1) \otimes \text{"interacting fields"} \otimes \text{free}(m_1) \otimes \text{free}(R^{13})$
- Contraction homotopy $k$: propagator

$\text{free} \circ \text{free}(m_1) \xrightarrow{m_1 = 0} \text{free} \circ \text{free}(m_1)$

$\text{interaction} \xrightarrow{\text{interaction}} \text{interaction}$
$D^o = P_0 \circ D_{\mu\nu} \circ o E$

$E = E_0 - H_0 \circ D_{\mu\nu} \circ o E$

$A(\phi_1, \phi_2, \ldots, \phi_k) = \langle \phi_1 \phi_2, \ldots, \phi_k \rangle$

$= \sum_{S \in S_k} \langle \phi_{\sigma(1)} \phi_{\sigma(2)} \cdots, \phi_{\sigma(k)} \rangle$

Construct currents $m_{2n}(\phi_1, \ldots, \phi_k)$

$4$-point for scalar $\phi$

$m_2(\phi, \phi) = \mu \phi \phi$

$m_3(\phi, \phi, \phi) = \lambda \phi \phi \phi$

$m_{2n}(\phi_1, \ldots, \phi_k) = \lambda \phi_1 \phi_2 \cdots$

$D^o = P_0 \circ D_{\mu\nu} \circ (E_0 - H_0 \circ D_{\mu\nu} \circ o E_0 + \cdots) = \cdots + m_{2n} + \cdots$

Note:
  1. Naturally Incorporated
  2. For YM, the recursion $E = E_0 - H_0 \circ D_{\mu\nu} \circ o E$
  4. BG-recursion $\rightarrow$ Parke-Taylor amplitude formula
  5. $\rightarrow$ home BH recursion for all gauge in $\mathbb{F}_{\mathbf{K}}$. 
\[ BV : \quad Q_{BV} = \frac{s}{2} S_{BV} - 3 \rightarrow \quad t \Delta + \frac{s}{2} S_{BV} - 3 \]
\[ Q_{BV}^2 = 0 \]
\[ \Delta : \frac{\phi^2}{\phi^* \phi^4} \rightarrow \quad \Delta^* : \text{creates field/anti-field pairs} \]
\[ \phi^* \phi^4 \]

\[ HPL : \quad D \Delta \rightarrow \quad D \Delta + t \Delta^* \]

Things change: \[ P, E \] no longer algebra morphisms

\[ \rightarrow \quad \text{quantum theory homotopy algebra.} \]

\[ E = E_0 + H_0(t \Delta^* + D \Delta) E_0 \]

\[ E_{ij} \in \text{inputs, outputs.} \]

\[ l : \# \text{ of loops, } \quad v : \# \text{ of vehicles} \]

\[ E_{l,v}^{i,j} = 5 \int 5 \int E_0^{i,j} - \frac{i}{\phi^*} - \sum_{k=1}^{\infty} D \delta^k \int H_0^k \Delta^k E_{l,v}^{i,j} \]

\[ \text{Recursion for } l \text{-loop } v \text{-vertex amplitudes & terminates.} \]

\[ g_1(N) \quad \text{large } N \text{ limit} \]

\[ \text{1-loop level:} \]

\[ \text{non-planar given by planar} \]

\[ L_{\infty} \quad A_0 \quad \text{closed} \quad \text{open} \]

\[ \text{double copy} \]

\[ \text{Gravit} \leftrightarrow \text{YM} \]

\[ L_{\infty} \quad A_0 \quad \text{closed} \quad \text{open} \]