Ricci tensor in graded geometry

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Goal: Find a simple interpretation, in the language of graded geometry, of the Ricci tensor for Courant algebroids.

Reasons:

- More conceptual and geometric.
- Can try to extend the framework to the case of U-duality.

Motivated by: [Aschieri, Bonechi, Deser 2019]

Motivation: Courant algebroids & Ricci tensor

Definition (Liu, Weinstein, Xu 1997)

Courant algebroid (CA): a vector bundle $E \rightarrow M$ with

- an inner product $\langle \cdot, \cdot
 angle$ on the fibers
- a vector bundle map $\rho \colon E \to TM$ (the anchor)
- an \mathbb{R} -bilinear bracket $[\cdot, \cdot] \colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

+ compatibility conditions (Jacobi, ...)

Example (Ševera 1998; exact Courant algebroid) $TM \oplus T^*M \to M$ $[u + \alpha, v + \beta] = [u, v] + \mathcal{L}_u\beta - \iota_v\alpha + \eta(u, v, \cdot) \qquad (\eta \in \Omega^3_{\text{closed}}(M))$

Motivation: Courant algebroids & Ricci tensor

Definition

Generalized metric: $V_+ \subset E$ with $\langle \cdot, \cdot \rangle|_{V_+}$ non-degenerate $(V_- := V_+^{\perp})$

Divergence operator:^{*a*} \mathbb{R} -linear map div: $\Gamma(E) \to C^{\infty}(M)$ s.t. ...

Generalized Ricci tensor:^b section $\operatorname{GRic}_{V_+,\operatorname{div}}$ of $V^*_+ \otimes V^*_-$ defined by

 $GRic_{V_+,div}(u,v) = div[v,u]_+ - \rho(v) \, div \, u - \operatorname{Tr}_{V_+}([[\cdot,v]_-,u]_+),$

 $u \in \Gamma(V_+)$, $v \in \Gamma(V_-)$, $(\cdot)_{\pm} : E \to V_{\pm}$ the orthogonal projections

^aAlexeev, Xu 2001; Garcia-Fernandez 2014 ^bCoimbra, Strickland-Constable, Waldram 2011; Garcia-Fernandez 2014; Jurčo, Vysoký 2015; Ševera, V. 2018

Motivation: Courant algebroids & Ricci tensor

Example

Exact Courant algebroid

 $V_+ \rightsquigarrow (\mathsf{metric}\ g, \mathsf{closed}\ \mathsf{3-form}\ \eta)$

take div the divergence w.r.t. the metric volume form

 $\mathrm{GRic}_{V_+,\mathrm{div}}$ recovers Ricci for the metric connection with torsion η

String theory:

ordinary Ricci tensor \longleftrightarrow string background equations

Courant algebroids \leftrightarrow Poisson-Lie T-duality

Can use ${\rm GRic}$ to show the compatibility of string background equations and Poisson-Lie T-duality.

Question: Is there a more conceptual definition of $\operatorname{GRic}\nolimits?$

NQ symplectic manifolds

Definition (Ševera 2005)

N manifold: non-negatively graded manifold \mathcal{E} ($C^{\infty}(\mathcal{E})$ is \mathbb{N} -graded) **NQ** manifold: N manifold with a vector field Q of degree 1, $Q^2 = 0$ **NQ** symplectic manifold of degree n: NQ manifold with a symplectic form of degree n, $\mathcal{L}_Q\omega = 0$

 $\implies Q$ is Hamiltonian for some $H \in C^{\infty}_{n+1}(\mathcal{E})$, $\{H, H\} = 0$

From now on $n = 2 \leftrightarrow$ Courant algebroids [Roytenberg 01, Ševera 98]

In coordinates

$$\underbrace{x^{i}}_{\deg 0}, \underbrace{e^{\alpha}}_{\deg 1}, \underbrace{p_{i}}_{\deg 2}, \quad \omega = dp_{i}dx^{i} + \underbrace{g_{\alpha\beta}}_{\operatorname{cst.}} de^{\alpha}de^{\beta}$$
$$H = \rho^{i}_{\alpha}(x)p_{i}e^{\alpha} - \frac{1}{6}c_{\alpha\beta\gamma}(x)e^{\alpha}e^{\beta}e^{\gamma}$$

Generalized metric

Sequence: $\mathcal{E} = \mathcal{E}_2 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0$ ($\mathcal{E}_1 \rightarrow \mathcal{E}_0$ is a vector bundle)

Definition

Generalized metric: symplectic involution ι on \mathcal{E} preserving \mathcal{E}_0 , i.e. $\iota^2 = id_{\mathcal{E}} \qquad \iota^* \omega = \omega \qquad \iota|_{\mathcal{E}_0} = id_{\mathcal{E}_0}$

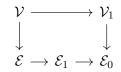
In coordinates

$$\underbrace{x^{i}}_{\deg 0}, \underbrace{e^{a}, e^{\dot{a}}}_{\deg 1}, \underbrace{p_{i}}_{\deg 2}, \quad \omega = dp_{i}dx^{i} + g_{ab}de^{a}de^{b} + g_{\dot{a}\dot{b}}de^{\dot{a}}de^{\dot{b}}$$
$$\iota^{*}x^{i} = x^{i}, \quad \iota^{*}p_{i} = p_{i}, \quad \iota^{*}e^{a} = e^{a}, \quad \iota^{*}e^{\dot{a}} = -e^{\dot{a}}$$

Generalized metric

Definition

- $\mathcal{V}_1 :$ the fixed point set of the ι action on \mathcal{E}_1
- $\mathcal{V}\colon$ the vector bundle pullback of \mathcal{V}_1 along $\mathcal{E}\to\mathcal{E}_0$



Remark: <i>i</i> determined b)у́	\mathcal{V}_1	$\subset \mathcal{E}_1$	
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In coordinates

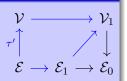
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Coordinates on
$$\mathcal{V}: \underbrace{x^i}_{\text{deg 0}}, \underbrace{e^a, e^{\dot{a}}, \xi^a}_{\text{deg 1}}, \underbrace{p_i}_{\text{deg 2}}$$

Definition (Aschieri, Bonechi, Deser 2019)

$$\tau: \mathcal{E} \to \mathcal{V}^*$$
 tautological section

$$=g_{ab}e^{a}\xi^{b}$$
 (under $\Gamma(\mathcal{V}^{*})\cong C^{\infty}_{\mathit{lin}}(\mathcal{V})$)



Connections, curvature and torsion

Definition (Aschieri, Bonechi, Deser 2019)Connection on \mathcal{V}^* :degree 1 vector field \hat{Q} on \mathcal{V} satisfying
 - \hat{Q} pushes forward to Q on \mathcal{E}
 - \hat{Q} preserves the space of sections $C_{lin}^{\infty}(\mathcal{V})$ Curvature: $R := \hat{Q}^2 = \frac{1}{2}[\hat{Q}, \hat{Q}]$ (degree 2 vector field on \mathcal{V})Torsion: $T := \hat{Q}\tau$ (section of \mathcal{V}^*)

In coordinates

$$\hat{Q} = Q + \psi^a{}_{b\alpha}(x) e^{\alpha} \xi^b \partial_{\xi^a}$$

Levi-Civita and Ricci

Definition

Levi-Civita condition:	$\iota^*T = T$ (invariance of torsion)
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Ricci tensor: $Ric = contraction(R_{anti-self-dual part})$

In coordinates

Levi-Civita
$$\leftrightarrow \psi^a_{\ b\dot{a}} = c^a_{\ b\dot{a}}$$
 ($\psi^a_{\ bc}$ undetermined)

Proposition (Ricci is GRicci)

 $\operatorname{Ric} = \operatorname{GRic}_{V_+,\operatorname{div}}(e_b, e_{\dot{a}}) \xi^b e^{\dot{a}} \quad (V_+ \leftrightarrow \mathcal{V}_1, \operatorname{div} e_b = -\psi^a{}_{ba})$

Ricci flow: $\operatorname{Ric} \in \Gamma(\mathcal{V}_1^* \otimes \mathcal{V}_1^{\perp *}) \cong \operatorname{Hom}_{\mathcal{E}_0}(\mathcal{V}_1, \mathcal{V}_1^{\perp}) \cong \text{deformations of } \mathcal{V}_1$

Scalar curvature:

Need to understand half-densities and divergences from the graded viewpoint.

U-duality:

Replace exact Courant algebroids $\leftrightarrow T^*[2]T[1]M$ by higher NQ symplectic manifolds: $T^*[3]T[1]M$, $T^*[6]T[1]M \times \mathbb{R}[3]$