Knizhnik - Zamolodchikov equation and its amazing properties

- Definition
- Topological interpretation
- Conformal invariance
- Multiple zeta - values
- Conformal field theory
- Poisson geometry : Hitchin Thm
- Back to topology : Turaev cobracket

Conformal field theory

2d topology

KZ equation

3d topology

Number theory

Background

- Lie algebra $\mathfrak{t}_n$ $\Leftarrow$ 3d topology

$$\mathfrak{t}_n = \text{free Lie}\left\langle t_{ij} = t_{ji} \,;\; i \neq j = 1, \dots, n \;\middle|\; \begin{array}{l} [t_{ij} + t_{ik}, t_{jk}] = 0 \\ [t_{ij}, t_{kl}] = 0 \end{array} \right\rangle$$

Examples: $k = \mathbb{R}, \mathbb{C}$

$\mathfrak{t}_2 = k\, t_{12}$ 1-dimensional Lie algebra

$\mathfrak{t}_3 = k t_{12} \oplus \text{free Lie}\langle t_{13}, t_{23} \rangle$

e.g. $[t_{12}, t_{13}] = [t_{13}, t_{23}]$

- another version $\tilde{\mathfrak{t}}_n$

add central generators $c_i = t_{ii}$

extra relations $\sum_j t_{ij} = 0$

KZ connection

$$\mathrm{Conf}_n(\mathbb{C}) = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \; ; \; z_i \neq z_j \}$$

for $i \neq j$

$$A \in \Omega^1(\mathrm{Conf}_n(\mathbb{C}), t_n)$$

$$\boxed{A = \kappa \sum_{i<j} t_{ij} \, d\log(z_i - z_j)}$$

$$\frac{dz_i - dz_j}{z_i - z_j}$$

Fact: A is flat

scaling

$$\boxed{\kappa \, dA + \frac{\kappa^2}{2}[A, A] = 0}$$

- $dA = 0$
- $[A, A] = 0$ ← exercise

$$\Uparrow$$

$$\frac{1}{(z_i - z_j)(z_j - z_k)} + \frac{1}{(z_j - z_k)(z_k - z_i)} + \frac{1}{(z_k - z_i)(z_i - z_j)} = 0$$

Topology of surfaces

Fact: $\frac{1}{2\pi i} \, d \log(z_i - z_j)$ is a basis of de Rham cohomology $H^1(\mathrm{Conf}_n)$

Interpretation: view t_{ij} as dual basis in $H_1(\mathrm{Conf}_n)$

$$A = \sum_{i<j} t_{ij} \, d \log(z_i - z_j)$$

represents the canonical element in

$$H_1(\mathrm{Conf}_n) \otimes H^1(\mathrm{Conf}_n)$$

Conformal transformations $\mathbb{C} \subset \mathbb{CP}^1$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad z \longmapsto \frac{az+b}{cz+d}$$

$$ad - bc = 1$$

$$g'(z) = (cz+d)^{-2}$$

Fact: $\quad z_i \longmapsto g(z_i) \quad \forall i$

$$g^* A = A - \sum_i d \log (cz_i + d) \sum_{j \neq i} t_{ij}$$

exercise

$$\underset{\text{in } \tilde{t}_n}{=} A - \frac{1}{2} \sum_i c_i \, d \log g'(z_i)$$

$$\Rightarrow \quad g^*(d + A) = F^{-1} (d + A) F$$

where $\quad F = \prod_i g'(z_i)^{-\frac{1}{2} c_i}$

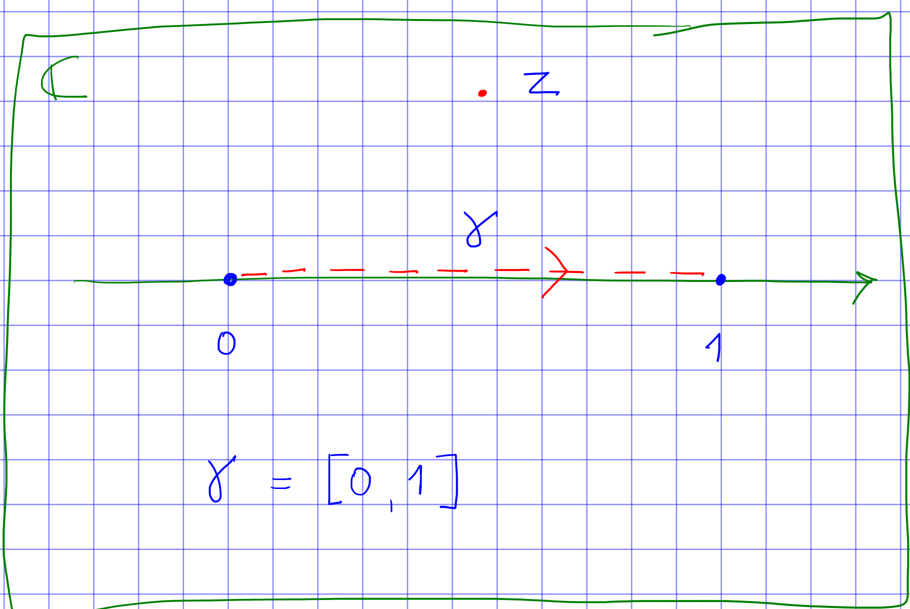
$\underline{n = 4}$ $\qquad \{z_1, z_2, z_3, z_4\} \subset \mathbb{C}P^1$

Fix $\quad z_1 = 0 \,, \quad z_2 = 1 \,, \quad z_3 = \infty$

$z_4 = z$ is free

Denote $\quad t_{14} = x \,, \quad t_{24} = y$

$$\boxed{A = \left(\frac{x}{z - \underset{\substack{\| \\ z_1}}{0}} + \frac{y}{z - \underset{\substack{\| \\ z_2}}{1}} \right) dz}$$



Parallel transport

~~Holonomy~~ along γ

$$\frac{d\psi}{dt} = A\psi \quad , \quad \psi(0) = 1 \qquad \in G,\ Ut_n$$

$$H = \psi(1) \qquad \text{" } P\exp \int A(t)\,dt \text{ "}$$

Recall:

$$H = 1 + \int_{1 \geq t_1 \geq 0} A(t_1)\,dt_1 + \int_{1 \geq t_1 \geq t_2 \geq 0} A(t_1)\,A(t_2)\,dt_1\,dt_2 + \dots$$

! H is ill defined because of the poles at 0 and 1 !

In low degrees:

$$H = 1 + x \underbrace{\int_0^1 \frac{dt_1}{t_1}}_{\text{"}=\text{" } 0} + y \underbrace{\int_0^1 \frac{dt_1}{t_1 - 1}}_{\text{"}=\text{" } 0}$$

$$+ \; xy \left[\int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{t_2 - 1} = \frac{\pi^2}{6} = \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \right]$$

$$+ \; yx \underbrace{\int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1 - 1} \frac{dt_2}{t_2}}_{= -\zeta(2)} + \ldots$$

$$= 1 + \zeta(2)[x, y] + \ldots$$

exercise; check it is gp.-like

$$\Phi = 1 + ?\, x + ?\, y$$

$$+ \; \zeta(2)\, xy + ?\, yx + \ldots$$

Theorem (Le-Murakami)

$\exists ! \; \boxed{\Phi} \in \mathbb{C}\langle\langle x, y\rangle\rangle$ non-commutative formal power series in x, y

3d top $\Rightarrow$ $\boxed{\Phi\Phi = \Phi\Phi\Phi}$

s.t.

- convergent coefficients coincide with those of H

- they are given by multiple zeta-values

$$\zeta(n_1, \dots, n_d) = \sum_{k_1 > k_2 > \dots > k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}}$$

converges iff $n_1 \geq 2$

- $\Delta \Phi = \Phi \otimes \Phi$

Δ = alg. homo

where $\Delta(x) = x \otimes 1 + 1 \otimes x$

$\Delta(y) = y \otimes 1 + 1 \otimes y$

Wess-Zumino-Witten models

$\mathfrak{g}$ = quadratic Lie algebra

$B : \mathfrak{g} \times \mathfrak{g} \to k$ invariant scalar product

$$B([x,y],z) + B(y,[x,z]) = 0$$

$\{e_a\}$ = a basis of $\mathfrak{g}$

$\{e^a\}$ = the dual basis

Consider

$$U\mathfrak{g}^{\otimes n} = \underbrace{U\mathfrak{g} \otimes U\mathfrak{g} \otimes \dots \otimes U\mathfrak{g}}_{n \text{ times}}$$

$$T_{ij} = \sum_a 1 \otimes \dots \otimes \underset{i}{e_a} \otimes \dots \otimes \underset{j}{e^a} \otimes 1 \dots \otimes 1$$

$$C_i = \sum_a 1 \otimes \dots \otimes \underset{i}{e_a e^a} \otimes \dots \otimes 1$$

Fact: $t_{ij} \longmapsto T_{ij}$
is a Lie homomorphism

Let $\rho_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$ be representations of $\mathfrak{g}$

Consider

$$V = (V_1 \otimes \dots \otimes V_n)^{\mathfrak{g}} \leftarrow \text{diagonal action}$$

Fact:

$$\rho(t_{ij}) = (\rho_i \otimes \rho_j)(T_{ij})$$

$$\rho(C_i) = \rho_i(C_i)$$

is a representation of $\tilde{t}_n$

Conformal blocks:

$$\Psi : \widetilde{\mathrm{Conf}}_n \longrightarrow V$$

$$\boxed{\Psi(z_1, \dots, z_n) \in V}$$

s.t.

F

F^{-1}

$$\boxed{(d + A))\Psi = 0}$$

F^{-1}

how to solve these eqn ?

Recall: $\quad g(z) = \dfrac{az+b}{cz+d}$

$$\Psi(g(z_1), \dots, g(z_n)) = F^{-1}\Psi(z_1, \dots, z_n)$$

where

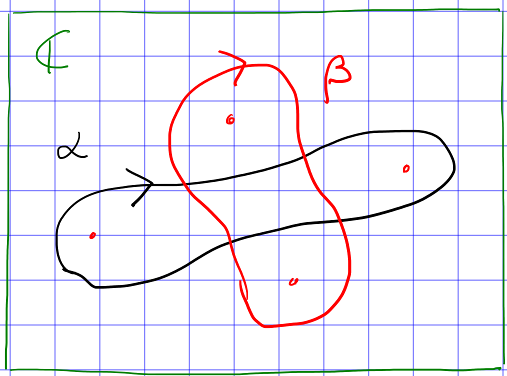
$$F^{-1} = \prod_i g'(z_i)^{\frac{\rho_i(C_i)}{2}}$$

"anomalous dimensions"

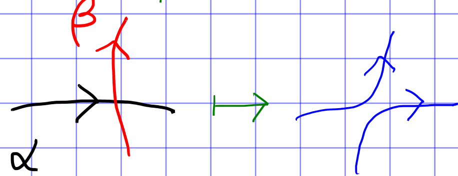
$$\langle \varphi(z_1) \dots \varphi(z_n) \rangle$$

Goldman bracket

$\mathfrak{g} = k\langle$ homotopy classes of free oriented loops on $\Sigma\rangle$



$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, (\alpha_p \beta)$$



Thm (Goldman) $[\cdot, \cdot]$ is well-defined and it is a Lie bracket

Denote $x_i = t_{i(n+1)}$, $z_{n+1} = z$

$$a = \sum_i x_i \, d \log (z - z_i)$$

Goldman functions:

$$F_\gamma = \mathrm{Cycl}\ \mathrm{Hol}(a, \gamma) =$$

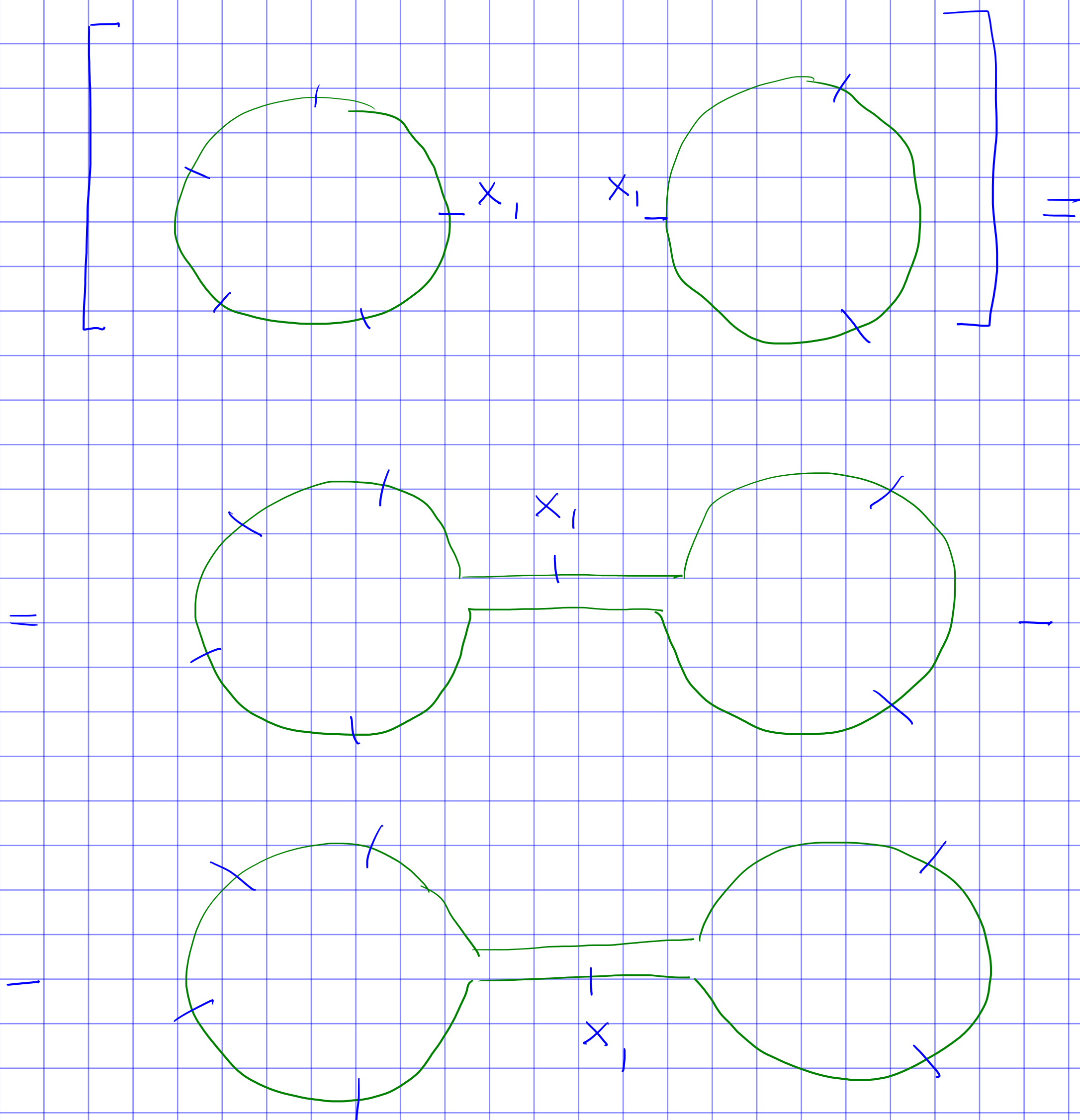
$$= 1 + \int_0^{2\pi} a(t)\, dt +$$

$$+ \int_{2\pi \geq t_1 \geq t_2 \geq 0} \mathrm{Cycl}\,(a(t_1) a(t_2))\, dt_1 dt_2 + \ldots$$

Thm (Hitchin) $\gamma \longmapsto F_\gamma$

is a Lie homomorphism under

$$[x_i, x_j] = \delta_{ij} (x_i \otimes 1 - 1 \otimes x_i)$$



$\mathrm{Cycl}(H_1(\Sigma)) \leftarrow$ Lie algebra $\Leftarrow$ Goldman

$TH_1 \leftarrow$ "double bracket"

Thank

you !