On conformal Killing fields and their trajectories

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Plan

- vector field $\rightsquigarrow$ flow lines $\rightsquigarrow$ curve invariants
- discussion for conformal Killing fields on conformal mflds
- conditions on special trajectories
- general discussion vs. special cases
- tensorial vs. tractorial approach

Conformal manifolds

M ... smooth manifold of dim n,

$\gamma_{ab} \in \mathcal{E}_{(ab)}[2]$ ... conformal metric (of riemannian signature),

$\sigma \in \mathcal{E}[1]$ ... conformal scale:

- $g_{ab} = \sigma^{-2}\gamma_{ab} \in \mathcal{E}_{(ab)}$ ... metric from the class,
- D_a ... Levi-Civita connection,
- P_{ab} ... Schouten tensor,

$\hat{\sigma} = f\sigma$ ... change of scale, $\Upsilon_a := f^{-1} D_a f$,

$$\widehat{D}_a \xi^b = D_a \xi^b + \Upsilon_a \xi^b - \xi_a \Upsilon^b + \xi^c \Upsilon_c \delta_a{}^b.$$

Conformal manifolds

$k^a \in \mathcal{E}^a$ ... conformal Killing field is an infinitesimal conformal symmetry,

i.e. $\mathcal{L}_k g \propto g$,

i.e. vanishing of the trace-free part of

$$D_{(a}k_{b)} \in \mathcal{E}_{(ab)}[2].$$

$\sigma \in \mathcal{E}[1]$ ... almost Einstein scale if $g_{ab} = \sigma^{-2}\gamma_{ab}$ is Einstein,

i.e. vanishing of the trace-free part of

$$D_a D_b \sigma + \mathrm{P}_{ab}\sigma \in \mathcal{E}_{ab}[1].$$

Standard tractors

$\mathcal{T} \to M$ ... standard tractor bundle (rank $n+2$),

given by the standard rep. of $O(n+1,1)$,

$$\begin{array}{rcll} & & \mathcal{E}[1] & \longleftarrow \text{projecting} \\ & & \oplus & \\ \mathcal{T} & \cong & \mathcal{E}^a[-1] & \\ & & \oplus & \\ & & \mathcal{E}[-1] & \longleftarrow \text{injecting} \end{array}$$

$\boldsymbol{X} : \mathcal{E}[-1] \hookrightarrow \mathcal{T}$ ... the insertion.

$\boldsymbol{\nabla}$ ... tractor connection,

$\cdot$ ... parallel bundle metric of signature $(n+1,1)$,

$$\boldsymbol{U} = \begin{pmatrix} \sigma \\ \phi^a \\ \rho \end{pmatrix}, \ \boldsymbol{V} = \begin{pmatrix} \tau \\ \psi^a \\ \pi \end{pmatrix} \mapsto \ \boldsymbol{U} \cdot \boldsymbol{V} = \sigma\pi + \phi^a \psi^b \gamma_{ab} + \rho\tau.$$

Standard tractors

$L : \mathcal{E}[1] \to \mathcal{T}$ ... BGG splitting operator,

$$\sigma \mapsto \begin{pmatrix} \sigma \\ D_a \sigma \\ -\frac{1}{n}(\Delta + J)\sigma \end{pmatrix} =: \boldsymbol{I}.$$

Theorem[1]

$\sigma =$ almost Einstein scale $\iff \boldsymbol{\nabla I} = 0$.

[1] T.N. Bailey, M.G. Eastwood, A.R. Gover, Thomas's structure bundle for conformal, projective and related structures, 1994

Adjoint tractors

$\mathcal{A} \to M$ ... adjoint tractor bundle,

given by the adjoint rep. of $O(n+1,1)$, $\mathcal{A} \cong \Lambda^2\mathcal{T}$,

$$\mathcal{A} \cong \begin{matrix} \mathcal{E}^a \\ \oplus \\ \mathcal{E}_{[ab]}[2] \oplus \mathcal{E} \\ \oplus \\ \mathcal{E}_a \end{matrix} \qquad \begin{matrix} \longleftarrow \text{projecting} \\ \\ \\ \\ \longleftarrow \text{injecting} \end{matrix}$$

∇ ... tractor connection,

$\bullet : \mathcal{A} \times \mathcal{T} \to \mathcal{T}$ ... standard algebraic action,

$$\mathcal{K} = \begin{pmatrix} k^a \\ \mu_{ab} \mid \nu \\ \rho_a \end{pmatrix}, \; \boldsymbol{V} = \begin{pmatrix} \tau \\ \psi^a \\ \pi \end{pmatrix} \mapsto \mathcal{K} \bullet \boldsymbol{V} = \begin{pmatrix} \nu\tau + \rho_a\psi^a \\ \tau k^a + \mu^a{}_b\psi^b - \rho^a\pi \\ -k_a\psi^a - \nu\pi \end{pmatrix}.$$

Adjoint tractors

$L : \mathcal{E}[1] \to \mathcal{A}$ ... BGG splitting operator,

$$k^a \mapsto \begin{pmatrix} k^a \\ D_{[a}k_{b]} \mid -\frac{1}{n}D_c k^c \\ -\frac{1}{n}(\Delta + J)k_a + \frac{2}{n}\mathrm{P}_{ac}k^c + \cdots \end{pmatrix} =: \mathcal{K}.$$

Theorem[2]

k^a = conformal Killing field $\iff \boldsymbol{\nabla}_a \mathcal{K} = k^c \boldsymbol{\Omega}_{ca}$,
where $\boldsymbol{\Omega}_{cd} \in \mathcal{E}_{[cd]} \otimes \mathcal{A}$ is the curvature of $\boldsymbol{\nabla}$.

[2] R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, 2005

Notation

For $\mathcal{K} = L(k)$,

$$\mathcal{K} =: \begin{pmatrix} k^a \\ \mu_{ab} \mid \nu \\ \rho_a \end{pmatrix},$$

where $\mu = \mathrm{d}k$, $\nu = -\frac{1}{n}\,\mathrm{div}\,k$, ...

k ... <u>conformal Killing</u>:

- $k' = \mu(k) + \nu k$.[3]

k ... <u>Killing</u>:

- $\nu = 0$,
- $k \perp k'$ and $\|k\| =$ const. along any flow line,
- $\rho \propto J\,k - 2\,\mathrm{P}(k)$.

[3] i.e. $k^c D_c k^a = \mu^a{}_c k^c + \nu k^a$

Curves

$C : \mathbb{R} \to M$ ... regular curve (with parameter t),

U, $U' := D_U U$, etc. ... tangent, acceleration, etc. vectors,

$u := \|U\| \in \mathcal{E}[1]$ ... key density along C.

$\boldsymbol{T} = u^{-1}\boldsymbol{X} \in \mathcal{T}$ ... tractor lift of C,

$\boldsymbol{U} := \boldsymbol{\nabla}_U \boldsymbol{T}$, $\boldsymbol{U}' := \boldsymbol{\nabla}_U \boldsymbol{U}$, etc. ... tangent, acceleration, etc. tractors.

Starting relations:

$\cdot$	$\boldsymbol{T}$	$\boldsymbol{U}$	$\boldsymbol{U}'$	$\boldsymbol{U}''$	$\boldsymbol{U}'''$
$\boldsymbol{T}$	0	0	-1	0	α
$\boldsymbol{U}$	0	1	0	$-\alpha$	$-\frac{3}{2}\alpha'$
$\boldsymbol{U}'$	-1	0	α	$\frac{1}{2}\alpha'$	$\frac{1}{2}\alpha'' - \beta$
$\boldsymbol{U}''$	0	$-\alpha$	$\frac{1}{2}\alpha'$	β	$\frac{1}{2}\beta'$
$\boldsymbol{U}'''$	α	$\frac{3}{2}\alpha'$	$\frac{1}{2}\alpha'' - \beta$	$\frac{1}{2}\beta'$	γ

Curves

Proposition[4]

$\alpha := \boldsymbol{U}' \cdot \boldsymbol{U}' = 0$ determines the projective pameters of the curve.

$\Delta_i := \det\left(\mathrm{Gram}\left(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots, \boldsymbol{U}^{(i-2)}\right)\right)$ ... key relative invariants,[5]

in particular,

$$\Delta_1 = \Delta_2 = 0, \quad \Delta_3 = -1, \quad \Delta_4 = -\beta + \alpha^2.$$

Cases:

- $\Delta_4 > 0 \rightsquigarrow$ conformal arc-length, absoulte invariants, ...
- $\Delta_4 = 0 \rightsquigarrow$ guess what? (s. 14)

[4] T.N. Bailey, M.G. Eastwood, A.R. Gover, Thomas's structure bundle for conformal, projective and related structures, 1994

[5] J. Šilhan, V. Žádník, Conformal theory of curves with tractors, 2019

General approach

k ... conformal Killing field,

$\mathcal{K} = L(k) \in \mathcal{A}$ ... the BGG splitting,

$\boldsymbol{K} \in \mathcal{T}[1]$ ... auxiliary tractor,

$$\boldsymbol{K} := \mathcal{K} \bullet \boldsymbol{X} = \begin{pmatrix} 0 \\ k^a \\ -\nu \end{pmatrix}.$$

Key lemma

$$\boldsymbol{K}' = \mathcal{K}^2 \bullet \boldsymbol{X}, \quad \boldsymbol{K}'' = \mathcal{K}^3 \bullet \boldsymbol{X}, \quad \boldsymbol{K}''' = \mathcal{K}^4 \bullet \boldsymbol{X}, \quad \text{etc.}$$

[first two slots easy, the last one needs some care]

First consequences

For particular flows,

$$\boldsymbol{T} = u^{-1}\boldsymbol{X}, \quad \boldsymbol{U} = u^{-1}\mathcal{K} \bullet \boldsymbol{X}, \quad \boldsymbol{U}' = u^{-1}\mathcal{K}^2 \bullet \boldsymbol{X}, \quad \text{etc.}$$

$\mathcal{K}^{\text{odd}} \in \Lambda^2\mathcal{T}$ and $\text{proj}(\Lambda^2\mathcal{T}) = \mathcal{E}^a$,

$\mathcal{K}^{\text{even}} \in S^2\mathcal{T}$ and $\text{proj}(S^2\mathcal{T}) = \mathcal{E}[2]$.

In particular,

$$\boldsymbol{U}^{(i)} \cdot \boldsymbol{U}^{(j)} = \begin{cases} u^{-2}\,\text{proj}\,\mathcal{K}^{i+j+2}, & \text{for } i+j = \text{even}, \\ 0, & \text{for } i+j = \text{odd}. \end{cases}$$

Proposition

Flows of k are parametrized by

- projective parameter $\iff$ $\text{proj}\,\mathcal{K}^4 = 0$,
- conformal arc-length $\iff$ $-u^2\,\text{proj}\,\mathcal{K}^6 + (\text{proj}\,\mathcal{K}^4)^2 = u^4$.

Conformal circles

As unparametrized curves, given in many equivalent ways:

- develops to ordinary circles in the model sphere,
- solutions to certain ODE (of 3rd order) on M,
- namely,[6]

$$\operatorname{mid} \boldsymbol{U}'' \propto \operatorname{mid} \boldsymbol{U},$$

- $\boldsymbol{U}'' \in \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \subset \mathcal{T}$, i.e. parallel rank 3 subbundle,
- solutions to tractor ODE,

$$\Delta_4 = -\boldsymbol{U}'' \cdot \boldsymbol{U}'' + (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 = 0,$$

- etc.[7] [8]

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[6] … indeed conformally invariant (since $\operatorname{proj} \boldsymbol{U}'' = \operatorname{proj} \boldsymbol{U} = 0$)

[7] R. Gover, D. Snell, A. Taghavi-Chabert, Distinguished curves and integrability in Riemannian, conformal, and projective geometry, 2019

[8] M. Eastwood, L. Zalabová, Special metrics and scales in parabolic geometry, 2020

Further consequences

k ... conformal Killing field,

$\mathcal{K} = L(k) \in \mathcal{A}$ ... the BGG splitting,

Proposition

Trajectories of k are conformal circles $\iff$ $\operatorname{proj} \mathcal{K}^3 \propto \operatorname{proj} \mathcal{K}$.

[$\operatorname{mid} \boldsymbol{U} \sim \operatorname{mid}(\mathcal{K} \bullet \boldsymbol{X}) \sim \operatorname{proj} \mathcal{K}, \dots$]

Specializations

k ... Killing field of Einstein metric:

$$\begin{aligned}
\operatorname{proj} \mathcal{K} &= k, \\
\operatorname{proj} \mathcal{K}^2 &= \|k\|^2, \\
\operatorname{proj} \mathcal{K}^3 &= c_1\, \mu^2(k) + (c_2 \operatorname{tr} \mu^2 + c_3\, \|k\|^2 + c_4) k, \\
\operatorname{proj} \mathcal{K}^4 &= c_5\, \mu^2(k) \cdot k + c_6 \operatorname{tr} \mu^2 + c_7, \quad \text{etc.}
\end{aligned}$$

In particular,

trajectories of k are conformal circles $\iff \mu^2(k) \propto k$.

Without Einstein assumption, the condition in box is replaced by

$$\mu^2(k) + \mathrm{P}(k) \propto k.$$

Remarks

Our characterization encompasses:

- Trajectories of a Killing field are metric geodesics $\iff \mu(k) \propto k$.

  [easy]

- Trajectories of hypersurface-OG Killing field are conformal circles.[9]

  $[\, k \wedge \mathrm{d}k = 0 \Longrightarrow \mu^2(k) \propto k$ and $\mathrm{P}(k) \propto k \,]$

- Trajectories of Killing field of Einstein metric on 4-mfld are conformal circles $\iff \mu(k) \lrcorner (k \wedge \mu) = 0$.[10]

  $[\,$indeed $\iff \mu^2(k) \propto k \,]$

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[9] P. Tod, Some examples of the behaviour of conformal geodesics, 2012

[10] M. Dunajski, P. Tod, Conformal geodesics on gravitational instantons, 2019

Remarks

Homogeneous setting allows many “simplifications”:

- above conditions formulated in terms of brackets,
- conformal circles as orbits of 1-parametric subgroups:
  - description of appropriate generators from the Lie algebra,
  - search for elements s.t. all trajectories are conformal circles,
- tested for Fubini–Study and few more examples…