BGG complexes in singular blocks of category $\mathcal{O}$

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Introduction

- BGG complexes: Bernstein-Gelfand-Gelfand '75, Lepowsky '77, Rocha-Charidi '80
- Invariant differential operators $\longleftrightarrow$ homomorphisms of (gen.) Verma modules
- Čap-Slovak-Souček '01 - geometric construction on curved spaces.
- Penrose transform: singular blocks of some maximal parabolic cases (Krump, Salač, Souček, Pandžić, Husadžić, M.) (cf. talks of L. Krump and V. Souček)
- This talk concerns singular blocks for the Borel case (minimal parabolic)

Category $\mathcal{O}$

- $\mathfrak{g}$ = semisimple Lie algebra (fin.dim., over $\mathbb{C}$)
- $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ ($\Leftrightarrow$ choosing $\mathfrak{h}$ and $R^+(\mathfrak{g}, \mathfrak{h})$)
- Category $\mathcal{O}$: Finitely generated $U(\mathfrak{g})$-modules, $\mathfrak{h}$-semisimple, locally-finite for $\mathfrak{n}^+$
- For $\lambda \in \mathfrak{h}^*$ we have:
  - the Verma module $\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$
  - the unique irreducible quotient $\Delta(\lambda) \twoheadrightarrow L(\lambda)$
  - the indecomposable projective cover $P(\lambda) \twoheadrightarrow \Delta(\lambda) \twoheadrightarrow L(\lambda)$

- $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+$, $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{u}^+$ gives the parabolic version of category $\mathcal{O}$, denoted by $\mathcal{O}^{\mathfrak{p}}$

Weyl group

A lot of structure of category $\mathcal{O}$ reduces to the combinatorics of the Weyl group.

- For each $\alpha \in R$ we have the reflection s_α on $\mathfrak{h}^*$ with respect to the hyperplane $\alpha^\perp$.
- The Weyl group W is the group generated by all s_α.
- Fact: It is enough to take only simple reflections.
- This gives us the length function $\mathtt{l}$, and the Bruhat order $\leq$ (subword order) on W.
- Write $u \to v$ if $u \leq v$ and $\mathtt{l}(v) = \mathtt{l}(u) + 1$ (so we consider W as a directed graph)

Blocks

- $\mathcal{O} \cong \bigoplus_{\chi\colon Z(U(\mathfrak{g}))\to\mathbb{C}} \mathcal{O}_\chi$
- $\{\chi\colon Z(U(\mathfrak{g})) \to \mathbb{C}\} \cong \mathfrak{h}^*/W$ (Harish-Chandra)
- Fact: $\Delta(\lambda)$, $L(\lambda)$, $P(\lambda)$ have χ corresponding to $\lambda + \rho$
- If $\lambda + \rho$ is dominant, denote the corresponding block by $\mathcal{O}_\lambda$
- Denote $w \cdot \lambda := w(\lambda + \rho) - \rho$
- $\mathcal{O} \cong \bigoplus_{\lambda+\rho \text{ dominant}} \mathcal{O}_\lambda$
  ($\mathcal{O}_\lambda$ contains $\Delta(w \cdot \lambda)$, $L(w \cdot \lambda)$, $P(w \cdot \lambda)$, $w \in W$)
- We always assume λ integral (i.e. $\frac{2\langle\lambda+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for $\alpha \in R^+$) (WLOG, Soergel)

Regular blocks

Assume that $\langle \lambda + \rho, \alpha \rangle > 0$ for all $\alpha \in R^+$

- There exists $\Delta(w \cdot \lambda) \to \Delta(v \cdot \lambda) \Leftrightarrow v \leq w$ (Verma, BGG) (If $\exists$, the map is unique up to scalar and injective)
- There is a choice of scalars for which

$$\ldots \to \bigoplus_{\mathtt{l}(x)=i+1} \Delta(x \cdot \lambda) \to \bigoplus_{\mathtt{l}(x)=i} \Delta(x \cdot \lambda) \to \ldots \to \Delta(\lambda) \to L(\lambda) \to 0$$

  is an exact sequence (BGG resolution)
- Analogous statement holds in $\mathcal{O}^{\mathfrak{p}}$ (Lepowsky, Rocha-Caridi)
- Fix $w \in W$. There is a choice of scalars for which

$$\ldots \to \bigoplus_{\substack{x \geq w \\ \mathtt{l}(w,x)=i+1}} \Delta(x \cdot \lambda) \to \bigoplus_{\substack{x \geq w \\ \mathtt{l}(w,x)=i}} \Delta(x \cdot \lambda) \to \ldots \to \Delta(w \cdot \lambda) \to L(w \cdot \lambda) \to 0$$

  is a chain complex. It is exact iff certain Kazhdan-Lusztig polynomials vanish (Boe-Hunziker, Enright)

Singular blocks

Assume $\langle \lambda + \rho, \alpha \rangle \geq 0$ for $\alpha \in R^+$

- $S := \{\alpha \text{ simple} : \langle \lambda + \rho, \alpha \rangle = 0\}$
- $W_\lambda :=$ the subgroup generated by $s_\alpha, \alpha \in S$
- Fact: $W_\lambda = \mathrm{Stab}_W(\lambda + \rho)$
- So, $L/\Delta/P$ in $\mathcal{O}_\lambda$ are parametrized by W/W_λ (and homomorphisms by the induced order on cosets)
- $\tilde{W}^\lambda :=$ set of the longest representatives of cosets W/W_λ

Translation to the wall

$\exists$ exact functor $T = T_0^\lambda \colon \mathcal{O}_0 \to \mathcal{O}_\lambda$ s.t. (BG, Jantzen):

- $T(\Delta(w \cdot 0)) = \Delta(w \cdot \lambda)$
- $T(L(w \cdot 0)) = \begin{cases} L(w \cdot \lambda), & \text{if } w \in \tilde{W}^\lambda \\ 0, & \text{otherwise} \end{cases}$

Fix $w \in \tilde{W}^\lambda$. We want to resolve $L(w \cdot \lambda) \in \mathcal{O}_\lambda$ by direct sums of Vermas.

- Start with the Boe-Hunziker complex in $\mathcal{O}_0$

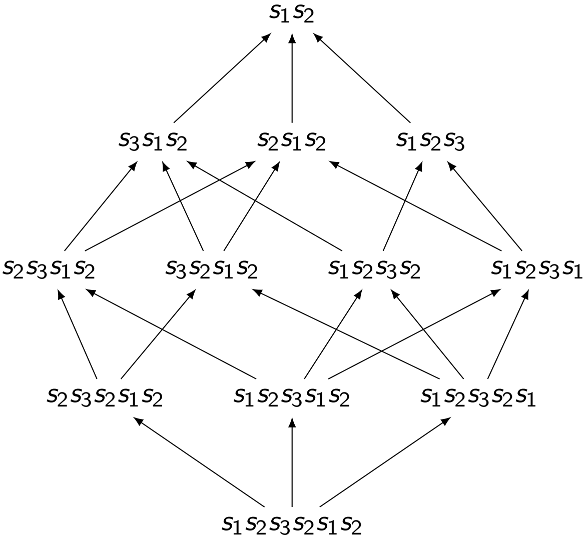
$$\mathbf{\Delta}_w := \ldots \to \bigoplus_{\substack{x \geq w \\ \mathtt{l}(x) = i}} \Delta(x \cdot 0) \to \ldots \to L(w \cdot 0) \to 0$$

- If $\mathbf{\Delta}_w$ is exact, then $T(\mathbf{\Delta}_w)$ provides an answer.
- Problems: $T(\mathbf{\Delta}_w)$ does not respect Bruhat order? What if $\mathbf{\Delta}_w$ is not exact? Uniqueness?

Example

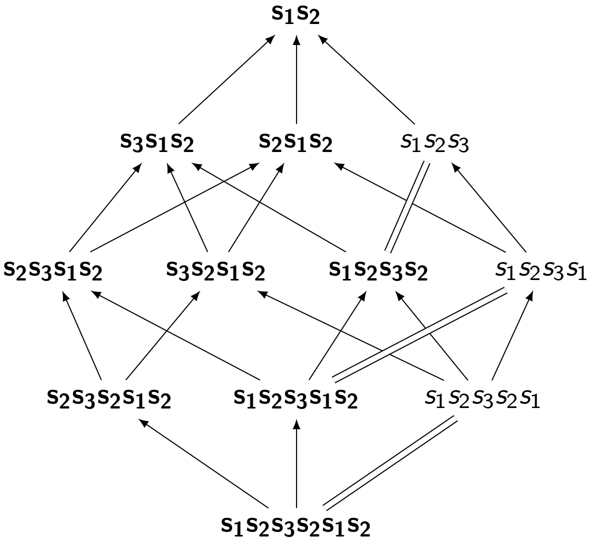
Type $A_3 = \circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$, $\lambda + \rho = (2,1,1,0)$, $S = \{\alpha_2\}$, $w = s_1 s_2$.

$\mathbf{\Delta}_w =$



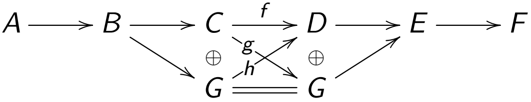
Example

$T(\boldsymbol{\Delta}_w) =$



Lemma

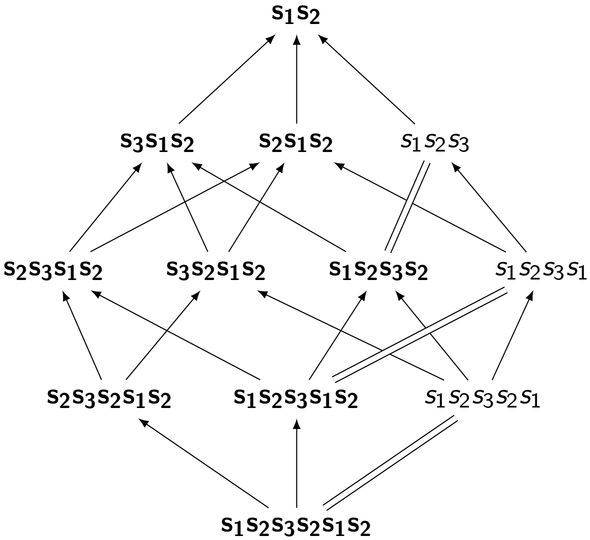
If a sequence of modules



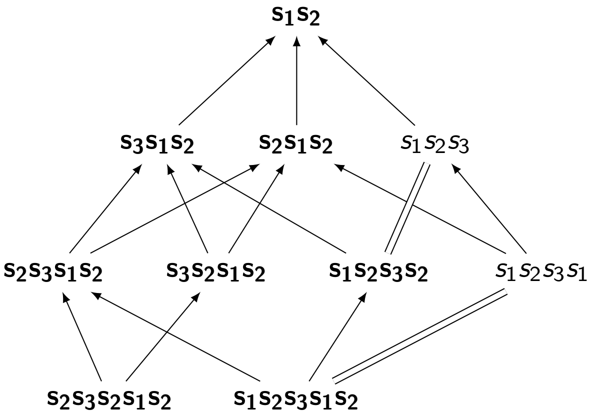
is a complex (resp. exact), then so is

$$A \longrightarrow B \longrightarrow C \xrightarrow{f-hg} D \longrightarrow E \longrightarrow F\ .$$

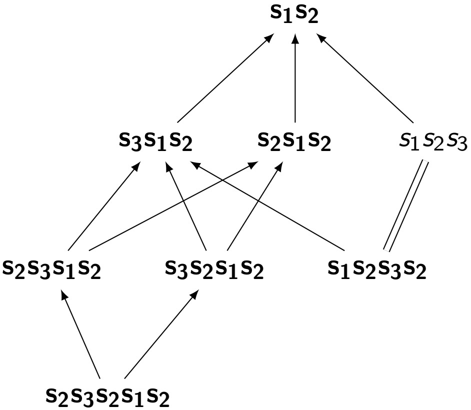
Example (continued)



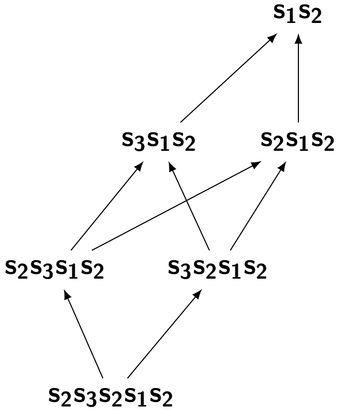
Example (continued)



Example (continued)

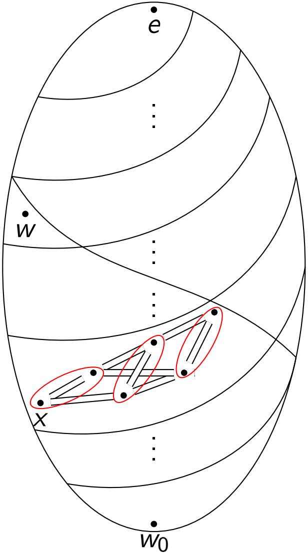


Example (continued)



The remaining homomorphisms are non-zero, and do not depend on the order of "cutting off" the isomorphisms.

In general



$(W, \leq)$, cosets W/W_λ

Fix $w \in \tilde{W}^\lambda$ and consider $[w, w_0]$
(= shape of $\mathbf{\Delta}_w$)

Choose $x \in \tilde{W}^\lambda$, $x \geq w$

Lemma
$xW_\lambda \cap [w, w_0]$ contains
a unique minimal element

$\Rightarrow$ The intersection is an interval

$\Rightarrow$ If not singleton, it can be partitioned into subsets of the form $\{\bullet = \bullet\}$

$\Rightarrow$ If not singleton, it can be completely "cleaned up"

Singular BGG complexes

Denote $X_w^i := \{x \in \tilde{W}^\lambda \colon \mathtt{l}(x) = \mathtt{l}(w) + i,\ |xW_\lambda \cap [w, w_0]| = 1\}$

Theorem

Let $\lambda + \rho$ be dominant and integral and $w \in \tilde{W}^\lambda$.

(a) *One can choose non-zero coefficients so that the singular BGG complex below is a cochain complex:*

$$\ldots \to \bigoplus_{x \in X_w^{i+1}} \Delta(x \cdot \lambda) \to \bigoplus_{x \in X_w^i} \Delta(x \cdot \lambda) \to \ldots \to \Delta(w \cdot \lambda) \to L(w \cdot \lambda) \to 0.$$

(b) *If the regular BGG complex of $L(w \cdot 0)$ is exact, then so is the singular BGG complex of $L(w \cdot \lambda)$.*

$$\mu(w, x) = \begin{cases} 0, & \text{if } \exists z \notin \tilde{W}^\lambda \text{ s.t. } w < z < x; \\ (-1)^{l(x) - l(w)}, & \text{otherwise.} \end{cases}$$

Proposition

$x \in X_w$ if and only if $\mu(x, w) \neq 0$.

Exactness

Theorem

For $w \in \tilde{W}^\lambda$, the following statements are equivalent:

(a) *The singular BGG complex of $L(w \cdot \lambda)$ is exact.*

(b) *For all $i \geq 0$, we have*

$$H^i\left(\mathfrak{n}^+, L(w \cdot \lambda)\right) = \bigoplus_{x \in X_w^i} \mathbb{C}_{x \cdot \lambda},$$

(c) *For all $i \geq 0$ and $x \in \tilde{W}^\lambda$, we have*

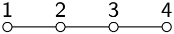
$$\dim \operatorname{Ext}^i_{\mathcal{O}}\left(\Delta(x \cdot \lambda), L(w \cdot \lambda)\right) = \begin{cases} 1 & : x \in X_w^i \\ 0 & : \text{otherwise.} \end{cases}$$

(d) *For all $x \in \tilde{W}^\lambda$ with $x \geq w$, we have $P^{w_0 \cdot \lambda}_{xw_0, ww_0}(q) = |\mu(w, x)|$.*

(Non-)Kostant modules in low rank

The calculations were performed in SageMath, version 8.4.

Type A_4	
Block (S)	Non-Kostant modules (w)
$\emptyset$	(3), (2), (32), (23), (42), (31), (41), (342), (232), (231), (423), (421), (312), (341), (431), (412), (2321), (2342), (4232), (4231), (3412), (1232), (3431), (4121), (42321), (23431), (34121), (12342), (34312), (41231), (343121), (123431)
$\{1\}$	(31), (41), (341), (431), (421), (2321), (3431), (4231), (4121), (42321), (23431), (41231), (343121), (123431)
$\{2\}$	(2), (32), (42), (342), (232), (312), (412), (2342), (4232), (4121), (3412), (1232), (34121), (12342), (34312), (343121)
$\{1,2\}$	(4121), (343121)
$\{1,3\}$	(31), (431), (2321), (3431), (4231), (23431), (42321), (41231), (123431)
$\{1,4\}$	(41), (341), (421), (3431), (4121), (343121), (123431)
$\{2,3\}$	(232), (4232), (1232)
$\{1,2,4\}$	(4121), (343121)

(up to the non-trivial isomorphism of 

)

Filtrations of projective modules

- Koszul duality: $\mathcal{O}_\lambda \longleftrightarrow \mathcal{O}_0^\lambda$ = regular block of the parabolic cat. (Beilinson-Ginzburg-Soergel)
- Objects in $\mathcal{O}_0^\lambda$ are parametrized by ${}^\lambda W$:= the shortest representatives of the cosets $W_\lambda \backslash W$
- $\exists$ bijection $\tilde{W}^\lambda \xrightarrow{\sim} {}^\lambda W$, given by $w \mapsto w^{-1} w_0$
- $\dim \mathrm{Ext}(\Delta, L) \longleftrightarrow (P : \Delta)$

Proposition

The singular BGG complex of $L(w \cdot \lambda) \in \mathcal{O}_\lambda$ is exact if and only if $P^\lambda(w^{-1} w_0 \cdot 0) \in \mathcal{O}_0^\lambda$ has a multiplicity free Verma flag with parameters in $(X_w)^{-1} w_0$.

Proposition

The (Boe-Hunziker) BGG complex of $L(w^{-1} w_0 \cdot 0) \in \mathcal{O}_0^\lambda$ is exact if and only if $P(w \cdot \lambda) \in \mathcal{O}_\lambda$ has a multiplicity free Verma flag. (Stroppel: if and only if $\mathrm{End}(P(w \cdot \lambda))$ is commutative.)

Uniqueness of BGG resolution

- $\mathcal{P}_\mu :=$ the minimal projective resolution of $L(\mu)$ in $\mathcal{O}_\lambda$
- $T(\mu) :=$ the indecomposable tilting module (self-dual, has a Verma flag) with $\Delta(\mu) \subseteq T(\mu)$
- $\mathcal{T}_\mu :=$ minimal complex of tilting modules $\cong L(\mu)$ in $D^b(\mathcal{O})$ ($\exists$, by Mazorchuk).
- If Δ is a BGG resolution of $L(\mu)$, then $\exists!$ quasi-isomorphisms

$$\begin{array}{ccc} \mathcal{P}_\mu & \xrightarrow{\ \xi\ } & \Delta \\ & \searrow^{\eta} & \downarrow{\varphi} \\ & & \mathcal{T}_\mu \end{array}$$

- Clearly $\Delta \cong \mathrm{Im}(\eta)$. We show that $\mathrm{Im}(\eta)$ does not depend on particular choice of Δ.
- It follows that all BGG resolutions of $L(\mu)$ are isomorphic as complexes to the canonical subcomplex of $\mathcal{T}_\mu$.

Thank you!

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