Cohomology rings of some oriented Grassmann manifolds

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Introduction

Let us denote $G_{n,k}$ the Grassmann manifold of k–dimensional vector subspaces in $\mathbb{R}^n$, i.e. the space $O(n)/(O(k) \times O(n-k))$.
Denote $\widetilde{G}_{n,k}$ the *oriented* Grassmann manifold of *oriented* k–dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n-k))$.
We may suppose that $k \leq n-k$ for both of them.

Introduction

The manifolds $G_{n,k}$ and $\widetilde{G}_{n,k}$ come equipped with their canonical k–plane bundles, which we denote $\gamma_{n,k}$ and $\widetilde{\gamma}_{n,k}$. We will denote $w_i = w_i(\gamma_{n,k})$ and $\widetilde{w}_i = w_i(\widetilde{\gamma}_{n,k})$ the Stiefel-Whitney classes of those vector bundles. Similarly, we will abbreviate $H^j(X;\mathbb{Z}_2)$ to $H^j(X)$.

Introduction

Cohomology ring of $G_{n,k}$

The cohomology ring of the Grassmann manifold $G_{n,k}$ is

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{n,k},$$

where $\mathrm{dim}(w_i) = i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \ldots, \bar{w}_n$, where each $\bar{w}_i$ denotes the i–dimensional component of the formal power series

$$(1+w_1+w_2+\cdots+w_k)^{-1} = 1+(w_1+w_2+\cdots+w_k)+(w_1+w_2+\cdots+w_k)^2+\cdots .$$

Cohomology ring of $\widetilde{G}_{n,2}$

The cohomology ring of $G_{n,k}$ is fully generated by the Stiefel-Whitney classes of its canonical bundle. However the same is not true for the oriented Grassmann manifolds $\widetilde{G}_{n,k}$. As an example, the cohomology ring of $\widetilde{G}_{n,2}$ is as follows.

Theorem

For n odd we have $H^*(\widetilde{G}_{n,2}) \cong \mathbb{Z}_2[\widetilde{w}_2]/(\widetilde{w}_2^{\frac{n-1}{2}}) \otimes \Lambda_{\mathbb{Z}_2}(a_{n-1})$, where $a_{n-1} \in H^{n-1}(\widetilde{G}_{n,2})$ is an anomalous class.
For $n \equiv 0 \pmod 4$ we have $H^*(\widetilde{G}_{n,2}) \cong \mathbb{Z}_2[\widetilde{w}_2]/(\widetilde{w}_2^{\frac{n}{2}}) \otimes \Lambda_{\mathbb{Z}_2}(a_{n-2})$.
For $n \equiv 2 \pmod 4$ the cohomology ring is generated by $\widetilde{w}_2$ and an anomalous class $a_{n-2} \in H^{n-2}(\widetilde{G}_{n,2})$ such that $a_{n-2}^2 = \widetilde{w}_2^{\frac{n-2}{2}} a_{n-2}$ is the generator of the top cohomology group $H^{2(n-2)}(\widetilde{G}_{n,2})$.

Cohomology ring of $\widetilde{G}_{n,2}$

There is a covering projection $p\colon \widetilde{G}_{n,k} \longrightarrow G_{n,k}$ which induces homomorphism $p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$ that maps each class w_i to $\widetilde{w}_i$.
Note that $H^1(\widetilde{G}_{n,k}) = 0$ and $\widetilde{w}_1 = 0$.
Denoting g_i the reduction of the polynomial $\bar{w}_i$ from the description of $H^*(G_{n,k})$ we see that $H^*(\widetilde{G}_{n,k})$ contains $\mathbb{Z}_2[\widetilde{w}_2, \ldots, \widetilde{w}_k]/J_{n,k}$, where $J_{n,k} = (g_{n-k+1}, \ldots, g_n)$.

Characteristic rank

The notion of characteristic rank quantifies the degree up to which the cohomology ring is generated by the Stiefel-Whitney classes.

Definition

Let X be a connected, finite CW–complex and ξ a vector bundle over X. The *characteristic rank* of the vector bundle ξ, denoted $\mathrm{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

Thus for $\widetilde{G}_{n,k}$ the degree in which the first anomalous class a_i appears is given by $i = \mathrm{charrank}(\widetilde{\gamma}_{n,k}) + 1$.

Cohomology ring of $\widetilde{G}_{n,3}$

Computing the characteristic rank will determine the first occurrence of an anomalous class in $H^*(X)$, but it does not determine the degrees of any other that might exist. In $H^*(\widetilde{G}_{n,3})$ we already encounter multiple different anomalous generators of the cohomology ring.

- For $n = 2^t$ there is one anomalous generator in degree $2^t - 1$.
- For $n = 2^t - 1, 2^t - 2, 2^t - 3$ there is one anomalous generator in degree $2^t - 4$.
- For $2^{t-1} < n \leq 2^t - 4$ there is one anomalous generator in degree $2^t - 4$ and one anomalous generator in degree $3n - 2^t - 1$.

Cohomology ring of $\widetilde{G}_{n,3}$

$H^*(\widetilde{G}_{6,3})$	$H^*(\widetilde{G}_{7,3})$	$H^*(\widetilde{G}_{8,3})$	$H^*(\widetilde{G}_{9,3})$	$H^*(\widetilde{G}_{10,3})$
a_4	a_4			
		a_7		
			a_{10}	
			a_{12}	a_{12}
				a_{13}

Cohomology ring of $\widetilde{G}_{n,3}$

Note that since dimension of $\widetilde{G}_{n,3}$ is $3n-9$, the Poincaré dual to anomalous generator a_{3n-2^t-1} is a class in degree 2^t-8. That is, there exists $v_{2^t-8} \in H^{2^t-8}(\widetilde{G}_{n,3})$ such that $a_{3n-2^t-1}v_{2^t-8} \neq 0$.
Moreover v_{2^t-8} is always from the "characteristic" part $p^*(H^{2^t-8}(G_{n,3}))$ and it appears to be "stable". For example, the Poincaré dual to a_7, a_{10}, a_{13} is $\widetilde{w}_2\widetilde{w}_3^2$.

Generators of $H^j(\widetilde{G}_{8,4})$

Theorem

We have the following generators of $H^j(\widetilde{G}_{8,4})$.

j	gen.	j	gen.
0	$\widetilde{w}_0$	9	$a_4\widetilde{w}_2\widetilde{w}_3,\ \widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
1	—	10	$a_4\widetilde{w}_2^3,\ a_4\widetilde{w}_2\widetilde{w}_4,\ \widetilde{w}_2^3\widetilde{w}_4$
2	$\widetilde{w}_2$	11	$a_4\widetilde{w}_3\widetilde{w}_4$
3	$\widetilde{w}_3$	12	$a_4\widetilde{w}_2^4,\ a_4\widetilde{w}_2^2\widetilde{w}_4,\ \widetilde{w}_2^4\widetilde{w}_4$
4	$a_4,\ \widetilde{w}_2^2,\ \widetilde{w}_4$	13	$a_4\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
5	$\widetilde{w}_2\widetilde{w}_3$	14	$a_4\widetilde{w}_2^3\widetilde{w}_4$
6	$a_4\widetilde{w}_2,\ \widetilde{w}_2^3,\ \widetilde{w}_2\widetilde{w}_4$	15	—
7	$a_4\widetilde{w}_3,\ \widetilde{w}_3\widetilde{w}_4$	16	$a_4\widetilde{w}_2^4\widetilde{w}_4$
8	$a_4\widetilde{w}_2^2,\ a_4\widetilde{w}_4,\ \widetilde{w}_2^4,\ \widetilde{w}_2^2\widetilde{w}_4$		

Generators of $H^j(\widetilde{G}_{9,4})$

Theorem

0	$\widetilde{w}_0$	11	$a_8\,\widetilde{w}_3$
1	—	12	$a_8\,\widetilde{w}_2^2,\ a_8\,\widetilde{w}_4,\ \widetilde{w}_2^4\,\widetilde{w}_4$
2	$\widetilde{w}_2$	13	$a_8\,\widetilde{w}_2\,\widetilde{w}_3$
3	$\widetilde{w}_3$	14	$a_8\,\widetilde{w}_2^3,\ a_8\,\widetilde{w}_2\,\widetilde{w}_4$
4	$\widetilde{w}_2^2,\ \widetilde{w}_4$	15	$a_8\,\widetilde{w}_3\,\widetilde{w}_4$
5	$\widetilde{w}_2\,\widetilde{w}_3$	16	$a_8\,\widetilde{w}_2^4,\ a_8\,\widetilde{w}_2^2\,\widetilde{w}_4$
6	$\widetilde{w}_2^3,\ \widetilde{w}_2\,\widetilde{w}_4$	17	$a_8\,\widetilde{w}_2\,\widetilde{w}_3\,\widetilde{w}_4$
7	$\widetilde{w}_3\,\widetilde{w}_4$	18	$a_8\,\widetilde{w}_2^3\,\widetilde{w}_4$
8	$a_8,\ \widetilde{w}_2^4,\ \widetilde{w}_2^2\,\widetilde{w}_4$	19	—
9	$\widetilde{w}_2\,\widetilde{w}_3\,\widetilde{w}_4$	20	$a_8\,\widetilde{w}_2^4\,\widetilde{w}_4$
10	$a_8\,\widetilde{w}_2,\ \widetilde{w}_2^3\,\widetilde{w}_4$		

$H^*(\widetilde{G}_{10,4})$ and $H^*(\widetilde{G}_{11,4})$

In $H^*(\widetilde{G}_{10,4})$ there are two anomalous generators a_{12} and b_{12} of degree 12. The dual to a_{12} is $\widetilde{w}_2^4\widetilde{w}_4$ and the dual to b_{12} is $\widetilde{w}_2^6$.

In $H^*(\widetilde{G}_{11,4})$ there are two anomalos generators a_{12} and a_{16}. The latter can be chosen such that $a_{16}\widetilde{w}_2^4\widetilde{w}_4 \neq 0$, $a_{16}\widetilde{w}_2^6 = 0$ and $a_{16}\widetilde{w}_2^3\widetilde{w}_3^2 = 0$.

Conclusion

We may formulate some conjectures about $H^*(\widetilde{G}_{n,4})$.
We observe there is one anomalous generator a_{2^t} in $H^{2^t}(\widetilde{G}_{2^t+1,4})$ for $t=3$ reflecting the general case for $H^*(\widetilde{G}_{2^t,3})$.
It appears that for $2^t+1 < n \leq 2^{t+1}-4$ there are at least two anomalous generators $a_{4n-3\cdot 2^t-4} \in H^{4n-3\cdot 2^t-4}(\widetilde{G}_{n,4})$ and $a_{2^{t+1}-4} \in H^{2^{t+1}-4}(\widetilde{G}_{n,4})$.
Note that previously mentioned a_{2^t} can be thought of as also being of the form $a_{4n-3\cdot 2^t-4}$ for $n = 2^t+1$.
From observing that the Poincaré dual to those $a_{4n-3\cdot 2^{t-1}-4}$ in our examples for $n = 9, 10, 11$ was always of the form $\widetilde{w}_2^4\widetilde{w}_4$, we may reasonably anticipate these duals will exhibit some kind of stability in general.

Conclusion

Thank you.