# Non-holonomic equations for the normal extremals in geometric control theory

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**following the joint paper with A. Rod Gover, arXiv:1909.06592**

January 12, 2020

Srní, Czech Republic

# Subriemannian geometry

**Definition**

Subriemannian geometry $(M, D, S)$ on a manifold $M$ is given by a distribution $D$, and (positive definite) metric $h$ on $D$.

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Sheaf $\mathcal{D}^{-1} = \mathcal{D}$ of vector fields valued in $D$ generates the filtration by sheafs

$$\mathcal{D}^j = \{[X, Y], X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \dots .$$

We say that $D$ is a bracket generating distribution if for some $k$, $\mathcal{D}^k$ is the sheaf of all vector fields on $M$.

Bracket generating distribution $D$ defines the filtration of subspaces

$$T_xM = D_x^k \supset \cdots \supset D_x^{-1}$$

at each point $x \in M$.
The associated graded tangent space

$$\mathrm{gr}\, T_xM = T_xM/D_x^{k+1} \oplus \cdots \oplus D_x^{-1}$$

comes equipped with the structure of a nilpotent Lie algebra.

The sub-Riemannian metric can be viewed as $h : T^*M \to TM$ with the image $D$.
There are the equivalent short exact sequences:

$$0 \to K \to T^*M \xrightarrow{h} D \to 0$$

$$0 \to D \to TM \xrightarrow{q} Q \to 0.$$

There is also the $D$-valued Levi-form defined by projecting the Lie bracket of vector fields in $D$

$$L : D \times D \to Q.$$

Splittings of the sequences correspond to splittings of $TM$ or $T^*M$.

A change of splitting from $s$ to another $\widehat{s} : Q \to TM$ may be naturally identified with a bundle map $f : Q \to D$.
Changes of splitting induce:

$$[TM]_s \ni [v]_s = \begin{pmatrix} \sigma^a \\ u^i \end{pmatrix}_s \mapsto \begin{pmatrix} \widehat{\sigma}^a \\ \widehat{u}^i \end{pmatrix}_{\widehat{s}} = \begin{pmatrix} \sigma^a \\ u^i - f_a^i \sigma^a \end{pmatrix}_{\widehat{s}} = [v]_{\widehat{s}} \in [TM]_{\widehat{s}}$$

and similarly

$$\begin{pmatrix} u^i \\ \nu_a \end{pmatrix} \mapsto \begin{pmatrix} u^i \\ \nu_a + f_a^i u_i \end{pmatrix} \qquad \text{where} \qquad u_i = h_{ij} u^j,$$

Non-holonomic Riemannian structure $(M, g, D, D^{\perp})$
Choose $E = TM$ and for $\alpha \geq 0$

$$\Phi_{\alpha} = \begin{cases} \mathrm{id}_D & \text{on } D \\ \alpha\,\mathrm{id}_{D^{\perp}} & \text{on } D^{\perp}. \end{cases}$$

When $\alpha$ approaches zero we charge each of the $D^{\perp}$ components of the velocities $\dot{c}(t)$ by a $1/\alpha$ multiple of its original size with respect to $g$. At the $\alpha = 0$ limit we obtain the original sub-Riemannian geometry.

### Theorem

*Given a sub-Riemannian geometry $(M, D, h)$, let $g$ be a Riemannian metric on $TM$ that restricts to $h$ on $D$ and write $D^\perp$ for the orthogonal complement of $D$. Then there is the unique metric connection $\nabla$ on $TM$ such that both $D$ and $D^\perp$ are preserved, and*

$$T^D_{DD} = 0$$

$$T^{D^\perp}_{D^\perp D^\perp} = 0$$

$$T^{D^\perp}_{DD^\perp} \text{ is symmetric with respect to } g_{|D^\perp}$$

$$T^{D}_{DD^\perp} \text{ is symmetric with respect to } g_{|D}.$$

*This connection $\nabla$ is invariant with respect to constant rescalings of $g$ on $D$ or $D^\perp$.*

Fix extension metric $g$ of the given sub-Riemannian metric $h$, write $TM = D \oplus D^{\perp}$, consider the family of metrics $g^{\epsilon}_{|D} = g$ and $g^{\epsilon}_{|D^{\perp}} = \epsilon g$. They all share the special connection $\nabla$. We rewrite the geodesic equation for the metric minimizers of $g^{\epsilon}$ in term of $\nabla$ and its torsion.

Write $D^{\epsilon}$ for the Levi Civita connection of $g^{\epsilon}$ and $A^{\epsilon} : TM \otimes TM \to TM$ be the contorsion tensor,

$$D^{\epsilon}_X Y = \nabla_X Y + A^{\epsilon}(X, Y).$$

Consider local non-holonomic frames spanning $D$ and $D^\perp$ and use indices $i, j, k, \dots$ and $a, b, c, \dots$ in relation to $D$ and $D^\perp$, respectively, i.e., $u = u^i + u^a$ is the tangent curve $u = \dot{c}$, $\nabla = \nabla_i + \nabla_a$.
Similarly, write $g_{ij}$ and $\epsilon g_{ab}$, and torsions

$$T^i{}_{jk} + T^i{}_{ja} + T^i{}_{ab} + T^a{}_{jk} + T^a{}_{jb} + T^a{}_{bc}.$$

The variational equations $D^{\epsilon}_{u} u = 0$ for the tangent curves $u = \dot{c}^{\epsilon}$ of the $g^{\epsilon}$ critical curves $c^{\epsilon}$ are

$$
\begin{aligned}
0 = g_{ij} u^k \nabla_k u^j + g_{ij} u^a \nabla_a u^j + g_{kj} u^k T^j{}_{ia} u^a \\
+ \epsilon g_{ab} u^a T^b{}_{ic} u^c + \epsilon g_{ab} u^a T^b{}_{ik} u^k \\
0 = \epsilon g_{ab} u^k \nabla_k u^b + \epsilon g_{ab} u^c \nabla_c u^b + g_{ij} u^i T^j{}_{ab} u^b \\
+ g_{ij} u^i T^j{}_{ak} u^k + \epsilon g_{cb} u^b T^c{}_{ak} u^k .
\end{aligned}
$$

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&\qquad + \epsilon g_{ab}u^a T^b{}_{ic}u^c + \epsilon g_{ab}u^a T^b{}_{ik}u^k \\
0 &= \epsilon g_{ab}u^k\nabla_k u^b + \epsilon g_{ab}u^c\nabla_c u^b + g_{ij}u^i T^j{}_{ab}u^b \\
&\qquad + g_{ij}u^i T^j{}_{ak}u^k + \epsilon g_{cb}u^b T^c{}_{ak}u^k.
\end{aligned}
$$

Now we "renormalize" the $D^{\perp}$ component $u^a$ as $u^a = \frac{1}{\epsilon}\nu^a$ and $\delta = 1/\epsilon$. In the limit $\delta = 0$ we arrive at

$$
\begin{aligned}
0 &= g_{ij}u^k\nabla_k u^j + g_{ab}\nu^a T^b{}_{ik}u^k \\
0 &= g_{ab}u^k\nabla_k \nu^b + g_{ij}u^i T^j{}_{ak}u^k + g_{cb}\nu^b T^c{}_{ak}u^k
\end{aligned}
$$

With the help of $g$, we can view this as equations coupling the components $(u^i) \in \mathcal{D}$ with $(\nu_a)$ in the annihilator of $\mathcal{D}$ in $T^*M$.

## Theorem

*For each set of initial conditions* $x \in M$, $u(0) \in \mathcal{D} \subset T_x M$, *and* $\nu(0) \in \mathcal{D}^\perp \subset T_x^* M$, *the component* $u(t)$ *of the unique solution of the equations*

$$
\begin{aligned}
0 &= u^k \nabla_k u^i + h^{ij} \nu_a L^a{}_{ik} u^k \\
0 &= u^k \nabla_k \nu_a + g_{ij} u^i T^j{}_{ak} u^k + \nu_b T^b{}_{ak} u^k
\end{aligned}
\qquad (1)
$$

*projects to a locally defined normal extremal* $c(t)$ *of the sub-Riemannian geometry with* $c(0) = x$ *and* $\dot{c}(t) = u(t)$.

# generalized Heisenberg in 5D

Holonomic coordinates $(x^1, x^2, x^3, x^4, z)$, $D$ spanned by

$$X_1 = \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial z} \qquad X_2 = \lambda\left(\frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial z}\right)$$

$$X_3 = \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial z} \qquad X_4 = \lambda\left(\frac{\partial}{\partial x^4} + x^2 \frac{\partial}{\partial z}\right)$$

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$$X_3 = \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial z} \qquad X_4 = \lambda\left(\frac{\partial}{\partial x^4} + x^2 \frac{\partial}{\partial z}\right)$$

The set of first 5 equations reads (here $u(t) = \alpha^i X_i$ in the non-holonomic frame)

$$\dot{x}^1 = \alpha^1, \qquad \dot{x}^2 = \lambda\alpha^2, \qquad \dot{x}^3 = \alpha^3, \qquad \dot{x}^4 = \lambda\alpha^4,$$
$$\dot{z} = x^1\alpha^3 - x^3\alpha^1 + \lambda x^2\alpha^4 - \lambda x^4\alpha^2,$$

Our "non-holonomic equations" then get

$$\begin{aligned}
\dot{\alpha}^1 &= \frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4) - \nu\alpha^3, \\
\dot{\alpha}^2 &= -\frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}\alpha^1\alpha^2 - \frac{\lambda_{x^3} + x_1\lambda_z}{\lambda}\alpha^2\alpha^3 - (\lambda_{x^3} + x^2\lambda_z)\alpha^2\alpha^4 \\
&\quad - (x^4\lambda_z - \lambda_{x^2})\alpha^4\alpha^4 - \lambda^2\nu\alpha^4, \\
\dot{\alpha}^3 &= \frac{\lambda_{x^3} - x^1\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4) - \nu\alpha^1, \\
\dot{\alpha}^4 &= -\frac{\lambda_{x^1} - x^3\lambda_z}{\lambda}\alpha^1\alpha^4 - \frac{\lambda_{x^3} + x_1\lambda_z}{\lambda}\alpha^3\alpha^4 + (\lambda_{x^4} + x^2\lambda_z)\alpha^2\alpha^2 \\
&\quad + (x^4\lambda_z - \lambda_{x^2})\alpha^2\alpha^4 + \lambda^2\nu\alpha^2, \\
\dot{\nu} &= \frac{2\lambda_z}{\lambda}(\alpha^2\alpha^2 + \alpha^4\alpha^4).
\end{aligned}$$

In particular, if $\lambda_z = 0$ then $\nu$ is a free constant parameter. These equations coincide with the standard ones if $\lambda$ is a constant function.