Quantum Homotopy Algebras and Homological Perturbation Lemma

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Space of formal functions

Modular operad $\mathcal{P}$

- collection $\{\mathcal{P}(C,G) \mid C \text{ a finite set}, G \in \mathbb{Z}, G \geq 0\}$ of dg vector spaces s.t. the **stability condition** is satisfied:

$$2(G-1) + \mathtt{card}(C) > 0$$

- three collections of deg 0 morphisms

$$\{\mathcal{P}(\rho) : \mathcal{P}(C,G) \to \mathcal{P}(D,G) \mid \rho : (C,G) \to (D,G) \text{ bijection}\}$$
$$\{\ {}_a\circ_b : \mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \to \mathcal{P}(C_1 \sqcup C_2, G_1 + G_2)\ \}$$
$$\{\ \circ_{ab} : \mathcal{P}(C \sqcup \{a,b\}, G) \to \mathcal{P}(C, G+1)\ \}$$

These data are required to satisfy axioms for Σ-module, equivariance, and associativity of composition maps.

Space of formal functions on V is a space of invariants under the diagonal action

$$\mathrm{Fun}(\mathcal{P}, V) = \prod_{n \geq 0} (\mathcal{P}\,(\,[n], G) \otimes (V^*)^{\otimes n})^{\Sigma_n}$$

where $[n] = \{1, 2, \ldots n\}$

To compare with the trvial case - for $\mathcal{P}$ a commutative operad

$$\mathrm{Fun}(V) = \prod_{g \geq 0} \prod_{n \geq 0} \mathbb{K}\hbar^g \otimes ((V^*)^{\otimes n})^{\Sigma_n} = \prod_{g \geq 0} \prod_{n \geq 0} \mathbb{K}\hbar^g \otimes (V^*)^{\odot n}$$

- necessary to introduce a *weight grading*: $w = 2g + n \geq 1$

Special deformation retract

A pair (V, d) and (W, e) of dg vector spaces, a pair p and i of their morphisms and a homotopy $k : V \to V$ between ip and 1_V

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (W, e)$$

that satisfy the following

$$\begin{aligned}
&d^2 = 0, \quad e^2 = 0, \quad |d| = |e| = 1, && \textit{... differentials}\\
&pd = ep, \quad ie = di, \quad |p| = |i| = 0, && \textit{... chain maps}\\
&ip - 1_V = kd + dk, \quad |k| = -1, && \textit{... homotopy map}\\
&pi - 1_W = 0, && \textit{... deformation retract}\\
&pk = 0, \quad ki = 0, \quad k^2 = 0 && \textit{... special deformation retract.}
\end{aligned}$$

From *(Harmonious) Hodge decomposition* $V \cong H(V) \oplus \mathrm{Im}(d) \oplus W$

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(V), 0)$$

We use the *tensor trick* to get SDR on our formal functions:

$$K \circlearrowright ((\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n}, D) \underset{I}{\overset{P}{\rightleftarrows}} ((\mathcal{P}(n, G) \otimes H(V^*)^{\otimes n})^{\Sigma_n}, 0)$$

$$D = \sum_{n \geq 1} \sum_{i=1}^{n} 1_{\mathcal{P}} \otimes (1^{\otimes i-1} \otimes d^* \otimes 1^{\otimes n-i})$$

$$I = \sum_{n \geq 1} 1_{\mathcal{P}} \otimes (p^*)^{\otimes n} \qquad P = \sum_{n \geq 1} 1_{\mathcal{P}} \otimes (i^*)^{\otimes n}$$

$$K = \sum_{n \geq 1} \sum_{\sigma \in \Sigma_n} \sum_{i=1}^{n} \frac{\sigma}{n!} 1_{\mathcal{P}} \otimes (1^{\otimes i-1} \otimes k^* \otimes (p^* i^*)^{\otimes n-i}).$$

Connected sum

Collections of degree 0 chain maps

$$\begin{aligned} \# : \mathcal{P}(C,G)\otimes\mathcal{P}(C',G') &\to \mathcal{P}(C\sqcup C', G+G'+1) \\ \# : \mathcal{P}(C,G) &\to \mathcal{P}(C, G+2) \end{aligned}$$

- $\#(1\otimes\#) = \#(\#\otimes 1), \qquad \#\tau = \#, \qquad (\sigma\sqcup\sigma')\# = \#(\sigma\otimes\sigma')$
- As maps $\mathcal{P}(C,G)\to\mathcal{P}(C-\{i,j\}, G+3)$: $\circ_{ij}\# = \#\circ_{ij}$
- As maps $\mathcal{P}(C,G)\otimes\mathcal{P}(C',G')\to\mathcal{P}(C\sqcup C'-\{i,j\}, G+G'+2)$,

$$\circ_{ij}\# = \begin{cases} \#(\circ_{ij}\otimes 1) & \dots\ i,j\in C \\ \#\,{}_i\circ_j & \dots\ i\in C, j\in C' \\ \vdots & \end{cases}$$

- As maps $\mathcal{P}(C,G)\otimes\mathcal{P}(C',G')\to\mathcal{P}(C\sqcup C'-\{i,j\}, G+G'+2)$:
  ${}_i\circ_j(\#\otimes 1) = \#\,{}_i\circ_j \qquad \dots\ i\in C, j\in C'$
- As maps
  $\mathcal{P}(C,G)\otimes\mathcal{P}(C',G')\otimes\mathcal{P}(C'',G'')\to\mathcal{P}(C\sqcup C'\sqcup C''-\{i,j\}, G+G'+G''+1)$,

$$ {}_i\circ_j(1\otimes\#) = \begin{cases} \#({}_i\circ_j\otimes 1) & \dots j\in C' \\ \#(1\otimes {}_i\circ_j)(\tau\otimes 1) & \dots j\in C'' \end{cases}$$

Example: **The Quantum Closed modular operad $\mathcal{QC}$**:

$\mathcal{QC}(C, G) := \mathrm{Span}_{\mathbb{K}}\{C^G\}$ ... homeomorphism class of a surface of genus g and set C of punctures in the interior, $G = 2g + \dfrac{\mathtt{card}(C)}{2} - 1$.

$$\begin{aligned} C_1^{G_1} \# \, C_2^{G_2} &= (C_1 \sqcup C_2)^{G_1+G_2+1} \\ \# \, (C^G) &= C^{G+2} \end{aligned}$$

Example: **The Quantum Open modular operad $\mathcal{QO}$**:

$$\mathcal{QO}(O, G) := \mathrm{Span}_{\mathbb{K}} \left\{ \{\mathbf{o}_1, \ldots \mathbf{o}_b\}^g \mid b, g \in \mathbb{N}_0, \mathbf{o}_i \text{ cycle in } O, \bigsqcup_{i=1}^{b} \mathbf{o}_i = O \right\}$$

$$G = 2g + b - 1$$

$$\begin{aligned} \{\mathbf{o}_1, \ldots \mathbf{o}_{b_1}\}^{g_1} \# \{\mathbf{o}'_1, \ldots \mathbf{o}'_{b_2}\}^{g_2} &= \{\mathbf{o}_1, \ldots \mathbf{o}_{b_1}, \mathbf{o}'_1 \ldots \mathbf{o}'_{b_2}\}^{g_1+g_2} \\ \# (\{\mathbf{o}_1, \ldots \mathbf{o}_b\}^g) &= \{\mathbf{o}_1, \ldots \mathbf{o}_b\}^{g+1} \end{aligned}$$

Endomorphism odd modular operad

(V, d) with odd symplectic form ω, $d(\omega) = 0$.

$$\mathcal{E}_V(C, G) = \bigotimes_C V^*$$

with structure maps of deg 1

$$_i\bullet_j(f \otimes g) := \sum_k (-1)^{|f|+|g||b_k|} f(\cdots \otimes \underbrace{a_k}_{i\text{-th}} \otimes \cdots) \cdot g(\cdots \otimes \underbrace{b_k}_{j\text{-th}} \otimes \cdots)$$

$$\bullet_{ij}(f) := \sum_k (-1)^{|f|} f(\cdots \otimes \underbrace{a_k}_{i\text{-th}} \otimes \cdots \otimes \underbrace{b_k}_{j\text{-th}} \otimes \cdots)$$

where $b_k = \sum_l (-1)^{|a_l|} \omega^{kl} a_l$

Connected sum is simply:

$$f \# g = (f) \cdot (g)$$

$$\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G) = (\mathcal{P}(n,G)\otimes\mathcal{Q}(n,G))^{\Sigma_n}$$

$$\mathrm{Con}(\mathcal{P},\mathcal{Q}) = \prod_{n,G}\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)$$

$$d = d_{\mathcal{P}}\otimes 1 - 1\otimes d_{\mathcal{Q}} \quad : \quad \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)$$

$$\Delta = (\circ_{ij}\otimes\bullet_{ij})(\theta\otimes\theta) \quad : \quad \mathrm{Con}(\mathcal{P},\mathcal{Q})(n+2,G)\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G+1)$$

$$\{X,Y\} = \frac{(-1)^{|X|}}{2}\sum_{C_1,C_2}(\#\otimes\#)({}_i\circ_j\otimes{}_i\bullet_j)(\theta_1\otimes\theta_2\otimes\theta_1\otimes\theta_2)(1\otimes\tau\otimes 1):$$

$$\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+1,G_1)\otimes\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_2+1,G_2)\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+n_2,G_1+G_2)$$

A **product**

$\bullet : \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1, G_1) \otimes \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_2, G_2) \to \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+n_2, G_1+G_2+1)$

$$\bullet = \sum_{C_1, C_2} (\# \otimes \#)(\theta_1 \otimes \theta_2 \otimes \theta_1 \otimes \theta_2)(1 \otimes \tau \otimes 1)$$

Theorem: $\mathrm{Con}(\mathcal{P},\mathcal{Q})$ with operations $d, \Delta, \{-,-\}$ and $\bullet$ is a **Batalin-Vilkovisky algebra**

1. $\bullet$ is a commutative associative product, i.e. on elements:

$$X \bullet Y = (-1)^{|X|\cdot|Y|} Y \bullet X \qquad \text{and} \qquad (X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$$

2. $\Delta \bullet = \bullet(\Delta \otimes 1) + \bullet(1 \otimes \Delta) + \{-,-\}$
3. $\{-,-\}(1 \otimes \bullet) = \bullet(\{-,-,\} \otimes 1) + \bullet(1 \otimes \{-,-,\})(\tau \otimes 1)$

Quantum Master Equation

In[1], Barannikov observed that every dg operad morphism from Feynman tranform of $\mathcal{P}$ to $\mathcal{Q}$, i.e. $\mathbb{F}ey(\mathcal{P}) \to \mathcal{Q}$, is equivalently given by a degree 0 solution $S \in \mathrm{Con}(\mathcal{P}, \mathcal{Q})$ of the *quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0$$

This is equivalent with condition that e^S is $(d + \Delta)$-closed

[1] S. Barannikov, "Modular operads and Batalin-Vilkovisky geometry", International Mathematics Research Notices, Oxford University Press (OUP), 2006, Vol. 2007,

Homological Perturbation Lemma

Theorem: Consider SDR

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (W, e)$$

and its **perturbation** $\delta : V \to V$, i.e. linear map of deg 1, $(d + \delta)^2 = 0$, which is *small* in the sense:

$$(1 - \delta k)^{-1} = \sum_{i=0}^{\infty} (\delta k)^i$$

Then we have SDR

$$k' \circlearrowright (V, d + \delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (W, e')$$

$$K_1 \circlearrowright (\mathrm{Fun}(\mathcal{P}, V), d + \Delta) \underset{I_1}{\overset{P_1}{\rightleftarrows}} (\mathrm{Fun}(\mathcal{P}, H(V)), E_1)$$

From Harmonious Hodge decomposition $V \cong H(V) \oplus (\mathrm{Im}(d) \oplus W)$

$$\omega = \omega' + \omega'' \qquad \Delta = \Delta' + \Delta''$$

And the perturbed maps

$$\begin{aligned} E_1 &= \Delta' \\ K_1 &= K \sum_{i \geq 0} (\Delta'' K)^i \\ I_1 &= I \\ P_1 &= P \sum_{i \geq 0} (\Delta'' K)^i \end{aligned}$$

The **effective action** $W \in \mathrm{Fun}(\mathcal{P}, V)$

$$e^W = P_1(e^S) = P(1 - \Delta K)^{-1} e^S$$

Since

$$P_1(d + \Delta)(e^S) = E_1 P_1(e^S) = \Delta' P_1(e^S) = 0$$

satisfies **master equation**

$$\Delta'(e^W) = (\Delta' W + \frac{1}{2}\{W, W\}')\, e^W$$

Thank you for your attention!