Initial data for closed conformal Killing-Yano 2-forms

(cf. arXiv:1912.04752 with A. García-Parrado)

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Killing-Yano and Conformal Killing-Yano Operators

Operator on 2-forms $Y_{[ab]} = Y_{ab}$ on a Lorentzian manifold (M, g_{ab}):

$$\mathrm{CYK}_{a:[bc]}[Y] = \mathrm{CYK}_{a:bc}[Y] := 2\nabla_a Y_{bc} - \nabla_b Y_{ca} + \nabla_c Y_{ba} + \frac{3}{n-1} g_{ab} \nabla^d Y_{cd} - \frac{3}{n-1} g_{ca} \nabla^d Y_{bd}$$

Most general covariant, first order operators:

$$(\ydiagram{1}\,\nabla_a)(\ydiagram{1,1}\,Y_{bc}) = \frac{1}{3}(\ydiagram{2,1}\,\mathrm{CYK}_{a:bc}[Y]) + \frac{1}{6}(\ydiagram{1,1,1}\,\mathrm{d}Y)_{abc} - \frac{2}{n-1} g_{a[b}(\ydiagram{1}\,\delta Y)_{c]}$$

Tensor fields are pointwise irreps of $O(g) \subset GL(n)$:

$\ydiagram{1}, \ydiagram{2}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2}$ — traceless, Young symmetrized tensors

Tensor product of irreps: $\ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{2,1} + \ydiagram{1,1,1} + \ydiagram{1}$

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Why Closed Conformal Killing-Yano tensors?

- Killing tensors generalize the Killing vector equation:
  - Killing-Stäckel (SK): symmetric k-tensors
  - (Conformal) Killing-Yano (YK, CYK): p-forms
- Geodesic motion:
  $\mathrm{SK}[S] = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}(S_{a_1\cdots a_k}\dot{x}^{a_1}\cdots\dot{x}^{a_k}) = 0$
- SUSY extension of geodesic motion (Gibbons *et al*, 1993):
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- Hodge duality: Killing-Yano $\leftrightarrow$ closed conformal Killing-Yano

$$\mathrm{YK}[Y] = 0 \iff \mathrm{CYK}[*Y] = 0,\ \mathrm{d}{*Y} = 0 \quad \text{(cCYK)}$$

- Higher dimensional, Einstein vacuum, rotating black holes:

$$R_{ab} = \frac{2\Lambda}{n-2} g_{ab},\ \begin{bmatrix} \mathrm{CYK}[Y] \\ \mathrm{d}Y \end{bmatrix} = 0,\ \exists (Y_{ab})^{-1} \iff \begin{array}{c}\text{locally} \\ \text{Kerr-NUT-(A)dS} \\ \text{metric}\end{array}$$

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Initial Data Conditions

- Observation (1970s): given an initial data set $(\Sigma \subset M, g_{AB})$, $\pi_{AB} = \partial_t g_{AB}|_{t=0}$ for the vacuum Einstein equations on (M, g_{ab}), there exists a linear PDE on Σ for v_0, v_A whose solutions are in bijection with the restriction $v_a|_\Sigma$ of a Killing vector on (M, g_{ab}).
- This Killing initial data (KID) system has found applications to
  - the rigidity of black hole solutions,
  - the study of singular strata of the manifold of admissible initial data for the Einstein equations,
  - the characterization of the initial data for special solutions.
- What other geometric PDEs have analogous initial data systems?
- **Today:** Killing vectors $\rightsquigarrow$ closed conformal Killing-Yano 2-forms

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Propagation Identity

Lemma (Propagation Identity)

Consider an Einstein vacuum spacetime (M, g_{ab}) *with Cauchy surface* Σ*. The existence of identities (modulo* $R_{ab} = \frac{2\Lambda}{n-2} g_{ab}$*)*

$$P[\ \psi\ ] = \quad 0 \quad , \qquad \dots$$

where $P[\psi] = 0$ *and* $\dots$ *have well-posed initial value problems (IVPs), guarantees that* $\psi \ = 0$ *on* M *with vanishing P-initial data*

$\psi|_\Sigma,$ $\partial_t \psi|_\Sigma, \partial_t^2 \psi|_\Sigma, \dots$.

The above vanishing conditions can be collected into a PDE system $E^\Sigma[\phi|_\Sigma, \partial_t \phi|_\Sigma, \dots] = 0$ intrinsic to $(\Sigma, g_{AB}, \pi_{AB})$, which we call an *E*-initial data system.

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where $P[\psi] = 0$ and $Q[\phi] = 0$ have well-posed initial value problems (IVPs), guarantees that $E[\phi] = 0$ *on* M *with* Q*-initial data* $\phi|_\Sigma, \partial_t \phi|_\Sigma, \partial_t^2 \phi|_\Sigma, \ldots$ *with vanishing* P*-initial data* $\psi = E[\phi]|_\Sigma, \partial_t \psi|_\Sigma, \partial_t^2 \psi|_\Sigma, \ldots$*.*

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Killing Equation

An infinitesimal isometry v^a satisfies ($E[\phi] = 0$)

$$\mathrm{K}_{ab}[v] := \nabla_a v_b + \nabla_b v_a = 0.$$

Propagation identity ($P[E[\phi]] = \sigma[Q[\phi]]$, $Q[\phi] = \rho[E[\phi]]$):

$$\Box \mathrm{K}_{ab}[v] - 2R^c{}_{ab}{}^d \mathrm{K}_{cd}[v] = \mathrm{K}_{ab}[\Box v],$$

$$\Box v_a = \nabla^b \mathrm{K}_{ab}[v] - \frac{1}{2}\nabla_a \mathrm{K}^b{}_b[v].$$

Both P and Q are normally hyperbolic ($\Box +$ l.o.t), hence have well-posed IVPs.

Killing Initial Data (KID)

In Gaussian normal coordinates $(x^a) = (x^0 = t, x^A)$, we have $\Sigma = \{t = 0\}$, a unit normal $n_a = (\mathrm{d}t)_a$, and the splittings

$$v_a \to \begin{bmatrix} v_0 \\ v_A \end{bmatrix}, \quad g_{ab} \to \begin{bmatrix} -1 & 0 \\ 0 & g_{AB} \end{bmatrix}.$$

Let D_A be the covariant derivative on (Σ, g_{AB}), $\pi_{AB} = \partial_t g_{AB}$ (second fundamental form) with $\pi = \pi_A{}^A$, $(\pi \cdot \pi)_{AB} = \pi_A{}^C \pi_{CB}$, and r_{ABCD}, r_{AB} spatial Riemann and Ricci tensors.

KID system (modulo $\mathrm{K}_{00}[v] = \mathrm{K}_{0B}[v] = 0$)

$$D_A v_B + D_B v_A - 2\pi_{AB} v_0 = 0,$$

$$D_A D_B v_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - r_{AB}) v_0 - 2\pi_{(B}{}^C D_{A)} v_C - (D^C \pi_{AB}) v_C = 0.$$

Old result: due to Berezdivin (1974), Moncrief (1975), Coll (1977).

cCYK Propagation Identity

- Propagation identities: $P[E[\phi]] = \sigma[Q[\phi]]$, $Q[\phi] = \rho[E[\phi]]$.
- **Goal:** exhaustive search for P, Q, ρ, σ
  - covariant in ∇_a, g_{ab}, R_{abcd}, $\nabla_e R_{abcd}$, ...
  - 1st order σ, ρ; 2nd (total) order P, Q
  - hyperbolic P, Q

  Total order: $|g_{ab}| = 0$, $|\nabla| = 1$, $|R_{abcd}| = |\Lambda| = 2$, respected by composition.
- Use representation theory of $O(g)$ to parametrize ansatz.

$$\begin{bmatrix} P^{1,2,3,4,5,6} & P^{7,8} \\ \hat{P}^{5,6} & \hat{P}^{1,2,3,4} \end{bmatrix} \begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} - \begin{bmatrix} \sigma^1 \\ \hat{\sigma}^1 \end{bmatrix} Q^{1,2,3,4} = \begin{bmatrix} T^{1,2,3,4,5,6,7,8} \\ \hat{T}^{1,2,3,4} \end{bmatrix},$$

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$$\begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} P^{1,2,3,4,5,6} & P^{7,8} \\ \hat{P}^{5,6} & \hat{P}^{1,2,3,4} \end{bmatrix}\begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

$$- \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \sigma^1 \\ \hat{\sigma}^1 \end{bmatrix} \square\, Q^{1,2,3,4} \,\square = \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} T^{1,2,3,4,5,6,7,8} \\ \hat{T}^{1,2,3,4} \end{bmatrix} \square,$$

$$\square\, Q^{1,2,3,4} \,\square = \square \begin{bmatrix} \rho^1 & \rho^2 \end{bmatrix} \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

cCYK Propagation Identity

- Propagation identities: $P[E[\phi]] = \sigma[Q[\phi]]$, $Q[\phi] = \rho[E[\phi]]$.
- **Goal:** exhaustive search for P, Q, ρ, σ
  - covariant in $\nabla_a, g_{ab}, R_{abcd}, \nabla_e R_{abcd}, \ldots$
  - 1st order σ, ρ; 2nd (total) order P, Q
  - hyperbolic P, Q

  Total order: $|g_{ab}| = 0$, $|\nabla| = 1$, $|R_{abcd}| = |\Lambda| = 2$, respected by composition.
- Use representation theory of $O(g)$ to parametrize ansatz.

$$\begin{matrix}\yng(2,1)\\ \yng(1,1)\end{matrix}\begin{bmatrix} P^{1,2,3,4,5,6} & P^{7,8} \\ \hat{P}^{5,6} & \hat{P}^{1,2,3,4} \end{bmatrix} \begin{matrix}\yng(2,1)\\ \yng(1,1)\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

$$- \begin{matrix}\yng(2,1)\\ \yng(1,1)\end{matrix}\begin{bmatrix} \sigma^1 \\ \hat{\sigma}^1 \end{bmatrix} \square\, Q^{1,2,3,4} \square = \begin{matrix}\yng(2,1)\\ \yng(1,1)\end{matrix}\begin{bmatrix} T^{1,2,3,4,5,6,7,8} \\ \hat{T}^{1,2,3,4} \end{bmatrix} \square ,$$

$$\square\, Q^{1,2,3,4} \square = \square \begin{bmatrix} \rho^1 & \rho^2 \end{bmatrix} \begin{matrix}\yng(2,1)\\ \yng(1,1)\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

Representation Theory

- Use $O(g)$ tensor product rules to build complete operator bases.
- Bianchi identities: $\yng(2,2)\, W_{ab:cd} = R_{abcd} - \text{traces}$, $\yng(3,2)\, \nabla_a W_{bc:de}$.
- $n = 1, 2, 3$: degenerate cases
- $n > 8$: valid by Littlewood's rule (1958)
- $n = \not{4}, 5, \not{6}, 7, 8$: checked by LiE/SAGE; $n = 6$: exception! ($\yng(1,1,1)$ reducible)

tot.ord. $= 1, 2$:

		$\yng(1,1)\, Y_{ab}$	$\yng(2,1)\, C_{a:bc}$	$\yng(1,1,1)\, \Xi_{abc}$
∇	$\yng(1)$	$\yng(2,1)\,\sigma^1 + \yng(1,1,1)\,\hat{\sigma}^1$	$\yng(1,1)\,\rho^1$	$\yng(1,1)\,\rho^2$
$\square$	$\mathbb{R}$	$\yng(1,1)\, Q^1$	$\yng(2,1)\, P^1$	$\yng(1,1,1)\, \hat{P}^1$
$\nabla\nabla$	$\yng(2)$	$\yng(1,1)\, Q^2$	$\yng(2,1)\, P^{2,3} + \yng(1,1,1)\, \hat{P}^5$	$\yng(2,1)\, P^7 + \yng(1,1,1)\, \hat{P}^2$
W	$\yng(2,2)$	$\yng(1,1)\, Q^3$	$\yng(2,1)\, P^{4,5} + \yng(1,1,1)\, \hat{P}^6$	$\yng(2,1)\, P^8 + \yng(1,1,1)\, \hat{P}^3$
Λ	$\mathbb{R}$	$\yng(1,1)\, Q^4$	$\yng(2,1)\, P^6$	$\yng(1,1,1)\, \hat{P}^4$

tot.ord. $= 3$:

			$(\yng(1)\,\yng(1,1))\nabla_a Y_{bc}$		
		$\yng(1,1)\, Y_{ab}$	$\yng(1)\,\delta Y$	$\yng(2,1)\,\mathrm{CYK}[Y]$	$\yng(1,1,1)\,\mathrm{d}Y$
$\square\nabla$	$\yng(1)$	$\yng(2,1)\, T^1 + \yng(1,1,1)\, \hat{T}^1$			
$\nabla\nabla\nabla$	$\yng(3)$	$\yng(2,1)\, T^2$			
∇W	$\yng(3,2)$	$\yng(2,1)\, T^7$			
W	$\yng(2,2)$		$\yng(2,1)\, T^6$	$\yng(2,1)\, T^{3,4} + \yng(1,1,1)\, \hat{T}^2$	$\yng(2,1)\, T^5 + \yng(1,1,1)\, \hat{T}^3$
Λ	$\mathbb{R}$			$\yng(2,1)\, T^8$	$\yng(1,1,1)\, \hat{T}^4$

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$\square$	$\mathbb{R}$	$\yng(1,1)\, Q^1$	$\yng(2,1)\, P^1$	$\yng(1,1,1)\, \hat P^1$
$\nabla\nabla$	$\yng(2)$	$\yng(1,1)\, Q^2$	$\yng(2,1)\, P^{2,3} + \yng(1,1,1)\, \hat P^5$	$\yng(2,1)\, P^7 + \yng(1,1,1)\, \hat P^2$
W	$\yng(2,2)$	$\yng(1,1)\, Q^3$	$\yng(2,1)\, P^{4,5} + \yng(1,1,1)\, \hat P^6$	$\yng(2,1)\, P^8 + \yng(1,1,1)\, \hat P^3$
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tot.ord. $= 3$:

			$(\yng(1)\,\yng(1,1))\nabla_a Y_{bc}$		
		$\yng(1,1)\, Y_{ab}$	$\yng(1)\,\delta Y$	$\yng(2,1)\,\mathrm{CYK}[Y]$	$\yng(1,1,1)\,\mathrm{d}Y$
$\square\nabla$	$\yng(1)$	$\yng(2,1)\, T^1 + \yng(1,1,1)\, \hat T^1$			
$\nabla\nabla\nabla$	$\yng(3)$	$\yng(2,1)\, T^2$			
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Λ	$\mathbb{R}$			$\yng(2,1)\, T^8$	$\yng(1,1,1)\, \hat T^4$

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		$\ydiagram{1,1}\, Y_{ab}$	$\ydiagram{2,1}\, C_{a:bc}$	$\ydiagram{1,1,1}\, \Xi_{abc}$
∇	$\ydiagram{1}$	$\ydiagram{2,1}\, \sigma^1 + \ydiagram{1,1,1}\, \hat{\sigma}^1$	$\ydiagram{1,1}\, \rho^1$	$\ydiagram{1,1}\, \rho^2$
$\square$	$\mathbb{R}$	$\ydiagram{1,1}\, Q^1$	$\ydiagram{2,1}\, P^1$	$\ydiagram{1,1,1}\, \hat{P}^1$
$\nabla\nabla$	$\ydiagram{2}$	$\ydiagram{1,1}\, Q^2$	$\ydiagram{2,1}\, P^{2,3} + \ydiagram{1,1,1}\, \hat{P}^5$	$\ydiagram{2,1}\, P^7 + \ydiagram{1,1,1}\, \hat{P}^2$
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tot.ord. $= 3$:

			$(\ydiagram{1}\,\ydiagram{1,1})\nabla_a Y_{bc}$		
		$\ydiagram{1,1}\, Y_{ab}$	$\ydiagram{1}\, \delta Y$	$\ydiagram{2,1}\, \mathrm{CYK}[Y]$	$\ydiagram{1,1,1}\, \mathrm{d}Y$
$\square\nabla$	$\ydiagram{1}$	$\ydiagram{2,1}\, T^1 + \ydiagram{1,1,1}\, \hat{T}^1$			
$\nabla\nabla\nabla$	$\ydiagram{3}$	$\ydiagram{2,1}\, T^2$			
∇W	$\ydiagram{3,2}$	$\ydiagram{2,1}\, T^7$			
W	$\ydiagram{2,2}$		$\ydiagram{2,1}\, T^6$	$\ydiagram{2,1}\, T^{3,4} + \ydiagram{1,1,1}\, \hat{T}^2$	$\ydiagram{2,1}\, T^5 + \ydiagram{1,1,1}\, \hat{T}^3$
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$\Box$	$\mathbb{R}$	$\yng(1,1)\, Q^1$	$\yng(2,1)\, P^1$	$\yng(1,1,1)\, \hat{P}^1$
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tot.ord. $= 3$:

			$(\yng(1)\, \yng(1,1))\nabla_a Y_{bc}$		
		$\yng(1,1)\, Y_{ab}$	$\yng(1)\, \delta Y$	$\yng(2,1)\, \mathrm{CYK}[Y]$	$\yng(1,1,1)\, \mathrm{d} Y$
$\Box\nabla$	$\yng(1)$	$\yng(2,1)\, T^1 + \yng(1,1,1)\, \hat{T}^1$			
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Generalized Normal Hyperbolicity

- Normally hyperbolic operators: 2nd order $P[\psi] = \Box\psi + \text{l.o.t}$
  Well studied IVP (e.g., Bär-Ginoux-Pfäffle, EMS 2007).
- Higher order normally hyperbolic: $P[\psi] = \Box^k\psi + \text{l.o.t}$
  IVP reduces to normally hyperbolic case.
- Generalized normally hyperbolic: $\exists P' : P' \circ P[\psi] = \Box^k\psi + \text{l.o.t}$
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- Example:

$$Q[Y] = s\,\Box Y - (s-t)\,\mathrm{d}\delta Y + \text{l.o.t}, \quad Q'[Y] = s'\,\Box Y - (s'-t')\,\mathrm{d}\delta Y,$$

$$Q' \circ Q[Y] = \Box^2 Y + \text{l.o.t} \iff s' = \frac{1}{s}, t' = \frac{1}{t}$$

Q is generalized normally hyperbolic iff $s \neq 0$, $t \neq 0$.

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Main Result

Solutions of $P \circ E - \sigma \circ \rho \circ E = 0 \leftrightarrow$ left kernel of this numerical matrix:

$$
\left[\begin{array}{c|c}
P^{1\text{–}6} \circ \mathrm{CYK} & \\ \hline
P^{7,8} \circ \mathrm{d} & \\ \hline
 & \hat{P}^{1\text{–}4} \circ \mathrm{d} \\ \hline
 & \hat{P}^{5,6} \circ \mathrm{CYK} \\ \hline
\sigma^1 \circ \rho^1 \circ \mathrm{CYK} & \hat{\sigma}^1 \circ \rho^1 \circ \mathrm{CYK} \\
\sigma^1 \circ \rho^2 \circ \mathrm{d} & \hat{\sigma}^1 \circ \rho^2 \circ \mathrm{d}
\end{array}\right]
=
\left[\begin{array}{cccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 3\frac{(n-2)}{(n-1)} & 0 & 0 & 0 & -\frac{9}{2}(n-2) & 0 & 0 & & & & \\
-2 & -3\frac{(n-4)}{(n-1)} & 0 & 2 & 3 & -\frac{3}{2}(n+8) & 3 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\ \hline
2 & -6 & 1 & -2 & 0 & -3(n-1) & 6 & -\frac{12}{n-2} & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\ \hline
 & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & -2 & 0 & -2 & \frac{12}{n-2} \\
 & & & & & & & & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 1 \\ \hline
 & & & & & & & & 1 & -1 & \frac{1}{2} & 0 \\
 & & & & & & & & 0 & 1 & 0 & 0 \\ \hline
2 & 3\frac{(n-4)}{(n-1)} & 0 & -2 & -3 & \frac{3}{2}(n+8) & -3 & 0 & 2 & -2 & 1 & 0 \\
2 & -6 & 1 & -2 & 0 & -3(n-1) & 6 & -\frac{12}{n-2} & 2 & 0 & 2 & -\frac{12}{n-2}
\end{array}\right]
\left[\begin{array}{c|c}
T^{1\text{–}8} & \\ \hline
 & \hat{T}^{1\text{–}4}
\end{array}\right]
$$

Theorem (IK & A. Garcá Parrado, 2019)

(a) There exists a 5-parameter cCYK propagation identity for $n > 3$.
(b) Generic P, Q are generalized normally hyperbolic for $n > 4$.
(c) It is exhaustive for $n > 4$ ($n \neq 6$).

cCYK Propagation Identity

- Propagation identities: $P[E[\phi]] = \sigma[Q[\phi]]$, $Q[\phi] = \rho[E[\phi]]$.
- **Goal:** exhaustive search for P, Q, ρ, σ
  - covariant in ∇_a, g_{ab}, R_{abcd}, $\nabla_e R_{abcd}$, ...
  - 1st order σ, ρ; 2nd (total) order P, Q
  - hyperbolic P, Q

  Total order: $|g_{ab}| = 0$, $|\nabla| = 1$, $|R_{abcd}| = |\Lambda| = 2$, respected by composition.
- Use representation theory of $O(g)$ to parametrize ansatz.

$$\begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} P^{1,2,3,4,5,6} & P^{7,8} \\ \hat{P}^{5,6} & \hat{P}^{1,2,3,4} \end{bmatrix} \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

$$- \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \sigma^1 \\ \hat{\sigma}^1 \end{bmatrix} \square\, Q^{1,2,3,4} \square = \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} T^{1,2,3,4,5,6,7,8} \\ \hat{T}^{1,2,3,4} \end{bmatrix} \square,$$

$$\square\, Q^{1,2,3,4} \square = \square \begin{bmatrix} \rho^1 & \rho^2 \end{bmatrix} \begin{matrix}\boxplus\\ \boxminus\end{matrix}\begin{bmatrix} \mathrm{CYK} \\ \mathrm{d} \end{bmatrix} \square$$

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\hline
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\hline
 & \hat{P}^{1\text{–}4} \circ \mathrm{d} \\
\hline
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\hline
\sigma^1 \circ \rho^1 \circ \mathrm{CYK} & \hat{\sigma}^1 \circ \rho^1 \circ \mathrm{CYK} \\
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=
\left[\begin{array}{cccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 3\frac{(n-2)}{(n-1)} & 0 & 0 & 0 & -\frac{9}{2}(n-2) & 0 & 0 & & & & \\
-2 & -3\frac{(n-4)}{(n-1)} & 0 & 2 & 3 & -\frac{3}{2}(n+8) & 3 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
\hline
2 & -6 & 1 & -2 & 0 & -3(n-1) & 6 & -\frac{12}{n-2} & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
\hline
 & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & -2 & 0 & -2 & \frac{12}{n-2} \\
 & & & & & & & & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 1 \\
\hline
 & & & & & & & & 1 & -1 & \frac{1}{2} & 0 \\
 & & & & & & & & 0 & 1 & 0 & 0 \\
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\end{array}\right]
\left[\begin{array}{c|c}
T^{1\text{–}8} & \\
\hline
 & \hat{T}^{1\text{–}4}
\end{array}\right]
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(a) There exists a 5-parameter cCYK propagation identity for $n > 3$.
(b) Generic P, Q are generalized normally hyperbolic for $n > 4$.
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Main Result

Solutions of $P \circ E - \sigma \circ \rho \circ E = 0 \leftrightarrow$ left kernel of this numerical matrix:

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P^{7,8} \circ \mathrm{d} & \\ \hline
 & \hat{P}^{1-4} \circ \mathrm{d} \\ \hline
 & \hat{P}^{5,6} \circ \mathrm{CYK} \\ \hline
\sigma^1 \circ \rho^1 \circ \mathrm{CYK} & \hat{\sigma}^1 \circ \rho^1 \circ \mathrm{CYK} \\
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 & & & & & & & & 1 & 0 & 0 & 0 \\
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Discussion

- $O(g)$ representation theory helps with exhaustive search for covariant propagation identities of geometric PDEs.
- We have succeeded for the cCYK equation in dimension $n > 4$, with a propagation identity of total order 2.
- In $n = 4$, a special trick produces a 4th order propagation identity for non-closed CYK.
- Can this trick be systematically extend to $n > 4$?
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- For which equations $E[\phi] = 0$ does there not exist a (covariant) propagation identity?

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Thank you for your attention!