Ricci tensor in graded geometry

Fridrich Valach

Charles University, Prague

Srní, 2020

Summary

Goal: Find a simple interpretation, in the language of graded geometry, of the Ricci tensor for Courant algebroids.

Reasons:

- More conceptual and geometric.
- Can try to extend the framework to the case of U-duality.

Motivated by: [Aschieri, Bonechi, Deser 2019]

Motivation: Courant algebroids & Ricci tensor

Definition (Liu, Weinstein, Xu 1997)

Courant algebroid (CA): a vector bundle $E \to M$ with

- an inner product $\langle \cdot, \cdot \rangle$ on the fibers
- a vector bundle map $\rho \colon E \to TM$ (the anchor)
- an $\mathbb{R}$-bilinear bracket $[\cdot, \cdot] \colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

$+$ compatibility conditions (Jacobi, . . .)

Example (Ševera 1998; exact Courant algebroid)

$TM \oplus T^*M \to M$

$[u + \alpha, v + \beta] = [u, v] + \mathcal{L}_u \beta - \iota_v \alpha + \eta(u, v, \cdot) \qquad (\eta \in \Omega^3_{\text{closed}}(M))$

Motivation: Courant algebroids & Ricci tensor

Definition

Generalized metric: $V_+ \subset E$ *with* $\langle \cdot, \cdot \rangle|_{V_+}$ *non-degenerate* ($V_- := V_+^\perp$)

Divergence operator:[a] $\mathbb{R}$-*linear map* $\operatorname{div} \colon \Gamma(E) \to C^\infty(M)$ *s.t.* ...

Generalized Ricci tensor:[b] *section* $\mathrm{GRic}_{V_+,\operatorname{div}}$ *of* $V_+^* \otimes V_-^*$ *defined by*

$$\mathrm{GRic}_{V_+,\operatorname{div}}(u, v) = \operatorname{div}[v, u]_+ - \rho(v) \operatorname{div} u - \mathrm{Tr}_{V_+}([[\cdot, v]_-, u]_+),$$

$u \in \Gamma(V_+)$, $v \in \Gamma(V_-)$, $(\cdot)_\pm \colon E \to V_\pm$ *the orthogonal projections*

[a] Alexeev, Xu 2001; Garcia-Fernandez 2014

[b] Coimbra, Strickland-Constable, Waldram 2011; Garcia-Fernandez 2014; Jurčo, Vysoký 2015; Ševera, V. 2018

Motivation: Courant algebroids & Ricci tensor

Example

Exact Courant algebroid

$V_+ \rightsquigarrow$ (metric g, closed 3-form η)

take div the divergence w.r.t. the metric volume form

$\mathrm{GRic}_{V_+,\mathrm{div}}$ recovers Ricci for the metric connection with torsion η

String theory:

ordinary Ricci tensor $\longleftrightarrow$ string background equations

Courant algebroids $\longleftrightarrow$ Poisson-Lie T-duality

Can use GRic to show the compatibility of string background equations and Poisson-Lie T-duality.

Question: Is there a more conceptual definition of GRic?

NQ symplectic manifolds

Definition (Ševera 2005)

N manifold: *non-negatively graded manifold* $\mathcal{E}$ *(*$C^\infty(\mathcal{E})$ *is* $\mathbb{N}$*-graded)*
NQ manifold: *N manifold with a vector field* Q *of degree 1,* $Q^2 = 0$
NQ symplectic manifold of degree n**:** *NQ manifold with a symplectic form of degree* n*,* $\mathcal{L}_Q\omega = 0$

$\implies$ Q is Hamiltonian for some $H \in C^\infty_{n+1}(\mathcal{E})$, $\quad \{H, H\} = 0$

From now on $n = 2 \leftrightarrow$ Courant algebroids [Roytenberg 01, Ševera 98]

In coordinates

$$\underbrace{x^i}_{\text{deg } 0}, \underbrace{e^\alpha}_{\text{deg } 1}, \underbrace{p_i}_{\text{deg } 2}, \quad \omega = dp_i dx^i + \underbrace{g_{\alpha\beta}}_{\text{cst.}} de^\alpha de^\beta$$

$$H = \rho^i_\alpha(x) p_i e^\alpha - \tfrac{1}{6} c_{\alpha\beta\gamma}(x) e^\alpha e^\beta e^\gamma$$

Generalized metric

Sequence: $\mathcal{E} = \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0$ ($\mathcal{E}_1 \to \mathcal{E}_0$ is a vector bundle)

Definition

Generalized metric: *symplectic involution ι on $\mathcal{E}$ preserving $\mathcal{E}_0$, i.e.*

$$\iota^2 = id_{\mathcal{E}} \qquad \iota^*\omega = \omega \qquad \iota|_{\mathcal{E}_0} = id_{\mathcal{E}_0}$$

In coordinates

$$\underbrace{x^i}_{\text{deg } 0}, \underbrace{e^a, e^{\dot{a}}}_{\text{deg } 1}, \underbrace{p_i}_{\text{deg } 2}, \qquad \omega = dp_i dx^i + g_{ab} de^a de^b + g_{\dot{a}\dot{b}} de^{\dot{a}} de^{\dot{b}}$$

$$\iota^* x^i = x^i, \quad \iota^* p_i = p_i, \quad \iota^* e^a = e^a, \quad \iota^* e^{\dot{a}} = -e^{\dot{a}}$$

Generalized metric

Definition

$\mathcal{V}_1$: the fixed point set of the ι action on $\mathcal{E}_1$

$\mathcal{V}$: the vector bundle pullback of $\mathcal{V}_1$ along $\mathcal{E} \to \mathcal{E}_0$

$$\begin{array}{ccccc} \mathcal{V} & & \longrightarrow & & \mathcal{V}_1 \\ \downarrow & & & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \end{array}$$

Remark: ι determined by $\mathcal{V}_1 \subset \mathcal{E}_1$

In coordinates

Coordinates on $\mathcal{V}$: $\underbrace{x^i}_{\text{deg } 0}, \underbrace{e^a, e^{\dot{a}}, \xi^a}_{\text{deg } 1}, \underbrace{p_i}_{\text{deg } 2}$

Definition (Aschieri, Bonechi, Deser 2019)

$\tau : \mathcal{E} \to \mathcal{V}^*$ **tautological section**

$\tau = g_{ab} e^a \xi^b$ *(under $\Gamma(\mathcal{V}^*) \cong C^\infty_{lin}(\mathcal{V})$)*

$$\begin{array}{ccccc} \mathcal{V} & & \longrightarrow & & \mathcal{V}_1 \\ \uparrow{\scriptstyle \tau'} & & & \nearrow & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \end{array}$$

Connections, curvature and torsion

Definition (Aschieri, Bonechi, Deser 2019)

Connection on $\mathcal{V}^*$: *degree 1 vector field $\hat{Q}$ on $\mathcal{V}$ satisfying*
- *$\hat{Q}$ pushes forward to Q on $\mathcal{E}$*
- *$\hat{Q}$ preserves the space of sections $C^\infty_{lin}(\mathcal{V})$*

Curvature: $R := \hat{Q}^2 = \frac{1}{2}[\hat{Q}, \hat{Q}]$ *(degree 2 vector field on $\mathcal{V}$)*

Torsion: $T := \hat{Q}\tau$ *(section of $\mathcal{V}^*$)*

In coordinates

$$\hat{Q} = Q + \psi^a{}_{b\alpha}(x) e^\alpha \xi^b \partial_{\xi^a}$$

Levi-Civita and Ricci

Definition

Levi-Civita condition: $\iota^* T = T$ *(invariance of torsion)*

Ricci tensor: $\mathrm{Ric} = \textit{contraction}(R_{\textit{anti-self-dual part}})$

In coordinates

Levi-Civita $\leftrightarrow$ ${\psi^a}_{b\dot{a}} = {c^a}_{b\dot{a}}$ (${\psi^a}_{bc}$ undetermined)

Proposition (Ricci is GRicci)

$\mathrm{Ric} = \mathrm{GRic}_{V_+,\mathrm{div}}(e_b, e_{\dot{a}})\xi^b e^{\dot{a}}$ *($V_+ \leftrightarrow \mathcal{V}_1$, $\mathrm{div}\, e_b = -{\psi^a}_{ba}$)*

Ricci flow: $\mathrm{Ric} \in \Gamma(\mathcal{V}_1^* \otimes \mathcal{V}_1^{\perp *}) \cong \mathrm{Hom}_{\mathcal{E}_0}(\mathcal{V}_1, \mathcal{V}_1^{\perp}) \cong$ deformations of $\mathcal{V}_1$

Outlook:

Scalar curvature:

Need to understand half-densities and divergences from the graded viewpoint.

U-duality:

Replace exact Courant algebroids $\leftrightarrow T^*[2]T[1]M$ by higher NQ symplectic manifolds: $T^*[3]T[1]M$, $T^*[6]T[1]M \times \mathbb{R}[3]$