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Perturbative Quantum Field Theory with Homotopy Algebras

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Abstract

The BV-formalism associates to any Lagrangian field theory an L_∞-algebra. Any L_∞-algebra comes with minimal models, and the minimal models of L_∞-algebras originating from Lagrangian field theories capture the scattering amplitudes of these. The minimal models are easily computed using the homological perturbation lemma, which leads to recursive formulas helpful for their studies.

Pointers to literature (containing references to the original papers):

[1]: Review, conventions, technical details, classical field theory

[2]: shorter version of above

[3]: tree level amplitudes, scalar and Yang–Mills theory, Berends–Giele recursion

[4]: loop level amplitudes, homological perturbation lemma, non-planar diagrams

1. Lecture I

1.1. Motivation: Scattering amplitudes

Particle Accelerator (LHC at CERN):

- Beams of protons at very high velocity ($0.99999999c$) circulate in ring with diameter 27km.
- Collision in a small area, particles deflected and decay into new particles ("scatter")
- Detector measures these particles and their properties
- Initial and final particles are asymptotic and "free," i.e. non-interacting, not feeling potential

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Transition from asymptotic incoming to asymptotic outgoing particle configurations via S-matrix ("scattering matrix"):

$$|\text{asymptotic out}\rangle = S\ |\text{asymptotic in}\rangle\ , \tag{1.1}$$

Note:

- $|\dots\rangle$ denote asymptotic (i.e. $t = \pm\infty$) configurations.
- $|\dots\rangle$ are vectors/rays in a Hilbert space $\mathcal{H}$
- $\mathcal{H}$ is infinite dimensional (labeled e.g. by momentum of particle, spin, ...)
- S is a unitary operator

Scattering amplitude (probability amplitude) and probability:

$$\mathcal{A} := \langle\text{asymptotic out}|S|\text{asymptotic in}\rangle \quad \text{and} \quad |\mathcal{A}|^2\ . \tag{1.2}$$

Scattering amplitudes $\mathcal{A}$

- are crucial to understand nature
- have interesting and surprising structures.
- computed via heuristics dubbed Quantum Field Theory

Various prescriptions for computing $\mathcal{A}$, but usually involving much machinery

- Classical action
- Quantization of fields
- Wick's theorem
- (amputated) Feynman diagrams
- Dyson series
- LSZ reduction
- ...

This is complicated, hard to explain to mathematicians. Better:

- field theory actions $\xleftrightarrow{\text{BV formalism}}$ L_∞-algebras
- L_∞-algebras have minimal models (from homological perturbation lemma)
- Minimal models encode scattering amplitudes

Also: exposes many structural results on scattering amplitudes

1.2. Homotopy algebras

(Name: generalizations of the classical algebras: associative, Lie, Leibniz, commutative, etc., in which the identities hold only "up to homotopy")

Strict L_∞-algebras: Differential graded Lie algebra $(\mathfrak{g}, [-,-], \mathrm{d})$:

- graded vector space: $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$
- $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$: $\mathrm{d}a \in \mathfrak{g}_{i+1}$ $\qquad [a,b] \in \mathfrak{g}_{i+j}$
- $[a,b] = (-1)^{ab}[b,a] \qquad [a,[b,c]] = [[a,b],c] + (-1)^{ab}[b,[a,c]]$
- $\mathrm{d}^2 = 0 \qquad \mathrm{d}[a,b] = [\mathrm{d}a,b] + (-1)^a[a,\mathrm{d}b]$
- Maurer–Cartan: $\mathrm{d}a + \frac{1}{2}[a,a]$ for $a \in \mathfrak{g}_1$

Metric/quadratic extension: ("cyclic")

- graded symmetric bilinear form $\langle -,-\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$
- compatibility: $\langle \mathrm{d}a, b\rangle + (-1)^a\langle \mathrm{d}b, a\rangle = 0 \qquad \langle [a,b],c\rangle + (-1)^{ab}\langle b,[a,c]\rangle = 0$
- Maurer–Cartan action: $S = \frac{1}{2}\langle a, \mathrm{d}a\rangle + \frac{1}{3!}\langle a,[a,a]\rangle$.

Alternative description I: Codifferential on cocommutative coalgebra:

- d of degree 1, $[-,-]$ of degree 2
- grade-shift: $\mathfrak{g}[1]$, defined via $(\mathfrak{g}[i])_j := \mathfrak{g}_{i+j}$, all have grade 1
- combine into codifferential $D = \mathrm{d} + [-,-]$ on $\odot^\bullet(\mathfrak{g}[1]) \qquad D^2 = 0$

Alternative description II, Dually: Differential graded commutative algebra:

- basis τ_α of $\mathfrak{g}$, coordinate functions $\xi^\alpha \in \mathfrak{g}[1]^*$, $\xi^\alpha : \mathfrak{g}[1] \to \mathbb{R}$
- $Q = D^*$ is differential on $\odot^\bullet(\mathfrak{g}[1]^*)$, $Q^2 = 0$
- actually: vector field on $\mathfrak{g}$: $Q = \mathrm{d}_\alpha^\beta\xi^\alpha\frac{\partial}{\partial\xi^\beta} + \frac{1}{2}f_{\alpha\beta}^\gamma\xi^\alpha\xi^\beta\frac{\partial}{\partial\xi^\gamma}$
- cyclic structure is a symplectic form ω on $\mathfrak{g}[1]$.
- "Chevalley–Eilenberg algebra $\mathsf{CE}(\mathfrak{g})$ of $\mathfrak{g}$"

Extensions:

- general L_∞-algebra: allow for arbitrary polynomial coefficients in Q
  get totally antisymmetric linear "products" $\mu_i : \mathfrak{g}^{\wedge i} \to \mathfrak{g}$
  $D = \mu_1 + \mu_2 + \mu_3 + \dots \qquad D^2 = 0 \;\Leftrightarrow\; \mu_1^2 = 0\ ,\dots,\mu_1\mu_3 = \mu_2\mu_2$
- L_∞-algebroids: allow for graded vector bundle over manifold
- A_∞-algebra: replace $\odot$ by $\otimes$ everywhere
  Analogue of matrix algebras for matrix Lie algebras, Antisymmetrization: L_∞

Special cases:

- trivial: $*$, $\mathfrak{g}_i = *$.
- strict: $\mu_i = 0$ for $i > 2$: differential graded Lie algebras
- skeletal: $\mu_1 = 0$
  example: $\mathfrak{g} = (\mathbb{R}[1] \xrightarrow{0} \mathfrak{su}(n))$, $\mu_1 = 0$, $\mu_2 = [-,-]$, $\mu_3 = \langle -, [-,-] \rangle$
  ("string Lie 2-algebra")
- linearly contractible: $\mu_i = 0$ for $i \geq 2$, $H^\bullet_{\mu_1}(\mathfrak{g})$ is trivial
  example: $\mathbb{R}[q] \xrightarrow{\mathrm{id}} \mathbb{R}[q-1]$

Relation of L_∞-algebras to other concepts:

- differential graded commutative algebras
- codifferential graded cocommutative coalgebras
- L_∞-algebras are useful models of ∞-categorified Lie algebras:
  $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0$ is a "Lie $(k+1)$-algebra.
- Differentiation ∞-groups/-oids yields L_∞-algebras/-oids.
- Strict L_∞-algebras crossed modules, hypercrossed modules of Lie algebras
- Relation to operads.

1.3. Quasi-isomorphisms

$\mathfrak{g}$, $\tilde{\mathfrak{g}}$: L_∞-algebras. Note: μ_1, $\tilde{\mu}_1$ are differentials!

Appropriate notion of morphisms:

- strict morphism: $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\tilde{\mu}_i(\phi(a_1), \ldots, \phi(a_i)) = \phi(\mu_i(a_1, \ldots, a_i))$
  corresponds to morphism $\phi^* : \mathsf{CE}(\tilde{\mathfrak{g}}) \to \mathsf{CE}(\mathfrak{g})$ linear in the generators ξ^α.
- general morphism: $\phi^* : \mathsf{CE}(\tilde{\mathfrak{g}}) \to \mathsf{CE}(\mathfrak{g})$ without linear restriction
  corresponds to morphism ϕ encoded in maps $\phi_i : \mathfrak{g}^{\wedge i} \to \mathfrak{g}$ of degree $1 - i$

Quasi-isomorphism (appropriate notion of isomorphism for L_∞-algebras):

- (General) morphism of L_∞-algebras, inducing isomorphism on cohomologies:
  $\phi_* : H^\bullet_{\mu_1}(\mathfrak{g}) \cong H^\bullet_{\tilde{\mu}_1}(\tilde{\mathfrak{g}})$

Structural theorems: Any L_∞/A_∞-algebra is

- isomorphic to linearly contractible$\oplus$skeletal
- quasi-isomorphic to a skeletal one ("minimal model")
- quasi-isomorphic to a strict one

Examples:

- $\mathbb{R}[q] \xrightarrow{\mathrm{id}} \mathbb{R}[q-1]$ is quasi-isomorphic to $*$ ("trivial pairs" in BV gauge fixing)
- $\mathbb{R}[1] \to \mathfrak{g}$ is quasi-isomorphic to $\hat{L}_0\mathfrak{g}[1] \to P_0\mathfrak{g}$

1.4. Higher Chern–Simons theory

Ingredients:

- compact $d \geq 3$-dimensional manifold M
- higher gauge algebra $\mathfrak{g} = \mathfrak{g}_{d-3} \oplus \cdots \oplus \mathfrak{g}_0$, $\langle -, - \rangle$
- Note: dgca $\otimes$ L_∞-algebra carries L_∞-structure:
  - $\hat{\mu}_1 = \mathrm{d} \otimes 1 + 1 \otimes \mu_1$
  - $\hat{\mu}_i = 1 \otimes \mu_i$
  - $\langle \alpha \otimes a, \beta \otimes b \rangle = \left(\int_M \alpha \wedge \beta\right) \langle a, b \rangle$
- Maurer–Cartan action:
  - action: $S = \sum_{k\geq 1} \frac{1}{(k+1)!}\langle \mathsf{a}, \mu_k(\mathsf{a}, \ldots, \mathsf{a})\rangle$, $\mathsf{a} \in \mathfrak{g}_1$ (not S-matrix!!)
  - Critical points: $\frac{\delta S}{\delta a} = \sum_{k\geq 1} \frac{1}{k!}\mu_k(\mathsf{a}, \ldots, \mathsf{a}) =: \mathsf{f}$
  - Invariance of action S under $\mathsf{a} \to \mathsf{a} + \delta\mathsf{a}$
    with $\delta\mathsf{a} = \sum_{k\geq 0} \frac{1}{k!}\mu_{k+1}(\mathsf{a}, \ldots, \mathsf{a}, c_0)$, $c_0 \in \mathfrak{g}_0$
- This defines higher Chern–Simons theory in any dimension!

Example: $d = 3$

- $\mathfrak{g} = \mathfrak{g}_0$ is Lie (1-)algebra, $\mu_2 = [-, -]$
- $\mathsf{a} = A \in \Omega^1(M, \mathfrak{g})$
- $\mathsf{f} = F = \mathrm{d}A + \frac{1}{2}[A, A]$
- $S = \int_M \frac{1}{2}\langle A, \mathrm{d}A\rangle_{\mathfrak{g}} + \frac{1}{3!}\langle A, [A, A]\rangle_{\mathfrak{g}} =: \int_M \mathrm{cs}(A)$
- $\mathrm{d}\,\mathrm{cs}(A) = \langle F, F\rangle$

Example: $d = 4$.

- $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$
- $\mathsf{a} \in \Omega^\bullet(M, \mathfrak{g})_1$: $\mathsf{a} = A + B \in \Omega^1(M, \mathfrak{g}_0) \oplus \Omega^2(M, \mathfrak{g}_{-1})$
- $\mathsf{f} = \mathrm{d}A + \frac{1}{2}[A, A] + \mu_1(B) + \mathrm{d}B + \mu_2(A, B) + \frac{1}{3!}\mu_3(A, A, A)$
- $S = \int_M \langle B, \mathrm{d}A + \frac{1}{2}\mu_2(A, A) + \frac{1}{2}\mu_1(B)\rangle_{\mathfrak{g}} + \frac{1}{4!}\langle \mu_3(A, A, A), A\rangle_{\mathfrak{g}} =: \int_M \mathrm{cs}(A, B)$.
- $\mathrm{d}\,\mathrm{cs}(A, B) = \langle F, H\rangle$

2. Lecture II

2.1. Field Theory

A classical field theory is given by:

- space-time M, e.g. $M = \mathbb{R}^{1,3}$, better M: compact.
- set of "fields" $\mathfrak{F}$ (connections on some principal fiber bundle or sections of associated vector bundles)
- action functional $S : \mathfrak{F} \to \mathbb{R}$ defined via a Lagrangian $L : \mathfrak{F} \to \Omega^{\mathrm{top}}(M)$:

$$S = \int_M L \, . \tag{2.1}$$

- Classical equations of motion: $\frac{\delta S}{\delta \phi} = 0$ for all $\phi \in \mathfrak{F}$.
- Example:
  - Scalar field $\varphi : \mathbb{R}^{1,3} \to \mathbb{R}$
  - Action:

$$S = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4 x \left(\underbrace{\tfrac{1}{2}\varphi(-\Box - m^2)\varphi}_{\text{kinetic term}} \quad \underbrace{-\frac{\kappa}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4}_{\text{interaction term}} \right) , \tag{2.2}$$

  $\kappa, \lambda \in \mathbb{R}$, $\lambda > 0$.
- Connections:
  - encoded in local, Lie-algebra valued 1-forms $A \in \Omega^1(M, \mathfrak{g})$
  - redundancy: gauge equivalence. Infinitesimal gauge transformations: $A \to A + \delta A$ with $\delta A = \mathrm{d}c + [A, c]$, $c \in \Omega^0(M, \mathfrak{g})$
  - Yang–Mills Action: $S = \int_{\mathbb{R}^{1,3}} \langle F, \star F \rangle_{\mathfrak{g}}$, "curvature" $F := \mathrm{d}A + \frac{1}{2}[A, A]$
- Observables: Functions on $\mathfrak{F}$ which satisfy the equations of motion and which are invariant under gauge transformations.

Quantum Field Theory:

- want to compute expressions like ("path integrals"):

$$\int \underbrace{\mathcal{D}\Phi}_{\text{measure on } \mathfrak{F}} \quad f(\Phi) \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} S[\Phi]} \tag{2.3}$$

- Not rigorously possible, heuristics from finite Gaussian integrals, stationary phase formula, perturbation theory.
- <u>Very</u> roughly: sum over all Feynman diagrams: graphs constructed from vertices given by the monomials in the interaction term.

2.2. The Batalin–Vilkovisky formalism – classical part

For QFT, we first need to describe observables. Thus:

- Divide out redundancy / gauge equivalence
- Impose classical equations of motion

Gauge equivalence via Chevalley–Eilenberg resolution:

- Bad idea: divide field space by gauge transformations
- Note: Fields+gauge transformations: action Lie groupoid
- Infinitesimal: Lie algebroid

$$\mathfrak{F}^{\mathrm{BRST}} = \underbrace{\mathfrak{F}^{\mathrm{BRST}}_{-1}}_{\text{gauge trafos}} \oplus \underbrace{\mathfrak{F}^{\mathrm{BRST}}_{0}}_{\text{fields}} \tag{2.4}$$

- Dually:
  - coordinate functions c and A (degrees 1 and 0)
  - differential:

$$Q_{\mathrm{BRST}}A = \delta A = \mathrm{d}c + [A,c] \quad \text{and} \quad Q_{\mathrm{BRST}}c = -\tfrac{1}{2}[c,c] \tag{2.5}$$

  - complex:

$$0 \longrightarrow C^\infty_0(\mathfrak{F}_{\mathrm{BRST}}) \xrightarrow{Q_{\mathrm{BRST}}} C^\infty_1(\mathfrak{F}_{\mathrm{BRST}}) \xrightarrow{Q_{\mathrm{BRST}}} \cdots , \tag{2.6}$$

- Chevalley–Eilenberg resolution: $C^\infty(\mathfrak{F}/\text{gauge}) \;\cong\; H^0(\mathfrak{F}_{\mathrm{BRST}})$

Equations of motion via Koszul–Tate resolution:

- Add to each field Φ^A an "anti-field" Φ^+_A
- symplectic form $\omega = \mathrm{d}\Phi^A \wedge \mathrm{d}\Phi^+_A \;\rightarrow\;$ Poisson bracket ("anti-bracket")
- $Q = \{ \underbrace{S}_{\text{original action}} + \underbrace{\ldots}_{\text{at least linear in } \Phi^+} , - \}$

Both put together: BV-complex $\mathfrak{F}_{\mathrm{BV}}$, ω_{BV}, S_{BV}, $Q_{\mathrm{BV}} := \{S_{\mathrm{BV}}, -\}$

2.3. Homotopy algebras from field theories

The BV-complex is a dgca. There is a dual L_∞-algebra:

$\cdots$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\cdots$
$\cdots$	gauge-of-gauge transf.	gauge transf.	physical fields	equations of motion	Noether identities	higher Noether	$\cdots$

More direct computation:

- Guess n-ary products to recover action as homotopy Maurer–Cartan
- Example:
$$S = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \left(\tfrac{1}{2}\varphi(-\Box - m^2)\varphi - \frac{\kappa}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4\right) , \tag{2.7}$$
has L_∞-algebra
$$\underbrace{*}_{=:\,\mathfrak{g}_0} \longrightarrow \underbrace{C^\infty(\mathbb{R}^{1,3})}_{=:\,\mathfrak{g}_1} \xrightarrow{-\Box-m^2} \underbrace{C^\infty(\mathbb{R}^{1,3})}_{=:\,\mathfrak{g}_2} \longrightarrow \underbrace{*}_{=:\,\mathfrak{g}_3} \tag{2.8a}$$
with higher products
$$\begin{gathered} \mu_1(\varphi_1) \;:=\; (-\Box - m^2)\varphi_1 \;, \quad \mu_2(\varphi_1,\varphi_2) \;:=\; -\kappa\varphi_1\varphi_2 \;, \\ \mu_3(\varphi_1,\varphi_2,\varphi_3) \;:=\; -\lambda\varphi_1\varphi_2\varphi_3 \end{gathered} \tag{2.8b}$$

Yang–Mills theory: often more useful to work with A_∞-algebras:

- Action: (gauge algebra: $\mathfrak{gl}(N,\mathbb{C})$
$$S = \int_M (F, \star F) = \int_M \mathrm{tr}(F \wedge \star F) \tag{2.9}$$
- Higher products (some examples):
$$\begin{aligned} \mathsf{m}_1(A) &= \mathrm{d}^\dagger \mathrm{d}A + \ldots \;, \\ \mathsf{m}_2(A_1,A_2) &= \mathrm{d}^\dagger(A_1 \wedge A_2) + \star(A_1 \wedge \star \mathrm{d}A_2) + \ldots \;, \\ \mathsf{m}_3(A_1,A_2,A_3) &= \star(A_1 + \star(A_2 \wedge A_3)) + \ldots \end{aligned} \tag{2.10}$$
- Full gauge fixed, suitable for A_∞:
$$\begin{aligned} S_{\mathrm{YM,gf}} := \int \mathrm{tr}\Big\{ &\tfrac{1}{2}F \wedge \star F - (A^+ + \mathrm{d}\bar{c}) \wedge \star\nabla c - \tfrac{\kappa}{2}c^+ \wedge \star[c,c] \\ &- b \wedge \star\big(\bar{c}^+ + \mathrm{d}^\dagger A - \tfrac{\xi}{2}b\big)\Big\}, \end{aligned} \tag{2.11}$$

2.4. First results

- Equivalent Field Theories $\longleftrightarrow$ Quasi-isomorphic L_∞-algebras
  Example: 1st/2nd order Yang–Mills theory. Easier than integrating out etc.
- Strictification theorem $\Rightarrow$ Any field theory is equivalent to a field theory with only cubic vertices.
  Example: 1st/2nd order Yang–Mills theory

3. Lecture III

3.1. Homological Perturbation Lemma

Scattering amplitudes: encoded in minimal models of theory:
$$\mathcal{A}(\phi_0,\ldots,\phi_k)=\langle\phi_0,\mu_k^\circ(\phi_1,\ldots,\phi_k)\rangle^\circ$$

How to compute? Homological Perturbation Lemma:

- Start from A_∞-algebra $\mathfrak{a}$.
- Focus: underlying complex $(\mathfrak{a},\mathsf{m}_1)$.
- Link to minimal model via contracting homotopy

$$\begin{gathered}\mathsf{h}\circlearrowright(\mathfrak{a},\mathsf{m}_1)\underset{\mathsf{e}}{\overset{\mathsf{p}}{\rightleftarrows}}(\mathfrak{a}^\circ,0),\\ 1=\mathsf{m}_1\circ\mathsf{h}+\mathsf{h}\circ\mathsf{m}_1+\mathsf{e}\circ\mathsf{p},\quad\mathsf{p}\circ\mathsf{e}=1,\\ \mathsf{p}\circ\mathsf{h}=\mathsf{h}\circ\mathsf{e}=\mathsf{h}\circ\mathsf{h}=\mathsf{p}\circ\mathsf{m}_1=\mathsf{m}_1\circ\mathsf{e}=0.\end{gathered}\tag{3.1}$$

- Lift to codifferential coassociative coalgebra $\mathsf{T}(\mathfrak{a}):=\otimes^\bullet(\mathfrak{a}[1])$, $\mathsf{D}_0=\mathsf{m}_1$:

$$\begin{gathered}\mathsf{H}_0\circlearrowright(\mathsf{T}(\mathfrak{a}),\mathsf{D}_0)\underset{\mathsf{E}_0}{\overset{\mathsf{P}_0}{\rightleftarrows}}(\mathsf{T}(\mathfrak{a}^\circ),0),\\ \mathsf{P}_0|_{\mathsf{T}^k(\mathfrak{a})}:=\mathsf{p}^{\otimes^k},\quad\mathsf{E}_0|_{\mathsf{T}^k(\mathfrak{a}^\circ)}:=\mathsf{e}^{\otimes^k},\quad\mathsf{H}_0|_{\mathsf{T}^k(\mathfrak{a})}:=\sum_{i+j=k-1}1^{\otimes^i}\otimes\mathsf{h}\otimes(\mathsf{e}\circ\mathsf{p})^{\otimes^j}.\end{gathered}\tag{3.2}$$

- $\mathsf{D}=\mu_1+\mu_2+\mu_3+\cdots=\mathsf{D}_0+\mathsf{D}_{\rm int}$, regard $D_{\rm int}$ as perturbation
- Homological perturbation lemma: $\mathsf{D}_{\rm int}$ yields deformations

$$\begin{gathered}\mathsf{P}=\mathsf{P}_0\circ(1+\mathsf{D}_{\rm int}\circ\mathsf{H}_0)^{-1},\quad\mathsf{H}=\mathsf{H}_0\circ(1+\mathsf{D}_{\rm int}\circ\mathsf{H}_0)^{-1},\\ \mathsf{E}=(1+\mathsf{H}_0\circ\mathsf{D}_{\rm int})^{-1}\circ\mathsf{E}_0,\quad\mathsf{D}^\circ=\mathsf{P}\circ\mathsf{D}_{\rm int}\circ\mathsf{E}_0.\end{gathered}\tag{3.3}$$

  Proof: Computation. Note: existence of $(1+\mathsf{D}_{\rm int}\circ\mathsf{H}_0)^{-1}$ for small $\mathsf{D}_{\rm int}$.
- Outlook:

$$\mathsf{D}^\circ=\mathsf{P}_0\circ\mathsf{D}_{\rm int}\circ\mathsf{E}\qquad\mathsf{E}=\mathsf{E}_0-\mathsf{H}_0\circ\mathsf{D}_{\rm int}\circ\mathsf{E}\tag{3.4}$$

  $\Rightarrow$ Recursion relations!

3.2. Field theory and Feynman diagrams

Recall:

- Field theory in A_∞-algebra $\mathfrak{a}$
- $\mathfrak{a}^\circ$: kernel of m_1, "free fields," $\quad \mathfrak{a}^\circ = \ker_c(\mathsf{m}_1)$
- $\mathfrak{a}$: all fields, some choice, e.g. $\quad \mathfrak{a} = \ker_c(\mathsf{m}_1) + \mathcal{S}(\mathbb{R}^{1,3})$
- contracting homotopy: inverse of m_1: "propagator"
- Claim: Amplitudes

$$\mathcal{A}(\phi_0, \ldots, \phi_k) = \langle \phi_0, \mu_k^\circ(\phi_1, \ldots, \phi_k)\rangle^\circ = \sum_{\sigma \in S_k} \langle \phi_0, \mathsf{m}_k^\circ(\phi_{\sigma(1)}, \ldots, \phi_{\sigma(k)})\rangle \tag{3.5}$$

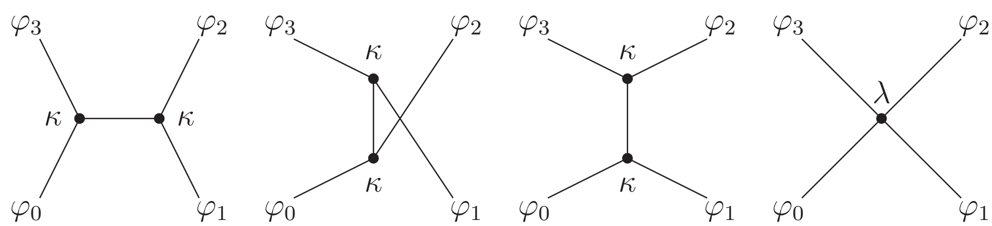
- Feynman diagrams from recursion relation:

$$\mathsf{D}^\circ = \mathsf{P}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E} \qquad \mathsf{E} = \mathsf{E}_0 - \mathsf{H}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E} \tag{3.6}$$

  Construct "current" $\mathsf{m}_k^\circ(\phi_1, \ldots, \phi_k)$ (by chopping off one leg of amplitude).

Example: Scalar fields, $D_{\text{int}} = \mathsf{m}_2 + \mathsf{m}_3$

- 4-point tree-level amplitude:



(3.7)

- $\mathsf{D}^\circ = P_0 \circ \mathsf{D}_{\text{int}} \circ (\mathsf{E}_0 - \mathsf{H}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E}_0 + \ldots) = \cdots + \mathsf{m}_3^\circ + \ldots$
  –sketch–

Note:

- Indeed amplitudes, "amputated diagrams"
- Recursion relation $\mathsf{E} = \mathsf{E}_0 - \mathsf{H}_0 \circ \mathsf{D}_{\text{int}} \circ \mathsf{E}$ was observed for Yang–Mills in 1988, Nucl. Phys. B, same journal as birthplace of L_∞-algebras!
- Berends–Giele recursion lead to Parke–Taylor formula, hugely important, inspired many things, etc. twistor strings.

3.3. Full quantum amplitudes

This all was "tree-level," i.e. our diagrams did not contain any loops. To get these:

- Recall in BV: classical master equation to quantum master equation:

$$Q_{\mathrm{BV}} := \{S_{\mathrm{BV}}, -\}\ ,\quad Q_{\mathrm{BV}}^2 = 0 \quad\longrightarrow\quad \hbar\Delta + \{S_{\mathrm{BV}}, -\}\ ,\quad 2\hbar\Delta S_{\mathrm{BV}} + \{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0 \tag{3.8}$$

  $\Delta = \omega^{AB}\frac{\delta}{\delta\Phi^A}\frac{\delta}{\delta\Phi^B} = \frac{\delta}{\delta\Phi^a}\frac{\delta}{\delta\Phi_a^+}$, "takes away" field-antifield pairs.
- In homological perturbation lemma, replace $\mathsf{D}_{\mathrm{int}}$ by $\mathsf{D}_{\mathrm{int}} + \hbar\Delta^*$
  Δ^* inserts field-antifield pairs in all possible ways.
- Structures in HPL get distorted:
  - P, E no longer algebra morphisms
  - minimal model as "quantum L_∞-algebra"
  - homotopy Jacobi identities distorted.
- Still: recursion relation

$$\mathsf{D}^\circ = \mathsf{P}_0 \circ \mathsf{D}_{\mathrm{int}} \circ \mathsf{E} \qquad \mathsf{E} = \mathsf{E}_0 - \mathsf{H}_0 \circ (\mathsf{D}_{\mathrm{int}} + \hbar\Delta^*) \circ \mathsf{E} \tag{3.9}$$

Example: Scalar field theory with $\mathsf{D}_{\mathrm{int}} = \mu_2 + \mu_3$:

- Restrict E, D to $\mathsf{E}^{i,j}$, $\mathsf{D}^{i,j}$ with i, j out/inputs.
- l: # of loops, v: # of vertices
- Recursion relation:

$$\mathsf{E}^{i,j}_{\ell,v} = \delta^0_\ell \delta^0_v \delta^{ij} \mathsf{E}^{i,i}_0 - \mathsf{H}_0 \circ \sum_{k=2}^{i+2} \mathsf{D}^{i,k}_{\mathrm{int}} \circ \mathsf{E}^{k,j}_{\ell,v-1} + \mathrm{i}\hbar\, \mathsf{H}_0 \circ \Delta^* \circ \mathsf{E}^{i-2,j}_{\ell-1,v} \tag{3.10}$$

— sketch : 1-loop correction to propagator —
Note: not properly amputated. Can be implemented, but irrelevant for combinatorics

Thus: We get Berends–Giele for any Lagrangian field theory and at loop level!

Applications

- Planar vs non-planar diagrams —sketch—
  ℓ-loop, n-point have maximally $t = \max\{\ell, n\}$ traces
  Thus: come with a factor of $N^{\ell-t+1}$
- Relation between planar and non-planar diagrams at 1-loop, knowing planar is sufficient!
- Hopefully: double copy
- Rewrite QFT books using homotopy algebras.

References

[1] B. Jurčo, L. Raspollini, C. Saemann, and M. Wolf, *L_∞-algebras of classical field theories and the Batalin–Vilkovisky formalism,* Fortsch. Phys. **67** (2019) 1900025 [1809.09899 [hep-th]].

[2] B. Jurčo, U. Schreiber, C. Saemann, and M. Wolf, *Higher Structures in M-Theory,* in: "Higher Structures in M-Theory," proceedings of the LMS/EPSRC Durham Symposium, 12-18 August 2018 [1903.02807 [hep-th]].

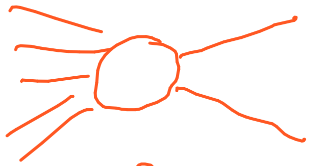
[3] T. Macrelli, C. Saemann, and M. Wolf, *Scattering amplitude recursion relations in BV quantisable theories,* Phys. Rev. D **100** (2019) 045017 [1903.05713 [hep-th]].

[4] B. Jurco, T. Macrelli, C. Saemann, and M. Wolf, *Loop Amplitudes and Quantum Homotopy Algebras,* 1912.06695 [hep-th].

Perturbative QFT & Homotopy Algebras.

Refs: arXiv 1809.09899, 1503.02887, 1903.05713,
1912.06695

Motivation: LHC  $d = 27\,\mathrm{km}$
$v = 0.99999999c$



$\mathcal{A} = \langle out | S | in \rangle$ scattering amplitude

$|\mathcal{A}|^2$: probability

$|out, free\rangle = S\,|in, free\rangle$

rays in $\mathcal{H}$

S-matrix, unitary operator

Scattering amplitudes:

- are crucial for understanding nature
- interesting & mysterious properties
- compute : heuristics "Quantum Field Theory"

Problem

Explain QFT : problematic & complicated

1) Field theory action $\xleftrightarrow[\text{BV-formalism}]{1:1}$ L_∞-algebra
2) L_∞-algebras have "minimal models"
3) Minimal models are given by scattering amplitudes.

Homotopy Algebras

dg: differential graded

dg Lie algebra: strict L_∞-algebra

- graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$
- $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$

  $da \in \mathfrak{g}_{i+1}$ $\qquad [a,b] \in \mathfrak{g}_{i+j}$
- $[a,b] = \pm [b,a]$ $\qquad [a,[b,c]] = \pm [[a,b],c] \pm [b,[a,c]]$
- $d^2 = 0$ $\qquad d[a,b] = [da,b] \pm [a,db]$
- Maurer–Cartan Equation $\quad a \in \mathfrak{g}_1 \quad da + \frac{1}{2}[a,a] = 0$

Metric / quadratic extension ("cyclic")

- graded sym. bilinear form $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$
- Compatibility: $\langle da, b \rangle = \pm \langle a, db \rangle$ etc.
- Maurer–Cartan action: $S = \frac{1}{2}\langle a, da\rangle + \frac{1}{3!}\langle a,[a,a]\rangle$

Alternative description I: Codiff cocomm. coalgebra.

- d, $[-,-]$ have "degrees" $1, 0$
- grade shift: $\mathfrak{g}[1]$ $\qquad (V[p])_q := V_{p+q}$
- Combine d $[-,-]$ codifferential $D = d + [-,-]$

  on $\odot^\bullet(\mathfrak{g}[1])$ $\qquad D^2 = 0$ of degree 1

Alternative II: dual dg commutative algebra

- basis τ_α of $\mathfrak{g}[1]$ coordinate functions $\xi^\alpha : \mathfrak{g}[1] \to \mathbb{R}$

  $\in (\mathfrak{g}[1])^*$
- $Q = D^*$ differential of degree 1 on $\odot^\bullet(\mathfrak{g}[1]^*)$ $\qquad Q^2 = 0$
- e.g. Lie algebra $\mathfrak{g}$, coord functions ξ^α of degree 1

  $Q = -\frac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \frac{\partial}{\partial \xi^\alpha}$ $\qquad Q^2 = 0 \longleftrightarrow$ Jacobi identity

Extensions:

- Allow coefficients in $\mathbb{Q}$ of arbitrary degree:
  $\to$ L_∞-algebras with products: $\mu_i : \mathfrak{g}^{\wedge i} \to \mathfrak{g}$

  μ_i are of degree $2-i$

  $$D = \mu_1 + \mu_2 + \mu_3 + \dots \qquad D^2 = 0 \qquad \mu_1^2 = 0$$

  $$\mu_2 \circ \mu_2 = \mu_1 \circ \mu_3 + \mu_3 \circ \mu_1$$

- L_∞-algebroids: graded vector bundles
- A_∞-algebras: $\odot \to \otimes$ $\qquad m_i : \mathfrak{g}^{\otimes i} \to \mathfrak{g}$

Examples:

trivial: $\mathfrak{g} = *$

strict: $\mu_i = 0$ for $i \geq 3$

skeletal: $\mu_1 = 0$

lin. contractible: $\mu_i = 0$ for $i \geq 2$ $\qquad H^\bullet_{\mu_1}(\mathfrak{g}) = 0$

Quasi-isomorphisms

morphism: L_∞-algebras have underlying complex
chain map $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$
+ compatible with higher products

strict

general morphisms: ϕ: actually morphisms of codiff cocom coalg
or, dually, dgca.

quasi-iso: $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$

(general) morphism of L_∞-algebras

$$\phi : H^\bullet_{\mu_1}(\mathfrak{g}) \xrightarrow{\cong} H^\bullet_{\mu_1}(\tilde{\mathfrak{g}})$$

Structural theorems:

Any L_∞-algebra
- is iso to skeletal $\oplus$ lin. contract.
- is q-iso to skeletal, minimal models
- is q-iso to a strict one

E.g. $\mathbb{R}[q] \xrightarrow{id} \mathbb{R}[q-1]$ qiso to $*$

1.4. Higher Chern-Simons theory

ingredients: $d \geq 3$-dim manifold M, compact

L_∞-algebra: $\mathfrak{g} = \mathfrak{g}_{3-d} \oplus \mathfrak{g}_{4-d} \oplus \dots \oplus \mathfrak{g}_0$

Note: dgca $\otimes$ L_∞-algebra has natural L_∞-algebra structure

- $\hat{\mu}_1 = d \otimes 1\!\!1 + 1\!\!1 \otimes \mu_1$
- $\hat{\mu}_i = 1\!\!1 \otimes \mu_i$

+ metric • $\widehat{\langle \alpha \otimes a, \beta \otimes b \rangle} = \langle \alpha, \beta \rangle \langle a, b \rangle$

Maurer Cartan action $S = \sum_{k \geq 1} \frac{1}{(k+1)!} \widehat{\langle a, \hat{\mu}_k(a, \dots, a) \rangle}$, $a \in \hat{\mathfrak{g}}_1$

• $\to$ higher Chern Simons for:

$\Omega^\bullet(M) \otimes \mathfrak{g}$

$d=3$: $\mathfrak{g} = \mathfrak{g}_0$

$\hat{\mathfrak{g}} = \Omega^\bullet(M) \otimes \mathfrak{g}_0$

$a \in \hat{\mathfrak{g}}_1$ $\quad a = A \in \Omega^1(M, \mathfrak{g})$

MC eqn: $F = dA + \frac{1}{2}[A, A] = 0$

$S = \int_M \frac{1}{2} \langle A, dA \rangle + \frac{1}{3!} \langle A, [A, A] \rangle = \int_M cs(A)$

$d\, cs(A) = \langle F, F \rangle$

$d=4$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$

$a \in \Omega^\bullet(M) \otimes \mathfrak{g}$ $\quad a = A + B \in \Omega^1(M, \mathfrak{g}_0) \oplus \Omega^2(M, \mathfrak{g}_{-1})$

$$S = \int_M \langle B, dA + \tfrac{1}{2}\mu_2(A, A) + \tfrac{1}{2}\mu_1(B) \rangle + \frac{1}{4!} \langle A, \mu_3(A, A, A) \rangle$$

$= \int_M cs(A, B)$ $\qquad d\, cs(A, B) = \langle F, H \rangle$

Perturbative QFT & Homotopy Algebras II

Recap:

$$\begin{tikzcd} \text{field theory actions} \arrow[r, leftrightarrow, "\mathrm{BV}"] & L_\infty\text{-algebras} \arrow[d, "\text{have}"] \\ \text{Scattering amplitudes} & \text{minimal models} \arrow[l, "\text{yield}"'] \end{tikzcd}$$

L_∞ - algebras

graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ $\quad$ Coalgebra pic: $\odot^\bullet \mathfrak{g}[1]$ $\quad$ dga-pic: $\odot^\bullet \mathfrak{g}[1]^*$

$\mu_i : \mathfrak{g}^{\wedge i} \to \mathfrak{g}$, $|\mu_i| = 2-i$ $\quad \longleftrightarrow \quad$ $D = \mu_1 + \mu_2 + \dots$ $\quad |D| = 1$ $\quad \longleftrightarrow \quad$ $Q = D^*$ $\quad |Q| = 1$

h Jacobi identities $\qquad D^2 = 0 \qquad Q^2 = 0$

A_∞ - algebras : replace $\odot \to \otimes$ $\qquad \underline{\mathfrak{a}} = \bigoplus_{i\in\mathbb{Z}} \underline{\mathfrak{a}}_i$, $m_i : \underline{\mathfrak{a}}^{\otimes i} \to \underline{\mathfrak{a}}$, etc.

homotopy Maurer - Cartan:

$$S = \sum_{k\geq 1} \frac{1}{(k+1)!} \langle a, \mu_k(a,\dots,a)\rangle \qquad a \in \mathfrak{g}_1$$

$$S = \sum_{k\geq 1} \frac{1}{k+1} \langle a, m_k(a,\dots,a)\rangle \qquad a \in \underline{\mathfrak{a}}_1$$

Field theory

classically: 1) spacetime M, $M = \mathbb{R}^{1,3}$, (M compact)
2) set of fields $\mathcal{F}$ (connections on principal bundles, sections of ass. vector bundles)
3) action functional $S : \mathcal{F} \to \mathbb{R}$, from Lagrangian
$L : \mathcal{F} \to \Omega^{top}(M)$

$$S = \int_M L$$

classical eom: $\underline{\dfrac{\delta S}{\delta \Phi} = 0}$ $\quad \Phi$: field

Example: scalar fields $\varphi : \mathbb{R}^{1,3} \to \mathbb{R}$

$$S = \int_{\mathbb{R}^{1,3}} d^4x \left(\underbrace{\tfrac{1}{2} \varphi (-\Box - m^2) \varphi}_{\text{kinetic term}} \underbrace{- \tfrac{\kappa}{3!} \varphi^3 - \tfrac{\lambda}{4!} \varphi^4}_{\text{interaction}} \right)$$

$\kappa, \lambda \in \mathbb{R}$ $\quad \lambda > 0$

Connections: $\quad \mathfrak{g}$: gauge Lie algebra
- encode in Lie algebra valued 1-form $A \in \Omega^1(M, \mathfrak{g})$
- gauge equivalence ($\Leftarrow$ bundle isomorphisms)
infinitesimally $c \in \Omega^0(M, \mathfrak{g})$ $\quad A \to A + \delta A$
$\delta A = dc + [A, c]$
- action : $S = \int d^4x \, (F, *F)$ $\quad F = dA + \frac{1}{2}[A, A]$

Observables: functions on $\mathcal{F}$. for elements satisfying eom which are invariant under gauge trafos

Quantum Field Theory:
Compute expressions like $\quad \int \underbrace{\mathcal{D}\Phi}_{\text{measure on } \mathcal{F}} f(\Phi) \, e^{-\frac{i}{\hbar} S[\Phi]}$

not rigorous, heuristics from Gaussian integration
→ perturbation theory
→ Feynman diagrams. $\quad \varphi^3 \leftrightarrow$  $\quad$  $\leftrightarrow \varphi^4$
graphs

BV-Formalism – classical part

need observables :
- factor out gauge symmetry ✓
- impose equations of motion

Gauge symmetry : Chevalley-Eilenberg resolution
- dividing spaces by group actions : bad idea

instead : action groupoid $\tilde{A} \xleftarrow{(g,A)} A$

action Lie algebroid

$$\mathcal{F}^{BRST} = \underbrace{\mathcal{F}^{BRST}_{-1}}_{\text{gauge trafos}} \oplus \underbrace{\mathcal{F}^{BRST}_{0}}_{\text{fields}}$$

Duality: coord. functions $A : \mathcal{F}_0^{BRST} \to \mathbb{R}$

$c : \mathcal{F}_{-1}^{BRST} \to \mathbb{R}$ ghost

BRST differential $Q_{BRST}\, A = \delta A = dc + [A,c]$

$Q_{BRST}\, c = -\frac{1}{2}[c,c]$

Complex $0 \to C_0^\infty(\mathcal{F}_{BRST}) \xrightarrow{Q} C_1^\infty(\mathcal{F}_{BRST}) \to \dots$

Resolution : $C^\infty(\mathcal{F}/_{gauge}) \leftrightarrow H^0(\mathcal{F}_{BRST})$

Koszul-Tate resolution :

Add to each field/ghost etc. Φ antifield Φ^+ of degree -1

symplectic form $\omega = d\Phi^A \wedge d\Phi^+_A$ A: runs over all types of fields

$$Q = \{ \overbrace{\underset{\uparrow}{S} \qquad + \underset{\uparrow}{\dots}}^{S_{BV}} , - \}$$

original action (+ghost) at least linear in Φ^+

$$\{S_{BV}, S_{BV}\} = 0$$

$\to$ BV complex $\mathcal{F}_{BV}$, ω_{BV}, S_{BV}, $Q_{BV} = \{S_{BV}, -\}$

$Q_{BV}^2 = 0$

$\to$ dgca.

$\{S_{BV}, \Phi^+\} = \text{eom}$

Homotopy algebras & Field theory

$\ldots$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\ldots$
	higher gauge trafos	ghosts	fields	antifields eom	Noether antifields for ghost	higher Noether af. for gh for gh	

More directly: "guess n-ary products such that action is hMC action of an L_∞-algebra"

E.g. $$S = \int_{\mathbb{R}^{1,3}} d^4x \left(\frac{1}{2}\varphi(-\Box - m^2)\varphi - \frac{\kappa}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 \right)$$

$$= \sum_{k\geq 1} \frac{1}{(k+1)!} \langle a, \mu_k(a, \ldots, a)\rangle$$

$$\cdots \longrightarrow \underset{\mathfrak{g}_0}{*} \longrightarrow \underset{\mathfrak{g}_1}{C^\infty(\mathbb{R}^{1,3})} \xrightarrow[\quad \ni \ \varphi_i]{\mu_1 = -\Box - m^2} \underset{\mathfrak{g}_2}{C^\infty(\mathbb{R}^{1,3})} \longrightarrow \underset{\mathfrak{g}_3}{*}$$

$$\mu_2(\varphi_1, \varphi_2) = -\kappa\, \varphi_1 \varphi_2$$

$$\mu_3(\varphi_1, \varphi_2, \varphi_3) = -\lambda\, \varphi_1 \varphi_2 \varphi_3$$

all other higher products trivial.

Yang-Mills (gauge: A, c

action: $$\int_{\mathbb{R}^{1,3}} d^4x\, (F, *F) = \sum_{k\geq 1} \frac{1}{k+1} \langle a, m_k(a, \ldots, a)\rangle$$

$a = A$ $\qquad m_1(A) = d^+ d A \qquad d^+ = *d*$

$$m_2(A_1, A_2) = d^+(A_1 \wedge A_2) + *(A_1 \wedge *dA_2) + \ldots$$

$$\underset{\Omega^1(M,\mathfrak{g})}{\mathfrak{g}_1} \xrightarrow{d^+ d A} \underset{\Omega^1(M,\mathfrak{g})}{\mathfrak{g}_2} \qquad m_3(A_1, A_2, A_3) = *(A_1 \wedge *(A_2 \wedge A_3)) + \ldots$$

incorporate ghosts, gauge fixing ... S_{BV} ....

Equivalence of Field Theories $\longleftrightarrow$ quasi-isos of L_∞-algebras

structural theorem: Any L_∞-alg q. iso to a strict one. $\mu_i = 0 \quad i \geq 3$

Perturbative QFT & HA III

Recap:

$$\text{field theory actions} \xleftrightarrow{BV} L_\infty\text{-algebras}$$

$$\Big\downarrow \text{have}$$

$$\text{Scattering amplitudes} \xleftarrow{\text{yield}} \text{minimal models}$$

Scalar field theory: $S = \int d^4x \left(\frac{1}{2}\varphi(-\Box - m^2)\varphi - \frac{\kappa}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 \right)$

L_∞-algebra

$$\mathfrak{g}_0 \to \mathfrak{g}_1 \to \mathfrak{g}_2 \to \mathfrak{g}_3$$

$$* \to C^\infty(\mathbb{R}^{1,3}) \to C^\infty(\mathbb{R}^{1,3}) \to *$$

$$\varphi \qquad\qquad \varphi^+$$

$$\mu_1(\varphi) = (-\Box - m^2)\varphi$$

$$\mu_2(\varphi_1, \varphi_2) = -\kappa\varphi_1\varphi_2$$

$$\mu_3(\varphi_1, \varphi_2, \varphi_3) = -\lambda\varphi_1\varphi_2\varphi_3$$

Equivalence of FT $\longleftrightarrow$ qiso L_∞-algebras

$\Rightarrow$ quick way of proving equivalence

Strictification theorem ($\mu_i = 0$ for $i \geq 3$) $\Rightarrow$ For any FT, $\exists$ equivalent FT with only cubic vertices $\curlywedge$.

e.g. 1st/2nd order YM theory

e.g. Yang-Mills:

$$\begin{array}{ccccccc} \mathfrak{g}_0 & & \mathfrak{g}_1 & & \mathfrak{g}_2 & & \mathfrak{g}_3 \\ \Omega^0(M, \mathfrak{gl}(N)) & \xrightarrow{d} & \Omega^1(M, \mathfrak{gl}(N)) & \xrightarrow{d^+ d} & \Omega^1(M, \mathfrak{gl}(N)) & \xrightarrow{d^+} & \Omega^0(M, \mathfrak{gl}(N)) \\ c & & A & & A^+ & & c^+ \end{array}$$

Homological Perturbation lemma:

A_∞-algebra $\underline{a}$, focus on underlying complex $\cdots \xrightarrow{m_1} \underline{a}_0 \xrightarrow{m_1} \underline{a}_1 \to ? \cdots$

$$h \circlearrowright (\underline{a}, m_1) \underset{e}{\overset{p}{\rightleftarrows}} (\underline{a}^\circ = H^\bullet_{m_1}(\underline{a}), 0) \qquad p \circ e = \mathrm{id}_{\underline{a}^\circ}$$

minimal model

$$\mathrm{id}_{\underline{a}} = e \circ p + m_1 \circ h + h \circ m_1 \qquad \text{Hodge Decomposition}$$

$$p \circ h = h \circ e = h \circ h = p \circ m_1 = m_1 \circ e = 0$$

Lift to codiff. coalgebra picture $D_0 = m_1$

$$H_0 \circlearrowright (\otimes^\bullet \underline{a}, D_0) \underset{E_0}{\overset{P_0}{\rightleftarrows}} (\otimes^\bullet \underline{a}^\circ, 0) \qquad P_0|_{\otimes^k \underline{a}} = p^{\otimes k}$$

E_0 same

$$H_0|_{\otimes^k \underline{a}} = \sum_{i+j=k-1} 1^{\otimes i} \otimes h \otimes (e \circ p)^{\otimes j}$$

$$D = m_1 + \underbrace{m_2 + m_3 + \cdots}_{\text{perturbation}} =: m_1 + D_{int}$$

HPL: $\quad P = P_0 \circ (1 + D_{int} \circ H_0)^{-1} \qquad H = H_0 \circ (1 + D_{int} \circ H_0)^{-1}$

$\quad E = (1 + H_0 \circ D_{int})^{-1} \circ E_0 \qquad D^\circ = P \circ D_{int} \circ E_0$

Useful: $\quad D^\circ = P_0 \circ D_{int} \circ E \quad , \quad E = E_0 - H_0 \circ D_{int} \circ E$

recursion relation $\qquad$ current

Field theory & Feynman diagrams

- FT in A_∞-algebra $\underline{a}$
- $\underline{a}^\circ = \ker(m_1) \qquad \underline{a}^\circ = \ker_c(m_1)$
- $\underline{a} = \ker_c(m_1) \oplus$ "interacting fields" $\qquad \ker_c(m_1) \oplus S(\mathbb{R}^{1,3})$
- contracting homotopy h : propagator

$$\begin{array}{ccc} \text{free} = \ker(m_1) & \xrightarrow{m_1 = 0} & \text{free} = \ker(m_1) \\ \oplus & & \oplus \\ \text{interacting} & \underset{h}{\overset{m_1}{\rightleftarrows}} & \text{interacting} \end{array}$$

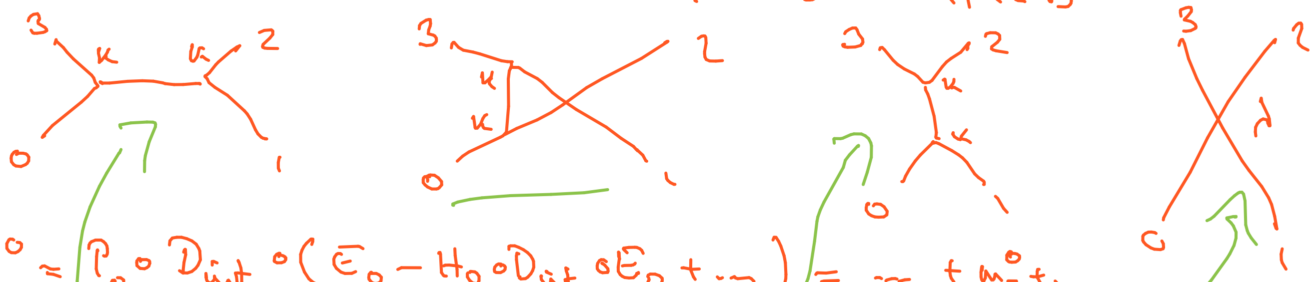
$$D^\circ = P_0 \circ D_{int} \circ E \qquad E = E_0 - H_0 \circ D_{int} \circ E$$

$$\mathcal{A}(\phi_0, \phi_1, \dots, \phi_k) = \langle \phi_0, \mu_k^\circ(\phi_1, \dots, \phi_k) \rangle$$
$$= \sum_{\sigma \in S_k} \langle \phi_0, m_k^\circ(\phi_{\sigma(1)}, \dots, \phi_{\sigma(k)}) \rangle$$

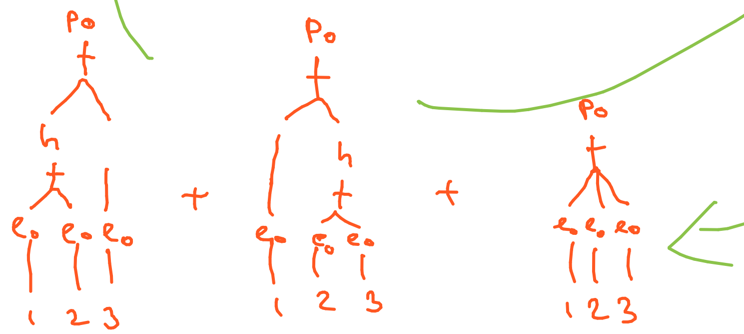
Construct currents $m_k^\circ(\phi_1, \dots, \phi_k)$

4-point for scalar FT $\qquad m_2(\varphi_1, \varphi_2) = \kappa\, \varphi_1 \varphi_2$

$\qquad m_3(\varphi_1, \varphi_2, \varphi_3) = \lambda\, \varphi_1 \varphi_2 \varphi_3$



$$D^\circ = P_0 \circ D_{int} \circ (E_0 - H_0 \circ D_{int} \circ E_0 + \dots) = \dots + m_3^\circ + \dots$$



Note:
- Naturally amputated
- For YM the recursion $E = E_0 - H_0 \circ D_{int} \circ E$ is Berends-Giele recursion for currents (1988)
- BG-recursion $\to$ Parke-Taylor amplitude formula.
- $\to$ have BG recursion for <u>all</u> Lagrangian FTs.

Quantum:

BV: $Q_{BV} = \{S_{BV}, -\}$ $\longrightarrow$ $\hbar\Delta + \{S_{BV}, -\}$

$Q_{BV}^2 = 0$ $\qquad$ $2\hbar\Delta S_{BV} + \{S_{BV}, S_{BV}\} = 0$

$\Delta : \dfrac{\delta^2}{\delta\phi^A \delta\phi^+_A}$ $\longrightarrow$ Δ^* : creates field/antifield pairs

$\Phi^A \otimes \dots \otimes \Phi^+_A$

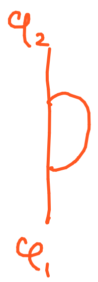
in any pos of tensor product.

HPL: $D_{int} \to D_{int} + \hbar\Delta^*$

things change: P, E no longer algebra morphisms

$\to$ quantum ~~to~~ homotopy algebra.

$E = E_0 + H_0(\hbar\Delta^* + D_{int})\circ E$



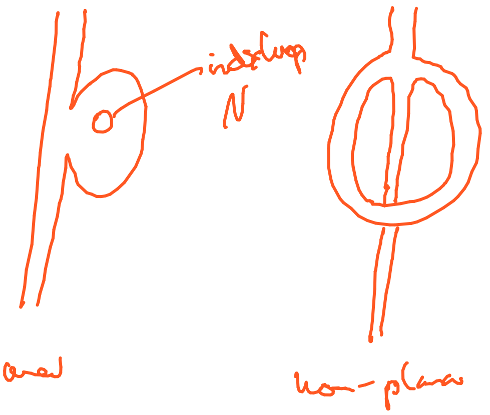
E^{ij} ← j inputs, i outputs.

l: # of loops, v: # vertices

$$E^{ij}_{l,v} = \delta^0_l \delta^0_v \delta^{ij} E^{ii}_0 - H_0 \circ \sum_{k=2}^{i+2} D^{i,k}_{int} \circ E^{k,j}_{l,v-1} + i\hbar H_0 \circ \Delta^* \circ E^{i-2,j}_{l-1,v}$$

recursion for l-loop v-vertex amplitude & terminates.

BG-recursion for <u>all</u> Lagrangian FT to all loop orders.



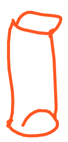
$gl(N)$ large N limit

1-loop level:

non-planar given by planar

L_∞ $\qquad$ A_∞

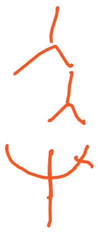
closed $\qquad$ open



double copy

Gravity $\longleftrightarrow$ YM

L_∞ $\qquad$ A_∞

 $=-$