Geometry of Pseudo-Riemanniamn Special Vinberg Cones of rank 2 and 3

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To the blessed memory of E. B. Vinberg

Definition of homogeneous convex cone

An open convex cone $\mathcal{V} \subset W$ without straight line of a vector space W is called
homogeneous , if the group of automorphisms

$$\mathrm{Aut}(\mathcal{V}) = \{\mathrm{A} \in \mathrm{GL}(\mathrm{V}), \mathrm{A}\mathcal{V} = \mathcal{V}\}$$

acts transitively in $\mathcal{V}$.
Then there exists a solvable subgroup $\mathrm{G} \subset \mathrm{Aut}(\mathcal{V})$ which acts simply transitively in $\mathcal{V}$.
The dual cone

$$\mathcal{V}^* = \{\xi \in \mathrm{W}^*, \xi(\mathrm{X}) > 0 \, \forall \mathrm{X} \in \mathcal{V}\}$$

is also a convex homogeneous cone w.r.t. action of $\mathrm{Aut}(\mathcal{V})$ in W^*.
A cone is called selfdual if there is an Euclidean metric $\mathrm{g} : \mathrm{W} \to \mathrm{W}^*$, which maps $\mathcal{V}$ onto $\mathcal{V}^*$.

Examples of selfdual cones

1. The Lorentz cone $\mathcal{V}(\mathbb{R}^{1,n+1})$ of future directed timelike vectors in the Minkowski space $\mathbb{R}^{1,n+1}$.
2. The cone $\mathcal{V}_n(\mathbb{K}) \subset \mathcal{H}_n(\mathbb{K})$ of Hermitian positively defined matrices over a division algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or for $n = 3$ over $\mathbb{O}$:

Köcher-Vinberg Theorem. The selfdual cones are cones of positive defined elements in Euclidean Jordan algebras. (positive = the square of an invertible element).

Euclidean Jordan algebras

We recall that Jordan algebras were defined by P. Jordan in 1933 in a paper devoted to the axiomatization of quantum mechanics as a notion which formalized the notion of an algebra of observables.
A Jordan algebra is a commutative algebra J with relation

$$[aba^2] := (ab)a^2 - a(ba^2) = 0.$$

It is called Euclidean if $< a, a > := \mathrm{tr}\, L_{a^2}$ is an Euclidean metric in J.

Jordan-vonNeumann-Wigner classification

Euclidean (or formally real) Jordan algebras were classified by P. Jordan, J. von Neumann and E. Wigner. Any such algebra is a direct sum of simple algebras.
All simple Euclidean Jordan algebras are exhausted by
i) the spin factor algebras $\mathcal{H}_2(\mathrm{V})$ and
ii) the algebras $\mathcal{H}_n(\mathbb{K})$ of rank n Hermitian matrices over division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (for $n = 3$).
The spin-factor algebra = rank 2 Vinberg Hermitian T-algebra is

$$\mathcal{H}_2(V) = \left\{ X = \begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix}, x_1, x_2 \in \mathbb{R}, \; v \in V = \mathbb{R}^n \right\}$$

Vinberg theory of matrix T-algebras

Vinberg reduced the classification of homogeneous convex cones to description of "compact rank n left symmetric normal algebras"(clans).

He developed a theory of clans and showed that they can be described in the framework of rank n matrix T-algebras $\mathcal{M}_n$ (Vinberg T-algebras).

Any homogeneous convex cone is the open orbit $G(\mathrm{Id}\,)$ of the identity matrix under some natural action of the upper triangular group $G \subset \mathcal{M}_n$ in the space of Hermitian T-matrices $\mathcal{H}_n \subset \mathcal{M}_n$.

The space $\mathcal{H}_n$ is a commutative algebra with respect to Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

In particular, the rank 2 Hermitian T-algebras are isomorphic to the (Euclidean) 'spinor factor" Jordan algebras and the associated cone is the Lorentz cone.

Definition of matrix T-algebras

A Vinberg rank n matrix T-algebra $\mathcal{M}_n$ consists of $n \times n$ matrices $X = ||x_{ij}||$ where $x_{ii} \in \mathbb{R}$, $x_{ij} \in V_{ij}$, $V_{ji} = V_{ij}^* = \mathrm{Hom}(V_{ij}, \mathbb{R})$, $i \neq j$ are Euclidean vector spaces.
To define the matrix multiplication, E. Vinberg postulated existence of isometric bilinear maps

$$V_{ij} \times V_{jk} \to V_{ik}, \ (x_{ij}, x_{jk}) \to x_{ij} \cdot x_{jk},$$

$$|x_{ij} \cdot x_{jk}| = |x_{ij}| \cdot |x_{jk}|.$$

Axioms of a T-algebra are reconstructed from the condition that the nilpotent upper triangular subalgebra

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & b_{12} & b_{13} & \cdots b_{1n} \\ 0 & 0 & b_{23} & \cdots b_{2n} \\ 0 & 0 & 0 & \cdots b_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots 0 \end{pmatrix} \right\}$$

(called N-algebra) is an associative algebra and the following condition holds:
If $< b_{ik}, \mathcal{N} b_{jk} >= 0$, then $< \mathcal{N} a_{ik}, \mathcal{N} b_{jk} >= 0$ for $i < j$.
Hence the algebra $\mathfrak{g} = \mathrm{diag} + \mathcal{N}$ of upper triangular metrices is also associative and the commutator defines in $\mathfrak{g}$ the structure of a solvable Lie algebra.

Vinberg solvable upper triangular group

The solvable connected and simply connected Lie group

$$G = G(\mathfrak{g}) = \left\{ \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots b_{1n} \\ 0 & b_{22} & b_{23} & \cdots b_{2n} \\ 0 & 0 & 0 & \cdots b_{3n} \\ \dots & \dots & \dots & \dots \\ & 0 & 0 & \cdots b_{nn} \end{pmatrix}, \, b_{ii} > 0 \right\}$$

generated by $\mathfrak{g}$ is called the Vinberg upper triangular group.

Action of Vinberg group on the space of Hermitian T-matrices and the main theorem

Denote by $\mathcal{H} = \{X = ||x_{ij}|| \in \mathcal{A}, x_{ji} = x_{ij}^*\}$ the subspace of Hermitian matrices. It is an algebra w.r.t. the Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX),\ X, Y \in \mathcal{H}$$

Vinberg Theorem

The solvable Lie algebra $\mathfrak{g}$ acts in the space $\mathcal{H}$ by

$$B : \mathcal{H} \ni X \mapsto B(X) := BX + XB^*.$$

The orbit

$$\mathcal{V} := G(\mathrm{Id}\ ) = \{e^B e^{B^*},\ B \in \mathfrak{g}\}$$

is a (homogeneous) convex cone with a simply transitive action of G.

Conversely, any homogeneous convex cone can be obtained by this construction.

Remark about isometric maps

The problem of classification of isometric maps

$$\varphi : \mathrm{U} \otimes \mathrm{V} \to \mathrm{W}$$

betwen Euclidean vector spaces is an important, but very complicated, open problem.
It was solved only in the following two cases:
i) dim $\mathrm{U} =$ dim $\mathrm{V} =$ dim W (A. Hurwitz 1898.)
ii) dim $\mathrm{U} =$ dim W Then the the problem reduces to
a) classification of Clifford $\mathrm{CL}(\mathrm{V})$- modules (M. F. Atiyah, R. Bott and A. S. Shapiro (1964)) and
b) classification of $\mathrm{Spin}(\mathrm{V})$-invariant metric in W (D. A. and V. Cortes) (1997). Both results are given for a pseudo-Euclidean space V of any signature.

Differential geometry of Vinberg cones. Characteristic function.

Vinberg developed the differential geometry of homogeneous convex cones $\mathcal{V} \subset \mathcal{H}$. In particular, he
1) Defined the characteristic function $\varphi(\mathrm{x})$ (called Koszul -Vinberg function). It is the density on the invariant measure. In particular, it is a relative invariant of the automorphism group.
Number of quotation «Vinberg characteristic function» in INTERNET is 680 000,
«Koszul characteristic function» - 136 000 and
«Koszul-Vinberg charcteristic function» - 5050.
2. Calculated the characteristic function for any Vinberg cone.
3. Associated to the characteristic function the invariant Riemannian metric $\mathrm{g} = -\partial^2 \log \varphi$, called canonical Vinberg-Koszul metric, and described its Levi-Civita connection.

4. Described the cone by inequalities.
5. Deascribed the full group of automorphisms of the cone.
6. Proved that any homogeneous convex domain can be obtained as a section of a cone by a hyperplane.

Comments. The section of a Vinberg cone by an affine hyperplane is a domain, and not necessarily homogeneous. However, Vinberg proved that any homogeneous convex domain can be realized as a codimension 1 section of a (classical, i.e. convex) Vinberg cone. A codimension 1 section through the origin, defines a cone.

About applications of Vinberg theory olf homogeneous convex cones

Vinberg theory has many applications to different parts of physics (quantum physics, supergravity, quantum field theory and renormalization), special Kähler and quaternionic Kähler geometry , harmonic analysis , information geometry, Souriau thermodynamics on Lie groups, statistics, convex optimization, combinatorics, numerical integration of differential equations etc.
From point of view of information geometry, Vinberg cones form an important class of manifolds of probability distributions (so-called "exponential family") and the canonical metric of the cone is identified with the Fisher-Rao metric. In Souriau thermodynamics, the Legendre transform of the logarithm of the characteristic function φ is interpreted as the entropy (F. Barbaresco). In convex optimization, the function φ is used as a barrier function, which allows to apply the Newton methods for the determination of the minimum of the given function.

Generalisation to indefinite case

Our goal is to generalized the Vinberg theory of rank 2 and special rank 3 homogeneous convex cone to the indefinite case, when matrix entries x_{ij} belong to pseudo-Euclidean vectors spaces V_{ij}. Most Vinberg results are naturally generalized to this case. The only difference is that the "cone $\mathcal{V}$ of positive defined matrices is not convex anymore.

We calculate also the cubic polynomial h, associated with the characteristic function. Its level set $h = 1$ is a projective special real manifold or very special real manifolds in the sense of De Wit and Van Proeyen.

It is the Riemannian scalar manifold of a supergravity theory in five dimensional Lorentzian space-time (when the metrics g_V, g_S are Euclidean).

Such manifolds give rise to projective special Kähler and quaternionic Kähler manifolds by dimensional reduction of the underlying supersymmetric field theories.

Rank 2 indefinite Vinber cones

Hermitian T-algebras $W = \mathcal{H}_2(V)$ where $V = \mathbb{R}^{p,q}$, is the Jordan algebra

$$\mathcal{H}_2(V) = \left\{ X = \begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} t + x_0 & v \\ v^* & t - x_0 \end{pmatrix} \right\} =$$

$$= \{X = (x_1, v, x_2)\} \simeq \mathbb{R}^{p+1,q+1}$$

Here $x_1, x_2 \in \mathbb{R}, v \in V$. The pseudo-Euclidean metric of signature $(p+1, q+1)$ is $g(X, X) = -\det X = -x_1 x_2 + |v|^2$.

Rank two Vinberg group G

The solvable Vinberg group of upper triangular matrices is a connected simply connected group

$$G = G_2(V) = \left\{ A = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix} \in \mathcal{M}_2 \middle| a_1 > 0, a_2 > 0, a_{12} \in V \right\}$$

It acts in the T-algebra $\mathcal{M}_2$ by

$$A : X \mapsto A \cdot X = AXA^*, \tag{1}$$

where

$$A^* = \begin{pmatrix} a_1 & 0 \\ a_{12}^* & a_2 \end{pmatrix}$$

is the conjugated to A lower triangular matrix.

Rank 2 Vinberg cone $\mathcal{V}_2 = \mathcal{V}_2(V)$

The group g preserves the space $W = \mathcal{H}_2$ of Hermitian matrices and acts simply transitively on the open but not convex Vinberg cone

$$\mathcal{V}_2(V) := G(Id\ ) = \{AA^*,\ A \in G\} \subset W.$$

It can be considered as the cone of formally positive definite matrices

$$\mathcal{V}_2(V) = \{X \in W \mid x_2 > 0, \det X = -g(X,X) > 0\}.$$

In the Euclidean case, it coincides with the Lorentz cone

$$\{X \in W \mid x_1 > 0, x_2 > 0, \det X > 0\} = \{X \in W, tr\ X > 0, , g(X,X) < 0$$

of future directed timelike vectors of the Minkowski space W.
Note that $(AX)A^* = A(XA^*)$, which is no longer true in the rank 3 case.
The Jordan algebra $\mathcal{H}_2 = \mathbb{R}Id\ + \mathcal{H}^0$, where $\mathcal{H}_2^0(V) = \hat{V} \simeq \mathbb{R} \oplus V$ is the traceless subspace of $\mathcal{H}_2$ with the metric $g(X,X) = -\det X$ of signature $(p+1,q)$.

The action of the Clifford algebra $Cl(\hat{V}, -g)$ on the graded $Cl(V)$-module $S = S_0 + S_1$

We denote by

$$\mu : V \otimes S \to S, \ (v, s) \mapsto \mu_v s \in S$$

the Clifford multiplication and consider elements $s = s_0 + s_1 \in S = S_0 \oplus S_1$ as column vectors $s = (s_1, s_0)^t$. Then we may define the product of a matrix $X \in W = \mathcal{H}_2^0(V)$ with a spin vector $s = \binom{s_1}{s_0}$ by

$$\gamma_X s = X \cdot s = \begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_0 \end{pmatrix} = \begin{pmatrix} x_1 s_1 + \mu_v s_0 \\ -\mu_v s_1 + x_2 s_0 \end{pmatrix} . \qquad (2)$$

Then for $X \in \mathcal{H}_2(V)^0 = \hat{V}$, we have

$$\gamma_X^2 = g(X, X) Id \ .$$

Hence, the $Cl(V)$-module $S = S_0 + S_1$ becomes a $Cl(\hat{V})$-module.

The case when $V = \mathbb{K}$ is a division algebra and the Baez-Huerta quantum-mechanical interpretation of Vinberg cone

J. C. Baez and J. Huerta gave a nice quantum mechanical interpretation of the rank 2 Vinberg cones in the case when $V = \mathbb{K}$ is a division algebra. In this case, the graded Clifford $CL(V)$-module $S = S_0 + S_1$ is identified with $\mathbb{K}^2 = \mathbb{K} + \mathbb{K}$, considered as the right $\mathbb{K}$-module. This allows to identify the Jordan algebra $\mathcal{H}_2(\mathbb{K})$ with the algebra $Herm(\mathbb{K}^2)$ of Hermitian operators in $\mathbb{K}^2$. and the Jordan multiplication $X \circ Y = \frac{1}{2}(XY + YX)$ defines on $\mathcal{H}_2(\mathbb{K})$ the structure of a Euclidean Jordan algebra with the Euclidean square norm

$$|X|^2 = \langle X, X \rangle = tr\ X^2 = (x_1^2 + x_2^2 + 2|v|^2).$$

Interpretation of Vinberg cone as the space of quantum observables for a quantum system on the spinor space

The irreducible graded $Cl(\mathbb{K})$-module $S = S_0 \oplus S_1$ (considered as a real Euclidean vector space) is identified with $\mathbb{K}^2 = \mathbb{K} \oplus \mathbb{K}$, where $\mathbb{K}^2$ is considered as the right module of the division algebra $\mathbb{K}$. Then the Jordan algebra $\mathcal{H}_2(\mathbb{K})$ is identified with the algebra of Hermitian operators in the right $\mathbb{K}$-module $S = \mathbb{K} \oplus \mathbb{K}$. Baez and Huerta proposed to identify the algebra $Herm(\mathbb{K}^2)$ with the algebra of quantum (mixed) states of the quantum system. The "positive states ρ, that is states which belong to cone $\mathcal{V}_2(\mathbb{K})$ and normalized by the condition $tr\ \rho = 1$, are considered as observable. The value $g(\rho, X)$ is considered as an expectation value of the observable $\rho \in \mathcal{V}_2$ in the state $X \in W = \mathcal{H}_2$. This interpretation is generalized to any special Vinberg cone, since the Hermitian T-algebra $\mathcal{H}_3(V, S)$ is identified with the algebra of selfdual operators in the space $W = S + V = S_0 + S_1 + V$.

Graded spinor modules associated to division algebras

n	$\mathbb{K}$	$Cl_{n,0}$	$Cl^0_{n,0}$	$S = S_0 \oplus S_1$	$Spin(n)$	$n(1,-1)$
1	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$	$\{Id\}$	1
2	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$	$U(1)$	1
4	$\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$	$SU(2) \times SU(2)$	1
8	$\mathbb{O}$	$\mathbb{R}(16)$	$\mathbb{R}(8)$	$\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$	$Spin(8)$	1

: Division algebras $\mathbb{K} \cong \mathbb{R}^n$, $n = 1, 2, 4, 8$, the Clifford algebra $Cl(\mathbb{K}) = Cl_{n,0}$, even Clifford algebra $Cl^0_{n,0}$, graded Clifford module $S = S_0 \oplus S_1 = \mathbb{K}^2$ and its decomposition $\mathbb{K} \oplus \mathbb{K}$ as a module of $Cl^0_{n,0}$, the spin group $Spin(n)$ and the number $n(1,-1) = n(\sigma,\tau)|_{\sigma=1,\tau=-1}$ of independent admissible symmetric ($\sigma = +1$) bilinear forms on $S = \mathbb{K}^2$ of type $\tau = -1$. Similar tables can be given for other dimension and signatures of V.

Special Rank 3 Vinberg algebras and cones

Let (V, g) be a pseudo-Euclidean vectort space and $(S = S_0 + S_1, g_S)$ a $\mathbb{Z}_2$-graded Clifford module with an admissible metric g_S i.e. a pseudo-Euclidean metric such that the Clifford multiplication $\mu_v : s \to v \cdot s$ is skew-symmetric and $g_S(S_0, S_1) = 0$. We denote by $* : v \to v^* = g_V \circ v$, $s \to s^* := g_S \circ s$ the metric isomorphism $V \to V^*, S \to S^*$.

The Vinberg Hermitian matrix algebra $\mathcal{H}_3 = \mathcal{H}_3(V, S)$ associated to (V, S) consists of the Hermitian matrices of the form

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{pmatrix} = \begin{pmatrix} x_1 & X_3 & X_2 \\ X_3^* & x_2 & X_1 \\ X_2^* & X_1^* & x_3 \end{pmatrix} = \begin{pmatrix} x_1 & v & s_1 \\ v^* & x_2 & s_0 \\ s_1^* & s_0^* & x_3 \end{pmatrix}$$

where $x_i \in \mathbb{R}$, $X_3 = X_{12} = v \in V$, $X_1 = X_{23} = s_0 \in S_0$, $X_2 = X_{13} = s_1 \in S_1$.

The Vinberg upper triangular group $G = G(V, S)$

The Vinberg group G consists of the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ 0 & \alpha_2 & a_{23} \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad (3)$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $a_{12} \in V$, $a_{23} \in S_0$, $a_{13} \in S_1$. It is a connected simply connected solvable Lie group with respect to matrix multiplication.

The dual low triangular group G^* consists of the transposed matrices of the form

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ a^*_{12} & \alpha_2 & 0 \\ a^*_{13} & a^*_{23} & \alpha_3 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{g}(V, S)$ of G consists of all upper triangular matrices

$$A = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ 0 & \alpha_2 & a_{23} \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

The Lie algebra acts on the space $\mathcal{H}_3$ by the formula

$$\mathfrak{g} \ni A : X \mapsto \frac{1}{2}(AX + XA).$$

It induces an action of the Lie group G in $\mathcal{H}_3$.

Homogeneous cone associated to indefinite special Hermitian T-algebra

The following theorem extends the result by Vinberg, who considered the positive definite case.

Theorem. The orbit $\mathcal{V} = G \cdot Id \ = \{AA^*, \ A \in G\}$ of the identity matrix is a homogeneous cone with the simply transitive action of G. If g_V and g_S are Euclidean metrics, then the cone is convex.

The cone $\mathcal{V}$ is called the special Vinberg cone (or cone of positively defined matrices) associated to a metric Clifford module (S, g_S).

Vinberg fundamental polynomials

Following Vinberg, consider the real-valued homogeneous polynomials of degree 1,2, and 4 , given by

$$
\begin{aligned}
p_3(X) = & \quad X_{33}, \\
p_2(X) = & \quad X_{33}X_{22} - |X_{23}|^2, \\
p_1(X) = & \quad (X_{33}X_{11} - |X_{13}|^2)(X_{33}X_{22} - |X_{23}|^2) \\
- & \quad (X_{33}X_{12} - X_{13}X_{32})(X_{33}X_{21} - X_{23}X_{31}) \\
= & \quad x_3[x_1x_2x_3 - x_1|X_1|^2 - x_2|X_2|^2 - x_3|X_3|^2 + 2(X_2 \cdot X_1^*) \cdot X_3^*]
\end{aligned}
$$

Lemma about simply transitive action of the Vinberg group on the cone

The following lemma shows that G acts simply transitively in $\mathcal{V}$ and elements $\alpha_1, \alpha_2, \alpha_3, a_{12}, a_{23}, a_{13}$ gives a global coordinates in the open cone $\mathcal{V}$.

Lemma i) If $X = AA^* \in \mathcal{V}, A \in G$, then

$$p_3(X) = \alpha_3^2,$$

$$p_2(X) = \alpha_2^2\alpha_3^2,$$

$$p_1(X) = \alpha_1^2\alpha_2^2\alpha_3^4.$$

ii) The matrix $A \in G$ is canonically reconstructed from $X = AA^* \in \mathcal{V}$ as follows

$$\alpha_3^2 = p_3(X),$$

$$\alpha_2^2 = \frac{p_2(X)}{p_3(X)}$$

$$\alpha_1^2 = \frac{p_1(X)}{p_2(X)p_3(X)}$$

$$a_{13} = \frac{X_{13}}{\sqrt{p_3(X)}},$$

$$a_{23} = \frac{X_{23}}{\sqrt{p_3(X)}},$$

$$a_{12} = \frac{\sqrt{p_3(X)}}{\sqrt{p_2(X)}} X_{12} - \frac{1}{\sqrt{p_2(X)p_3(X)}} X_{13} \cdot X_{23}^*.$$

Vinberg-Koszul characteristic function and invariant metric in special Vinberg cones

We consider $\mathcal{H} = \mathcal{H}_3 = \mathbb{R}^3 \oplus V \oplus S = \mathbb{R}^3 \oplus W$ as a pseudo-Euclidean vector space with the metric given by the standard Euclidean metric of $\mathbb{R}^3$ and the metric $g_W = g_V \oplus g_S$ on $W = V \oplus S$.

The group $G = G(V, S)$ acts as a linear group on the real vector space $\mathcal{H}$. We denote by $\det_{\mathcal{H}} A$ the determinant of an element $A \in G$, considered as a linear transformation of the real vector space $\mathcal{H}$. It depends only on the diagonal part $\Lambda = diag(\alpha_1, \alpha_2, \alpha_3)$ of A. More precisely,

$$det_{\mathcal{H}} A = \alpha_1^{2+n+N} \alpha_2^{2+n+N} \alpha_3^{2+2N}$$

where $n = \dim V$, $N = \dim S$.

In the case $n = 0$, the dimensions $N_0 = \dim S_0$ and $N_1 = \dim S_1$ do not necessarily coincide. In that case,

$\det_{\mathcal{H}} A = \alpha_1^{2+N_1} \alpha_2^{2+N_0} \alpha_3^{2+N_0+N_1}$.

The Vinberg-Koszul characteristic function of the cone $\mathcal{V}$ is defined as

$$\chi(X) = (det_{\mathcal{H}} A)^{-1} \text{ where } X = AA^*.$$

The functions $\alpha_1^2, \alpha_2^2, \alpha_3^2$ can be written as

$$\begin{aligned} \alpha_3^2 &= p_3(X) = x_3 \\ \alpha_2^2 &= \frac{p_2(X)}{p_3(X)} = \frac{x_2x_3 - |X_1|^2}{x_3} \\ \alpha_1^2 &= \frac{p_1(X)}{p_2(X)p_3(X)}, \end{aligned} \tag{4}$$

where

$$p_1(X) = x_3[x_1x_2x_3 - x_1|X_1|^2 - x_2|X_2|^2 - x_3|X_3|^2 + 2(X_2 \cdot X_1^*) \cdot X_3^*]$$

$$p_2(X) = x_2x_3 - |X_1|^2,$$

$$p_3(X) = x_3.$$

The cubic polynomial

$$h(X) = \frac{p_1(X)}{p_3(X)} = (\alpha_1\alpha_2\alpha_3)^2$$

is called the cubic potential of the cone $\mathcal{V}$ and the hypersurface $\mathcal{V}_1 = \{h(X) = 1\} \subset \mathcal{V}$ is called the canonical or determinant hypersurface of the cone.

Theorem

i) The Vinberg cone $\mathcal{V}$ is described in terms of the Vinberg polynomials by the inequalities

$$p_i(X) > 0,\ i = 1, 2, 3,$$

or, equivalently, in terms of the cubic potential $h(X)$ by the the three inequalities

$$\begin{aligned} h(x) &= x_1x_2x_3 - x_1|X_1|^2 - x_2|X_2|^2 - x_3|X_3|^2 + 2(X_2 \cdot X_1^*) \cdot X_3^* > 0, \\ & x_2x_3 - |X_1|^2 > 0 \quad \text{and} \quad x_3 > 0. \end{aligned}$$

ii) The connected component of the level set $\{X \in Herm(W) \mid h(X) = 1\}$ which contains the identity matrix coincides with the canonical hypersurface $\mathcal{V}_1 = \{X \in \mathcal{V} \mid h(X) = 1\}$ of the cone and $\mathcal{V} = \mathbb{R}^+ \cdot \mathcal{V}_1$.
iii) The characteristic function $\chi(X)$ is given by

$$\chi^{-1}(X) = \alpha_1^{2+n+N}\alpha_2^{2+n+N}\alpha_3^{2+2N} = h(X)^{1+\frac{1}{2}(n+N)}p_3(X)^{\frac{1}{2}(N-n)}.$$

Vinberg cones as a psudo-Riemannian homogeneous manifold

Theorem

i)A Vinberg cone $\mathcal{V}$ is a homogeneous pseudo-Riemannian G-manifold with the metric $g_{\mathcal{V}} := -\frac{1}{3}\partial^2 \log h$.

ii) The canonical hypersurface $\mathcal{V}_1$ with induced metric $g_{\mathcal{V}_1} = -\frac{1}{3}\partial^2 h|_{\mathcal{V}_1}$ is a homogeneous pseudo-Riemannian manifold of the unimodular group $G_1 = \{\alpha_1\alpha_2\alpha_3 = 1\} \subset G$.

iii) If the scalar product $g_W = g_V \oplus g_S$ is positive definite, the metrics $g_{\mathcal{V}}$ and $g_{\mathcal{V}_1}$ are complete Riemannian metrics, the domain $\mathcal{V}$ is convex and the canonical hypersurface is strictly convex.

The homogeneous projective special real manifolds classified by De Wit and Van Proyen and Cortes are precisely the homogeneous canonical hypersurface $\mathcal{V}_1$ of special Vinberg cones of rank 3 and Vinberg cones of rank 2. They are of interest in high energy theoretical physics as they are exactly the homogeneous scalar manifolds of supergravity coupled to vector multiplets in five space-time dimensions.

Image of the canonical hypersurface of a special cone under r-map and c-map

Under the supergravity r-map the above homogeneous projective special real manifolds give rise to homogeneous projective special Kähler manifolds, which in turn give rise to homogeneous quaternionic Kähler manifolds with solvable transitive group of isometries. Moreover, any homogeneous quaternionic Kähler manifold of a solvable transitive group, classified by D.A., admits such realisation, with the exception of the quaternion Lobachevski space $Sp(1,n)/Sp(1) \times Sp(n)$.