From gluons to gravitons via homotopy algebras: Einstein as Yang–Mills Squared

Leron Borsten

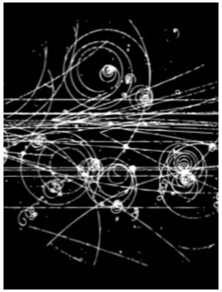
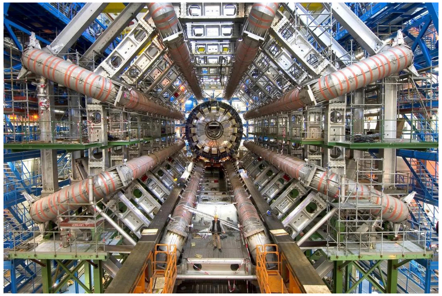
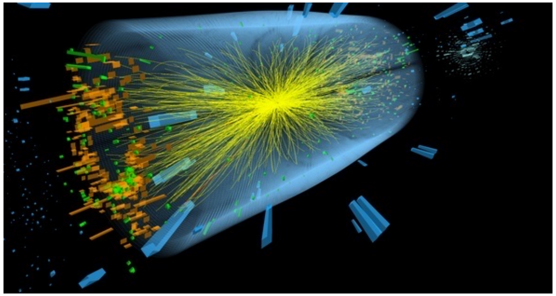
Maxwell Institute for Mathematical Sciences, Edinburgh
Heriot–Watt University, Edinburgh
Imperial College London

42th Winter School Geometry And Physics
Czech Republic, Srní, 15-22 January 2022

Twin Pillars of XX-century Physics

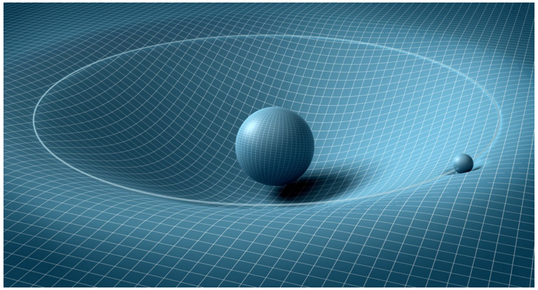
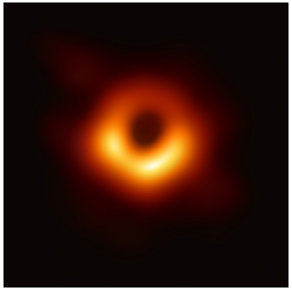
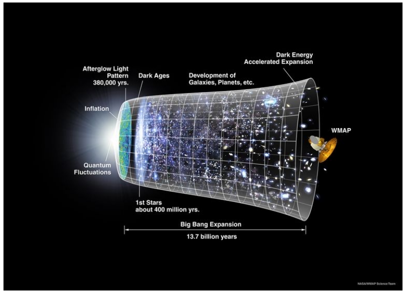
Quantum (field) theory

- Elementary particles and their fundamental interactions (excluding gravity)

General relativity

- Gravity: planetary orbits, black holes and the evolution of the entire universe itself

The fundamental forces of Nature are ostensibly described by two distinct frameworks

The Standard Model of Particle Physics

See Konrad's notes!

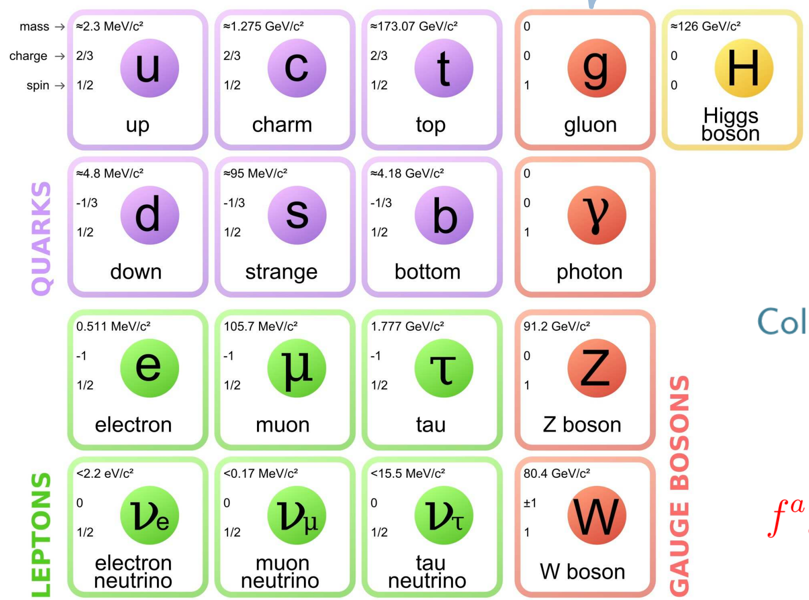
- The electroweak and strong forces: Maxwell/Yang–Mills gauge theories

Yang–Mills

Forces described by gauge fields of Yang-Mills theory

$$A_\mu{}^a$$

Quantum particles: gluons



$\mu = 0, 1 \ldots 3$

SO$(1,3)$ spacetime Lorentz index

$a = 1, 2 \ldots 8$

SU(3) colour index of strong force

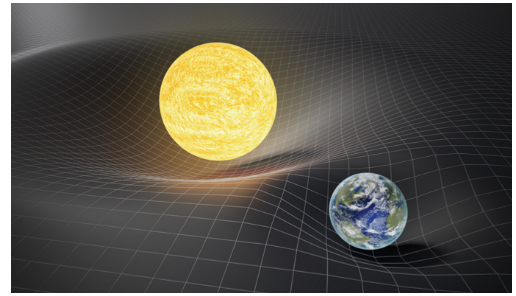
Colour Lie algebra: $[t_a, t_b] = f_{ab}{}^c t_c$

$$f^a{}_{be} f^e{}_{cd} + f^a{}_{ce} f^e{}_{db} + f^a{}_{de} f^e{}_{bc} = 0$$

Jacobi identity

General Relativity

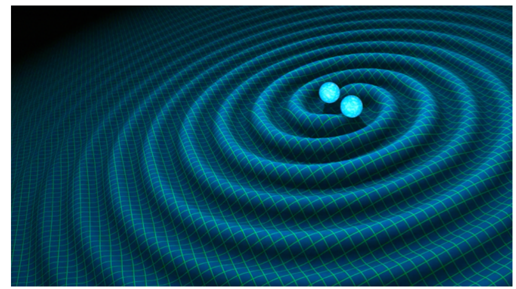
- The Standard Model plays out on fixed stage of spacetime
- Gravity is the stage itself: gravity is geometry



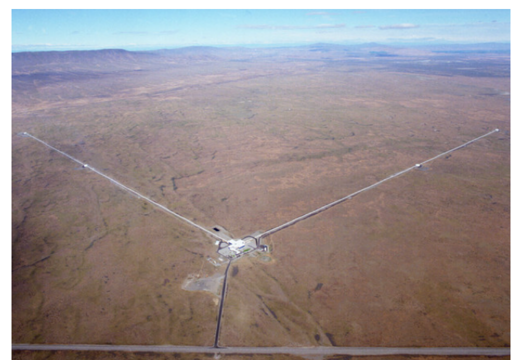
Metric: curvature of spacetime

$g_{\mu\nu}$

Perturbations/quanta



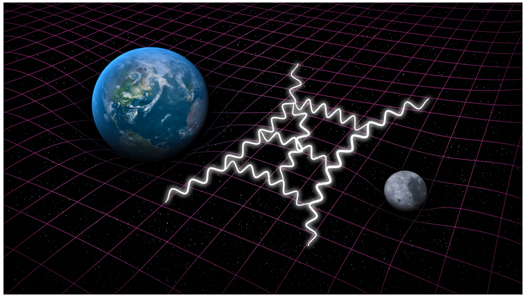
Classical gravity waves
Quantum particles: gravitons



- 2017 Nobel: gravity waves detected LIGO/VIRGO

A Schism in the Fundamental Forces of Nature

- General relativity is naively incompatible with quantum theory



Diverges at two loops [Goroff-Sagnotti '85]

- Black holes challenge the very foundations of quantum theory

Hawking radiation appears to violate unitarity [Hawking '74]

- The problem of quantum gravity - can the forces be reunited?

Gravity and gauge theory

- Gravity as a gauge theory:
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence ['t Hooft '93; Susskind '94; Maldacena '97]

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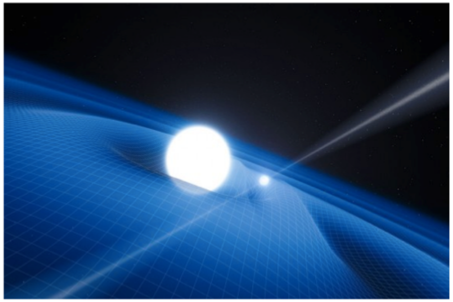
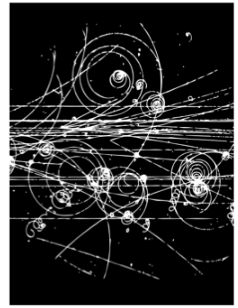
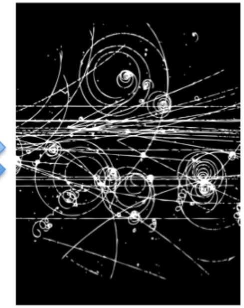
  - Holographic principle - AdS/CFT correspondence ['t Hooft '93; Susskind '94; Maldacena '97]
- Here, we appeal to a third and (superficially) independent perspective:

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

Gravity = Gauge × Gauge

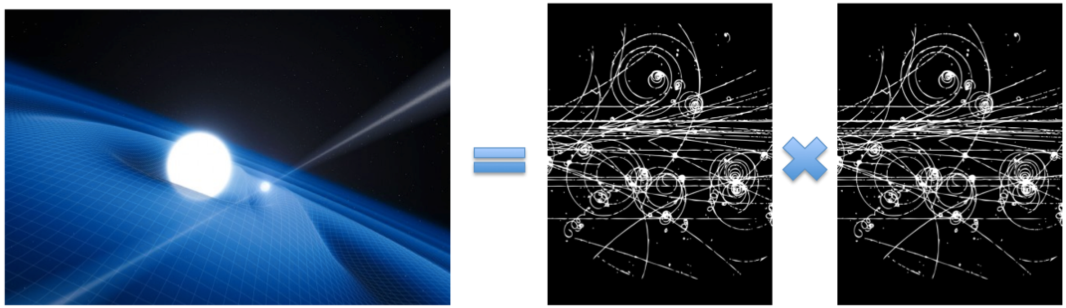
 =  × 

$$g_{\mu\nu} \qquad A_\mu{}^a \qquad A_\nu{}^b$$

- Is gravity the double copy of the other fundamental forces of Nature?

- Long history, many guises [Feynman, Papini, Kawai–Lewellen–Tye, Bern, Hodges. . .]

Gravity = Gauge × Gauge



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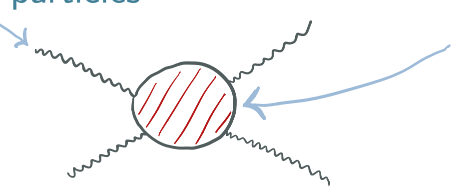
- Long history, many guises [Feynman, Papini, Kawai–Lewellen–Tye, Bern, Hodges. . .]

- Renaissance: Colour–Kinematics (CK) duality conjecture and double copy of gauge theory and gravity scattering amplitudes [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Scattering Amplitudes

→ Physical observables tested at particle accelerators (e.g. Large Hadron Collider)

$$\mathcal{A}^4_{\text{gluons}}$$



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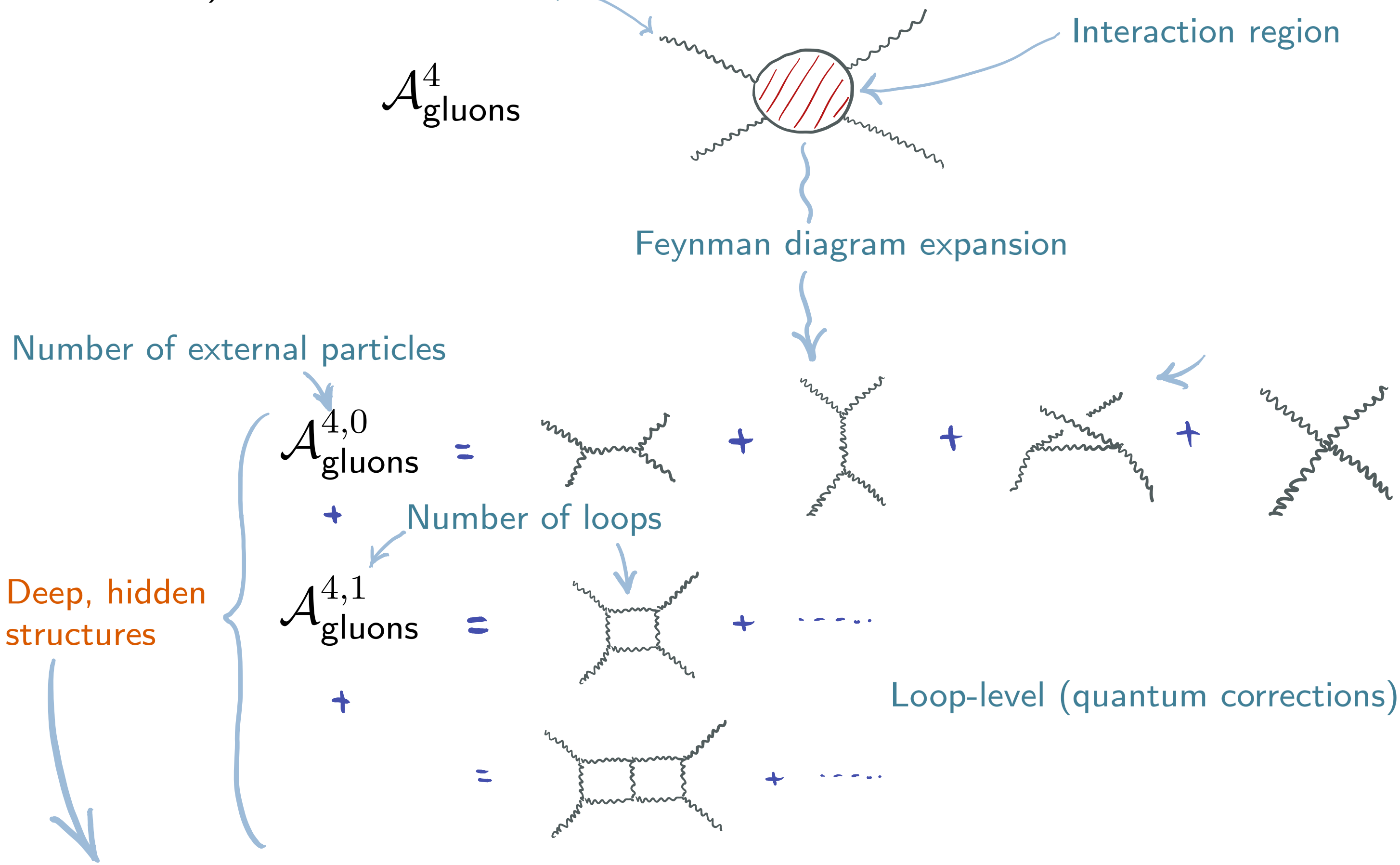


→ New insights into the underlying theories themselves

Colour–Kinematics Duality

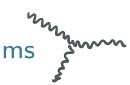
- Amplitude for gluons to scatter (very) schematically:

Colour numerators $c \sim f_{ab}{}^{c} f_{cd}{}^{e} \cdots$
colour/gauge group data of gluons

Kinematic numerators $n \sim \varepsilon_{\mu} p^{\mu} \cdots$
polarisation and momentum data of gluons

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_{i} \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Sum over cubic diagrams



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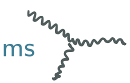
Sum over cubic diagrams 

Kinematic Jacobi identity

- Bern-Carrasco-Johansson CK duality conjecture 2008:

Jacobi identity $$c_i + c_j + c_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0$$

- Proven at tree level (zeroth order in $\hbar$)
- Conjectured at loop level (but see later!) with highly non-trivial examples

The Double Copy Prescription

- Assuming CK duality is realised, gravity comes for free:

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_i \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Double copy kinematics

$$c_i \longrightarrow n_i$$

$$\mathcal{A}^{n,L}_{\text{gravitons}} \sim \sum_i \int_{\text{loops}} \frac{n_i n_i}{d_i}$$

- 'Gluons for (almost) nothing, gravitons for free' JJ Carrasco

Implications and Applications

Computationally powerful: facilitates previously intractable calculations

- Miraculous cancellations: perturbatively finite quantum field theory of gravity?

- Black holes collisions and gravity wave astronomy: pushing the precision frontier

Implications and Applications

Computationally powerful: facilitates previously intractable calculations

- Miraculous cancellations: perturbatively finite quantum field theory of gravity?
- Black holes collisions and gravity wave astronomy: pushing the precision frontier

Conceptually provocative: is gravity really the square of gauge theory?

- Does CK duality and the double copy actually hold?
- What *is* CK duality?
- Can it be taken beyond amplitudes?

$\longrightarrow$ Lift CK duality and double copy to field theory?

Field Theory Colour-Kinematics Duality and Double Copy

- Introduce field theory realisation of CK duality the double copy
  [LB-Hughes-Duff-Nagy '14; Anastasiou-LB-Hughes-Duff-Nagy '14, 18'. . .]

Kinematic algebra index

Colour algebra index

Nakanishi-Lautrup field

$$A^{\mathbb{A}a} = (\, A_\mu{}^a, \;\; c^a, \;\; \bar{c}^a, \;\; b^a, \;\; B_{\mu\nu}{}^a \ldots)$$

Auxiliary fields

Faddeev-Popov ghosts

Field Theory Colour-Kinematics Duality and Double Copy

- CK duality: can be realised as an infinite dimensional anomalous symmetry of Yang–Mills Batalin–Vilkovisky (BV) action [LB, Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann, Martin Wolf (BJKMSW) '21]

$$S^{\mathrm{YM}}_{\mathrm{BV}}[A^{Aa}, A^{+}{}_{Aa}] \quad \Rightarrow \quad F_{AB}{}^{C} \quad \leftrightarrow \quad f_{ab}{}^{c}$$

YM BV-action

Anti-fields

Infinite dimensional kinematic Lie algebra

Field Theory Colour-Kinematics Duality and Double Copy

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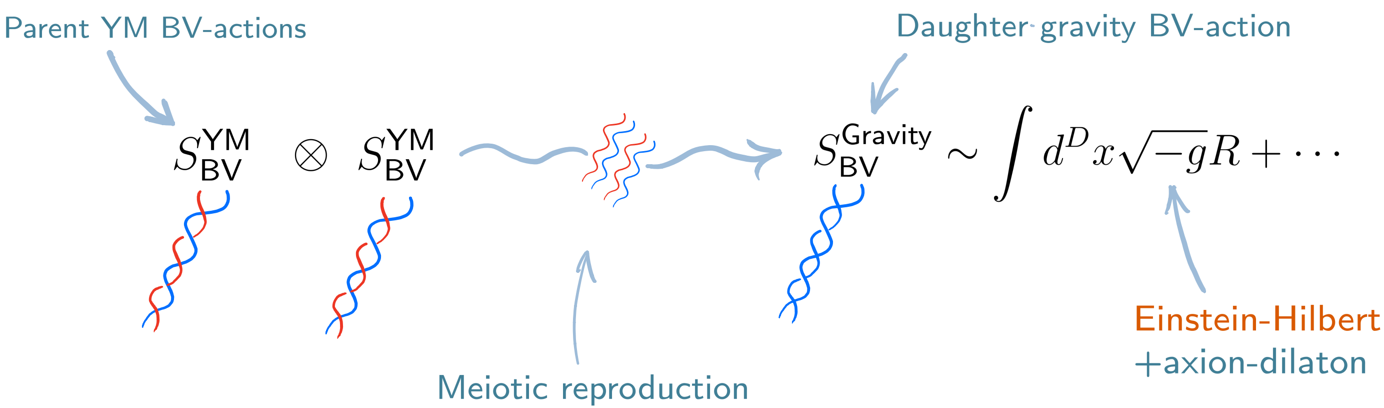
- BV-action double copy [BJKMSW '20; '21]



- Double copy origin of symmetries to all orders in perturbation theory:

$$\underbrace{(\text{gauge, global susy, R-sym}\ldots)}_{\text{(super) Yang–Mills symmetries}} \longrightarrow \underbrace{(\text{diffeomorphism, local susy, U-duality}\ldots)}_{\text{(super)gravity symmetries}}$$

Field Theory Colour-Kinematics Duality and Double Copy

- The double copy holds to all loops [BJKMSW '20]

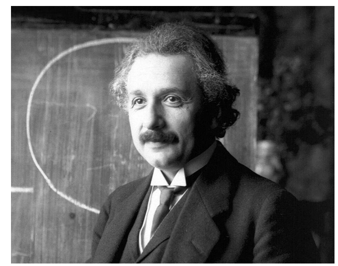
Quantum gravity is Yang-Mills theory squared!
(Well, perturbatively and coupled to the axion-dilaton)

Yang-Mills

$\otimes$

Yang-Mills

$=$

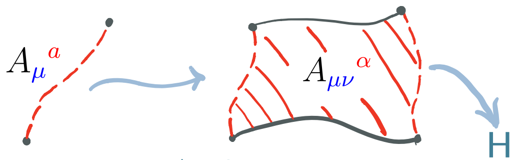


Einstein

- Revealed mathematical structure: homotopy algebras [BJKMSW '21]

Homotopy Algebras

- Higher symmetry and gauge theory is everywhere: condensed matter, M-theory... (see lectures of Konrad Waldorf)



Wilson line/surface

Higher symmetries

$$A_\mu{}^a \longrightarrow A_\mu{}^a, A_{\mu\nu}{}^\alpha, A_{\mu\nu\rho}{}^i, \ldots$$

Tower of higher gauge fields

- Higher symmetry $\longrightarrow$ homotopy algebras: intersection of category theory, topology, geometry and algebra

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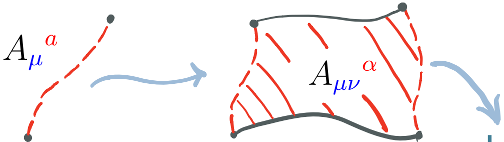


Higher symmetries

$$A_\mu{}^a \longrightarrow A_\mu{}^a, A_{\mu\nu}{}^\alpha, A_{\mu\nu\rho}{}^i, \ldots$$

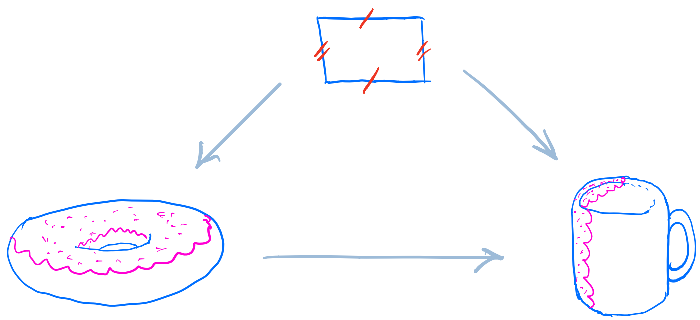
Tower of higher gauge fields

- Higher symmetry $\longrightarrow$ homotopy algebras: intersection of category theory, topology, geometry and algebra

- Generalise familiar (matrix, Lie. . .) algebras to include higher products:

$$[-,-] \qquad \longrightarrow \qquad [-],\ \ [-,-],\ \ [-,-,-],\ \ [-,-,-,-]\cdots$$

Jacobi relation — Homotopy Jacobi relations

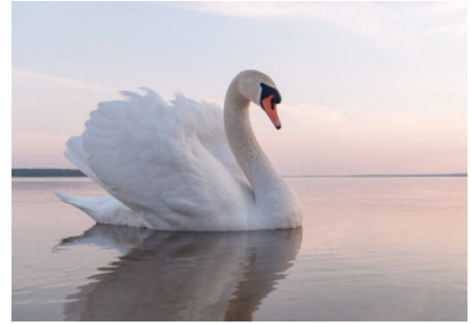


- Homotopy Lie L_∞-algebras: string field theory, quantum field theory, condensed matter/higher Berry connections. . .

The Homotopy Algebra of Colour-Kinematics Duality

- CK duality: kinematic algebra
  Hands on quantum field theory
- Q: but what is it?

The Homotopy Algebra of Colour-Kinematics Duality



- CK duality: kinematic algebra
  Hands on quantum field theory

- A: $BV_\infty^\square$ homotopy algebra
  Abstract mathematics [BJKMSW '22 (to appear)]

Gluons, quarks. . .
Scattering amplitudes
Symmetries of Nature

Michel Reiterer '19

$BV_\infty^\square$-algebras — Homotopy transfer — Exact CK duality

$BV_\infty^\square$-operad — Koszul resolution — $BV^\square$-operad

Cat(Hodge)

Symmetric monoidal category

Twisted tensor product of (co)algebras

Homotopy algebras
Koszul duality
Operads
Symmetric monoidal categories

$H^\square$ — $H^\square$-modules — Hodge Complexes

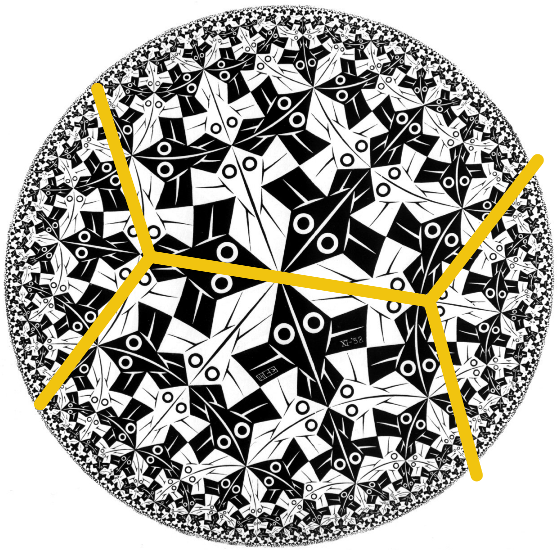
Hopf algebras with central element $\square$

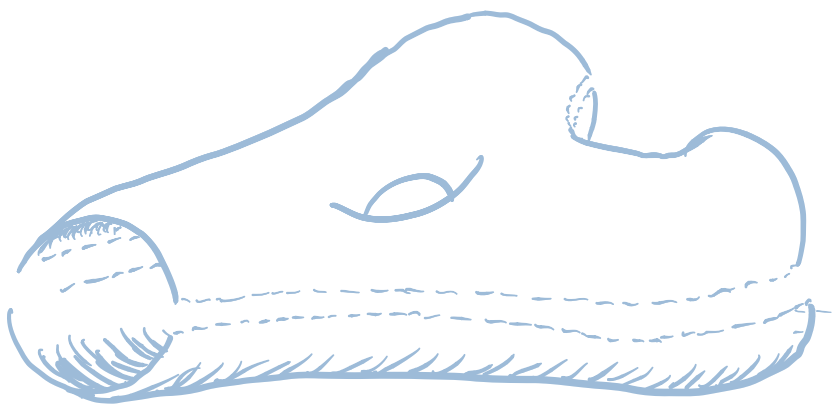
Natural generalisation of familiar Hodge–de Rham complex

- CK duality: a symmetry of Nature as a mug is a donut!

Future Directions

- Computational advantages for applications
- Zoology of CK dual and double copy theories,
  e.g. Bagger–Lambert–Gustavsson Chern–Simons-matter theories
- Curved backgrounds, e.g. (anti) de Sitter
- 'Closed $=$ Open $\times$ Open' string (field) theory





Order of Events

Lecture 1 Colour–kinematics duality and the double-copy

Lecture 2 Off-shell colour–kinematics and the BRST Lagrangian double copy

Lecture 3 The homotopy algebra of colour–kinematics duality

Lecture 1: Colour–kinematics duality and the double-copy

Yang–Mills theory

Connection on princ. G-bundle
See K. Waldorf
(here trivial w/ $M \cong \mathbb{R}^{d,1}$)

- Classical Yang–Mills action

$$S^{\mathrm{YM}}_{\mathrm{classical}} := \frac{1}{2g^2}\int \mathrm{tr} F \wedge \star F$$
$$= \int \mathrm{d}^d x \left\{ -\tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} \right\}$$

$$F := dA + A \wedge A, \qquad F^a_{\mu\nu} \;:=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a g A^b_\mu A^c_\nu, \quad A = g A^a_\mu dx^\mu \otimes t_a$$

- Invariant under gauge transformations

$$\delta_\theta A = \nabla\theta = d\theta + [A, \theta], \qquad (\nabla_\mu \theta)^a \;=\; \partial_\mu \theta^a + g f_{bc}{}^a A^b_\mu \theta^c$$

- Expanding:

$$S^{\mathrm{YM}}_{\mathrm{classical}} \sim \int \mathrm{d}^d x \left\{ A(\Box - \partial\partial)A + gAA\partial A + g^2 AAAA \right\}$$

- Kinematic operator $(\Box - \partial\partial)$ has kernel

Yang–Mills theory

- Gauge fix using BRST formalism $\theta \to c$, $Q^2 = 0$:

$$QA = \nabla c, \quad Qc = [c, c], \quad Q\bar{c} = b, \quad Qb = 0$$

with gauge-fixing fermion $\Psi = -\mathrm{tr}\bar{c}(\frac{\xi}{2}b - G[A])$:

$$S^{\mathrm{YM}}_{\mathrm{BRST}} = S^{\mathrm{YM}}_{\mathrm{classical}} + \int \star Q\Psi$$
$$= \int \mathrm{d}^d x \left\{ -\tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \bar{c}_a \partial^\mu (\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\}$$

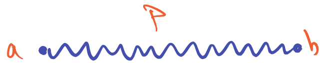
- Physical states in Q-cohomology $\longrightarrow$ asymptotic Hilbert space with unitary S-matrix

- Extended psuedo-Hillbert BRST space

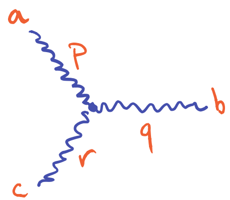
$$A_{\mathrm{phys}}, \quad A_{\mathrm{forward}}, \quad A_{\mathrm{backward}} \equiv b, \quad c, \quad \bar{c}$$

Yang–Mills Feynman diagrams

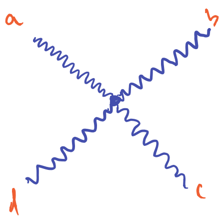
Functionally varying $\frac{\delta^n S}{(\delta A)^n}\Big|_{A=0}$ for n-point Feynman rules (Feynman gauge):



$$= -\frac{i\eta_{\mu\nu}\delta^{ab}}{p^2}$$



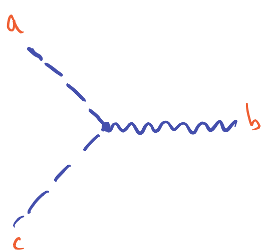
$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$



$$= -ig^2\left[f^{abx}f_x{}^{cd}\eta^{\mu\nu\rho\sigma} + f^{adx}f_x{}^{bc}\eta^{\mu\sigma\nu\rho} + f^{acx}f_x{}^{db}\eta^{\mu\rho\nu\sigma}\right]$$

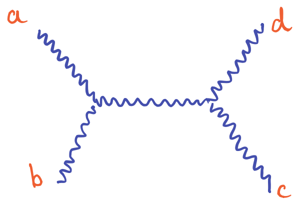
$\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}$

$$= \frac{i\delta^{ab}}{p^2}$$

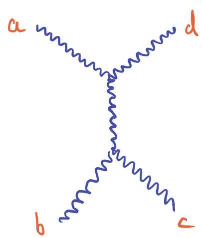


$$= -gf^{abc}p^\mu$$

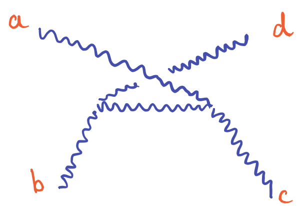
Tree-level 4-point colour–kinematics duality



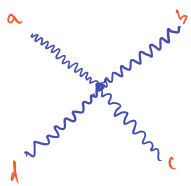
$$= \; -\mathrm{i}g^2 \frac{f^{abe} f_e{}^{cd} \mathrm{n}_s}{s} \; =: \; -\mathrm{i}g^2 \frac{\mathrm{c}_s \mathrm{n}_s}{s}$$



$$= \; -\mathrm{i}g^2 \frac{f^{aed} f_e{}^{bc} \mathrm{n}_t}{t} \; =: \; -\mathrm{i}g^2 \frac{\mathrm{c}_t \mathrm{n}_t}{t}$$



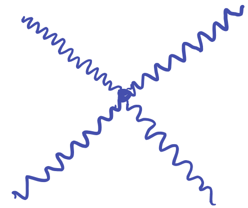
$$= \; -\mathrm{i}g^2 \frac{f^{aec} f_e{}^{db} \mathrm{n}_u}{u} \; =: \; -\mathrm{i}g^2 \frac{\mathrm{c}_u \mathrm{n}_u}{u}$$



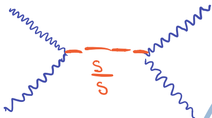
$$= \; -\mathrm{i}g^2 \left(\mathrm{c}_s \mathrm{n}_s^{(4)} - \mathrm{c}_t \mathrm{n}_t^{(4)} - \mathrm{c}_u \mathrm{n}_u^{(4)} \right)$$

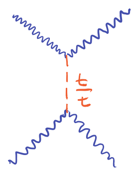
$$\mathrm{n}_s \; = \; 4\left[(\varepsilon_1 \cdot p_2)\varepsilon_2 - (\varepsilon_2 \cdot p_1)\varepsilon_1 + \tfrac{1}{2}(\varepsilon_1 \cdot \varepsilon_2) p_{12}\right] \cdot \left[(\varepsilon_3 \cdot p_4)\varepsilon_4 - (\varepsilon_4 \cdot p_3)\varepsilon_3 + \tfrac{1}{2}(\varepsilon_3 \cdot \varepsilon_4) p_{34}\right]$$

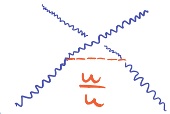
Tree-level 4-point colour–kinematics duality



$$\frac{1}{s} c_s s n_s^{(4)} - \frac{1}{t} c_t t n_t^{(4)} - \frac{1}{u} c_u u n_u^{(4)}$$





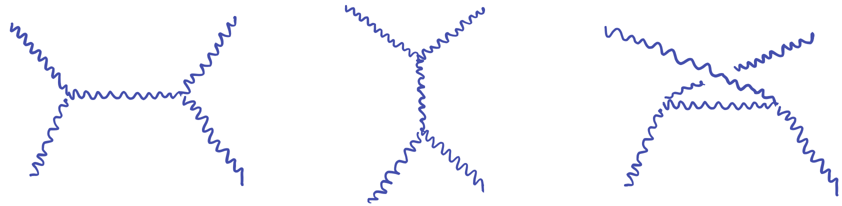


$$\frac{c_s\left(n_s + s n_s^{(4)}\right)}{s} \qquad \frac{c_t\left(n_t - t n_t^{(4)}\right)}{t} \qquad \frac{c_u\left(n_u - u n_u^{(4)}\right)}{u}$$

Tree-level 4-point colour–kinematics duality

- Amplitude now sum over purely trivalent diagrams:

$$A_4 \propto \frac{c_s \tilde{n}_s}{s} + \frac{c_t \tilde{n}_t}{t} + \frac{c_u \tilde{n}_u}{u}$$



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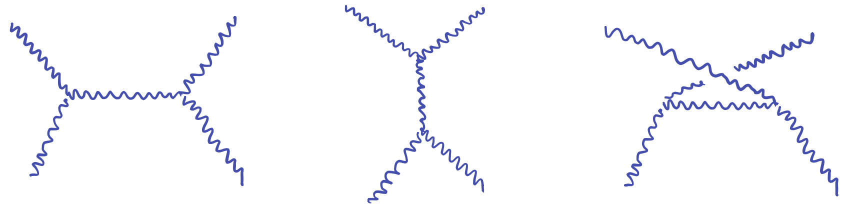


- Obvious (by Jacobi): $c_s - c_t - c_u = 3 f^{ea[b} f_e{}^{cd]} \; = \; 0$

- Exercise: show that [Zhu '80]

$$\tilde{n}_s - \tilde{n}_t - \tilde{n}_u = 0$$

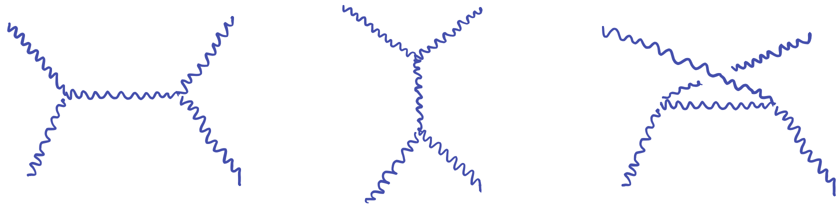
  Hint: Recall $p_i \cdot \varepsilon_i(p_i, q_i) = p_i^2 = 0$ and there is freedom in the choice of reference vectors q_i

- Kinematics appears to be playing by the same rules as the colour!

Amplitudes and cubic diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$



- c_i: colour numerator, built from f^{abc}, read off diagram i
- n_i: kinematic numerator, built from p, ε ← Non-unique
- d_i: propagator, $\prod_{\text{int. lines}} p^2$, read off diagram i

Amplitudes and cubic diagrams

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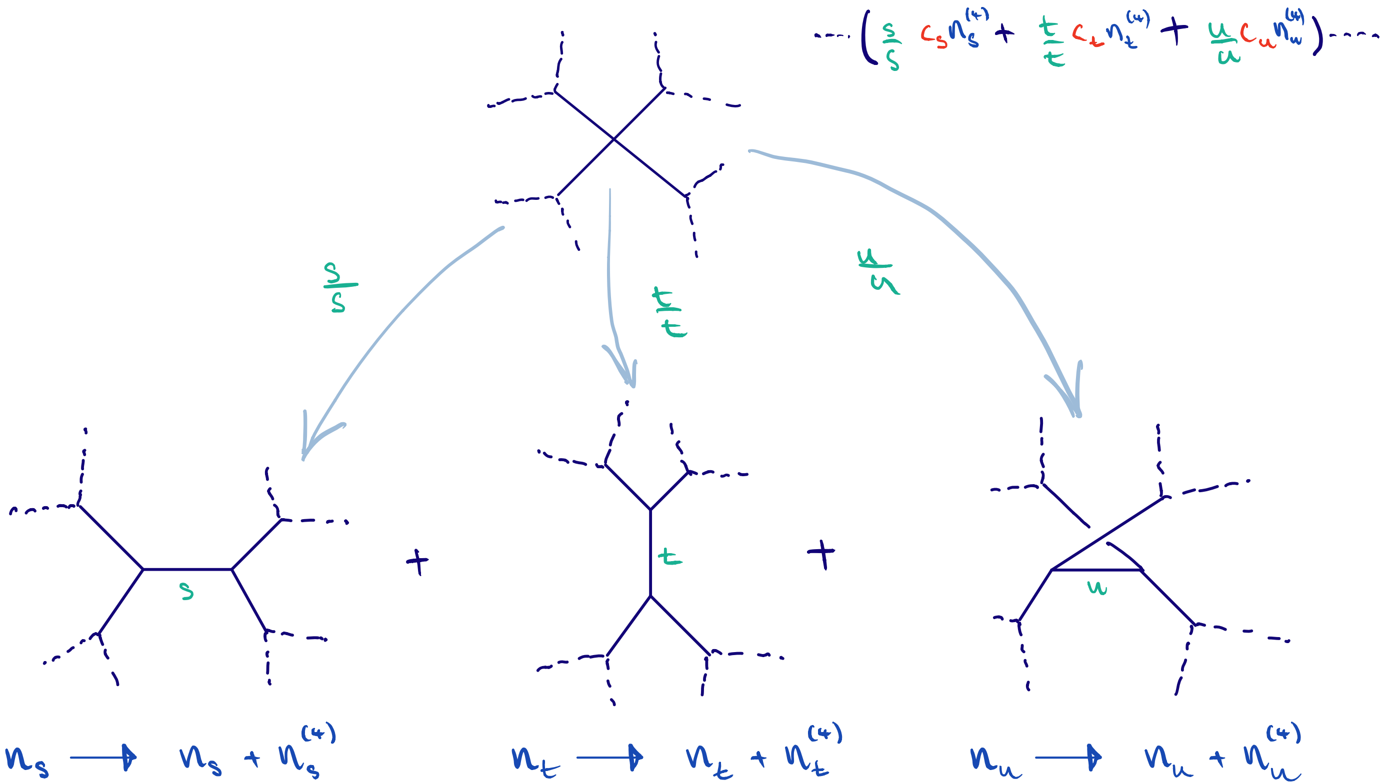


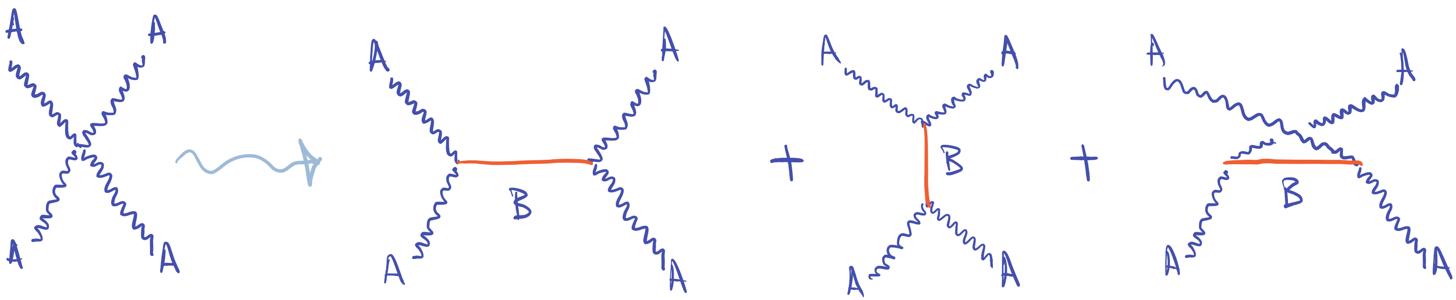
Amplitudes and cubic diagrams

- Can be realised in the YM Lagrangian through auxiliary fields:

$$g^2 [A_\mu, A_\nu][A^\mu, A^\nu] \longrightarrow \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu} [A^\mu, A^\nu]$$

[Bern-Dennen-Huang-Kiermaier '10]

$B \sim \frac{1}{\sqrt{2}\,\Box} g\, \partial [A, A]$



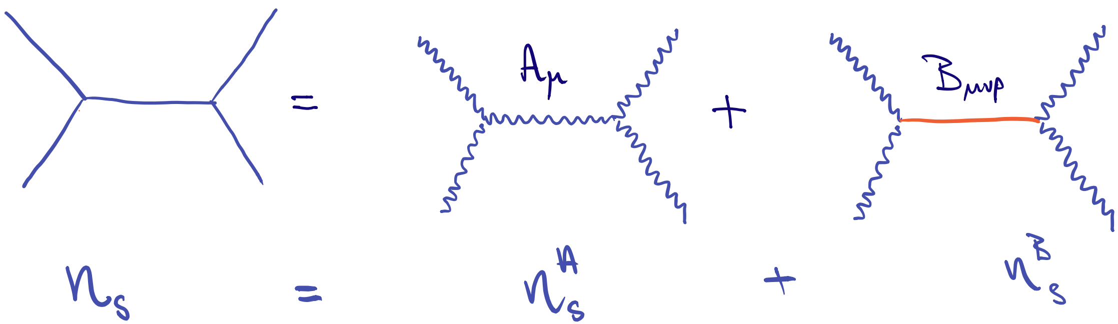
Amplitudes and cubic diagrams

- Feynman diagrams give 'cubic' amplitudes directly:

$$A^{n,L}_{\text{YM}} = \sum_{\alpha \in \textbf{Feynman diag}} \int_L \frac{c_\alpha n_\alpha}{S_\alpha d_\alpha} = \sum_{i \in \textbf{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

$$n_i \to \sum_\varphi n_i^\varphi$$

- Example: 4-point s-channel diagram



BCJ colour-kinematic duality conjecture

- There is an organisation of the n-point L-loop gluon amplitude:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\boxed{\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}}$$

[Bern-Carrasco-Johansson '08]

BCJ colour-kinematic duality conjecture

- There is an organisation of the n-point L-loop gluon amplitude:

Colour Jacobi

$$A^{n,L}_{\text{YM}} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

Kinematic Jacobi (BCJ numerators)

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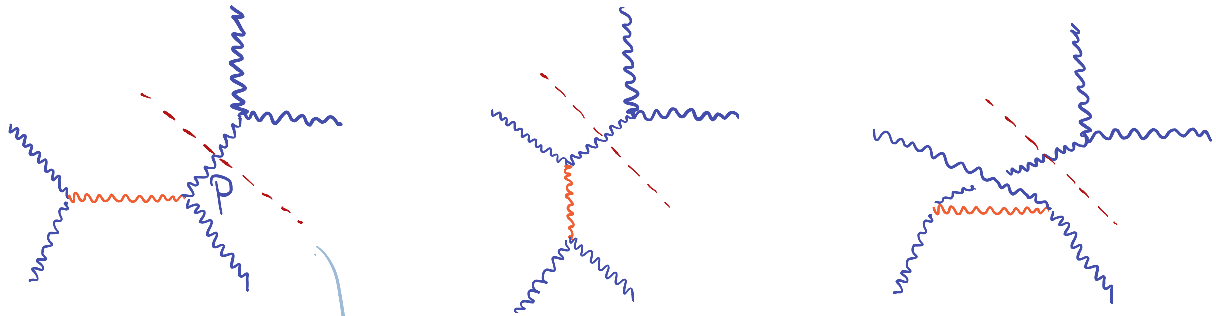
[Bern-Carrasco-Johansson '08]

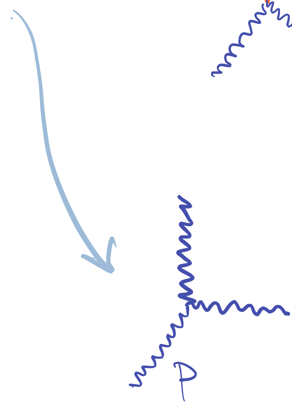
- CK duality established at tree-level
  [Stieberger '09, Bjerrum-Bohr–Damgaard–Vanhove '09; Mizera '19; Reiterer '19]

- Significant evidence up to 4 loops in various (super)YM theories
  [Carrasco–Johansson '11; Bern–Davies–Dennen–Huang–Nohle '13; Bern-Davies-Dennen '14. . .]

- Quickly becomes difficult to check: remains conjectural at the loop level
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

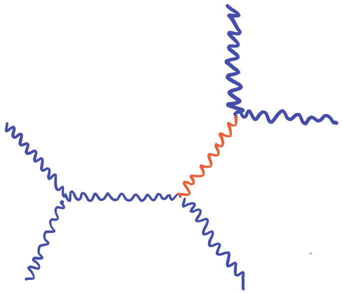
5-point example: why life isn't that easy

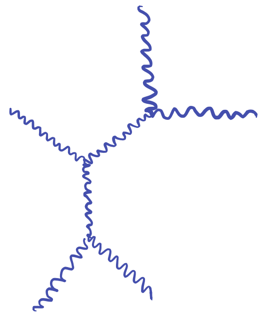


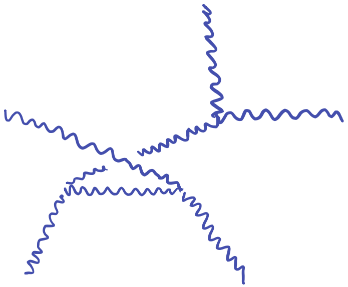


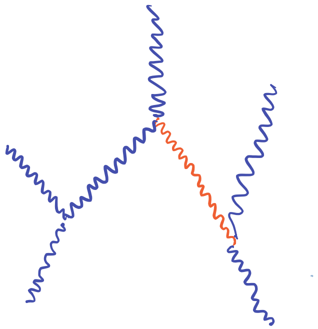
Not $\varepsilon(p)$ s.t. $p^2 = 0 = \varepsilon \cdot p$

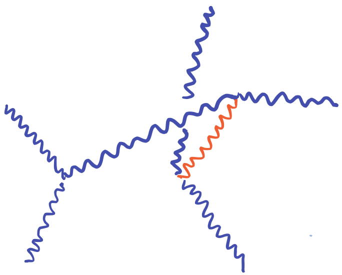
5-point example: why life isn't that easy











Tree-level statement

- Reformulate tree-level statement:

$$A_{\mathrm{YM}}^{n,0} = \mathsf{c}^T \mathsf{D} \mathsf{n}, \quad \mathsf{D}_{ij} = \frac{\delta_{ij}}{d_j},$$

$i = 1, 2, \ldots (2n-5)!!$

- Jacobi implies only $(n-2)!$ linearly independent, choose c_m and corresponding n_m:

$$\mathsf{c} = \mathsf{J}\mathsf{c}_m, \qquad \mathsf{n} = \mathsf{J}\mathsf{n}_m$$

where J is $(2n-5)!! \times (n-2)!$

Tree-level statement

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  where J is $(2n-5)!! \times (n-2)!$

- Non-trivial condition

$$\mathsf{A} = \mathsf{P}\mathsf{n} = (\mathsf{P}\mathsf{J})\mathsf{n}_m$$

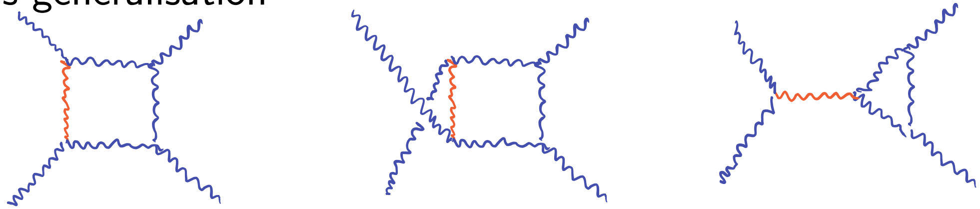
  where A is the $(n-2)!$-vector of colour-ordered partial amplitudes and P is an $(n-2)! \times (2n-5)!!$ matrix of signed propagators

- $\det(\mathsf{P}\mathsf{J}) = 0$, but can solve via Gaussian elimination

  → only $(n-3)!$ independent partial amplitudes (BCJ relations)

Going to loops

- Obvious generalisation



- Quickly becomes very difficult since CK duality relations become functional due the graph automorphisms $\Rightarrow$ numerator ansatz:
  1. The numerators are polynomials in momenta and polarization vectors
  2. Power-counting matches those of Feynman-gauge Feynman rules
  3. Diagrams with only trivalent vertices
  4. Relabeling maps numerators to numerators
  5. Diagram symmetries respected
  6. The cuts of ansatz match a spanning set of unitarity cuts for Yang–Mills
  7. CK duality manifest in integrand

- 1-6 are are manifested by Feynman diagrams (3 requires aux fields)

- 6 $\Leftrightarrow$ unitary theory + ansatz verification

- 1-7 cannot be satisfied at two loops: something has to give [Bern-Davies-Nohle '15]

Gravity

- Einstein–Hilbert action: perturbatively expanded $g = \eta + \kappa h$

$$S^{\mathrm{EH}}_{\mathrm{classical}} = \frac{1}{2\kappa^2}\int \star R$$

All order interactions

$$\sim \int d^D x \left\{ \partial\partial hh + \kappa h \partial\partial hh + \kappa^2 hh\partial\partial hh + \kappa^3 hhh\partial\partial hh \cdots \right\}$$

- Invariant under gauge transformations (remnant of diffeo)

$$\delta_\theta h = \frac{1}{\kappa}\mathcal{L}_\theta(g)|_{\kappa=0}, \qquad \delta_\theta h_{\mu\nu} = 2\nabla_{(\mu}\theta_{\mu)} = \nabla_\mu\theta_\nu + \nabla_\nu\theta_\nu$$

Gravity

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- Gauge fix using BRST formalism $\theta \to X$:

$$Qh = \nabla X, \quad QX = \mathcal{L}_X X, \quad Q\bar{X} = \pi, \quad Q\pi = 0$$

with gauge-fixing fermion $\Psi = -\bar{X}\cdot(\tfrac{\xi}{2}\pi - G[h])$:

$$S^{\rm EH}_{\rm BRST} = S^{\rm EH}_{\rm classical} + \int \star Q\Psi$$

Gravity Feynman diagrams

- Recall, gluon three point vertex:

$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$

Gravity Feynman diagrams

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$$= g f_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$

- Compare graviton three point [De Wit '69; Carrasco '15 (TASI lectures)]:

$$
\begin{aligned}
&\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1^{\lambda}k_1^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1^{\lambda}k_1^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1^{\lambda}k_1^{\rho} + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}k_1^{\sigma}k_1^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1^{\tau}k_1^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}k_2^{\lambda}k_1^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_2^{\lambda}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2^{\mu}k_1^{\rho} + \\
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&\eta^{\lambda\nu}\eta^{\mu\tau}k_2^{\rho}k_3^{\sigma} + \eta^{\lambda\mu}\eta^{\nu\tau}k_2^{\rho}k_3^{\sigma} + 2\eta^{\lambda\rho}\eta^{\nu\tau}k_3^{\mu}k_3^{\sigma} + 2\eta^{\lambda\rho}\eta^{\mu\tau}k_3^{\nu}k_3^{\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}k_1^{\lambda}k_3^{\tau} + \\
&\eta^{\mu\rho}\eta^{\nu\sigma}k_1^{\lambda}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\mu\rho}k_1^{\sigma}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\nu\rho}k_1^{\sigma}k_3^{\tau} + \eta^{\mu\sigma}\eta^{\nu\rho}k_2^{\lambda}k_3^{\tau} + \eta^{\mu\rho}\eta^{\nu\sigma}k_2^{\lambda}k_3^{\tau} - \\
&\eta^{\mu\nu}\eta^{\rho\sigma}k_2^{\lambda}k_3^{\tau} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_2^{\mu}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\rho\sigma}k_2^{\mu}k_3^{\tau} + \eta^{\lambda\sigma}\eta^{\mu\rho}k_2^{\nu}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\rho\sigma}k_2^{\nu}k_3^{\tau} - \\
&\eta^{\lambda\sigma}\eta^{\mu\nu}k_2^{\rho}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_2^{\rho}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_2^{\rho}k_3^{\tau} + 2\eta^{\lambda\rho}\eta^{\nu\sigma}k_3^{\mu}k_3^{\tau} + 2\eta^{\lambda\rho}\eta^{\mu\sigma}k_3^{\nu}k_3^{\tau} - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}k_3^{\sigma}k_3^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_3^{\sigma}k_3^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_3^{\sigma}k_3^{\tau} - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot \\
&k_2 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_2 + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_2 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_2 - \\
&\eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_2 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_2 + \\
&2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_2 - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_3 + \\
&2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_3 - \\
&\eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_3 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - \\
&\eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_3 - \\
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\end{aligned}
$$

Gravity Feynman diagrams

- Oh-shell three-point amplitude $(--+)$:

$$A^{3,0}_{\mathrm{graviton}} = 4[(\varepsilon_1 \cdot \varepsilon_3)(p \cdot \varepsilon_2) - (\varepsilon_2 \cdot \varepsilon_3)(q \cdot \varepsilon_1)]^2 = i2[A^{3,0}_{\mathrm{gluon}}]^2$$

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- Vast simplification $\longrightarrow$ hidden structure

Gravity $=$ Gauge $\times$ Gauge

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A_{\text{YM}}^{n,L} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

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- Gives an amplitude of $\mathcal{N} = 0$ supergravity

$$A_{\mathcal{N}=0}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{n_i n_i}{S_i d_i}$$

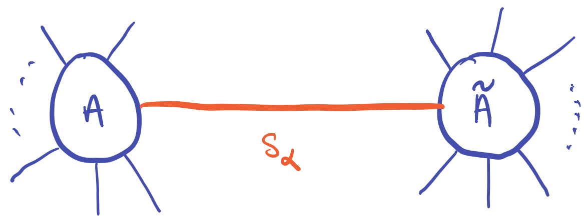
$$S_{\mathcal{N}=0} = \frac{1}{2\kappa^2} \int \star R - \frac{1}{d-2} d\varphi \wedge \star d\varphi - \frac{1}{2} \mathrm{e}^{-\frac{4}{d-2}\varphi} dB \wedge \star dB$$

where B is the Kalb-Ramond 2-form [See K. Waldorf lectures], φ is the dilaton [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

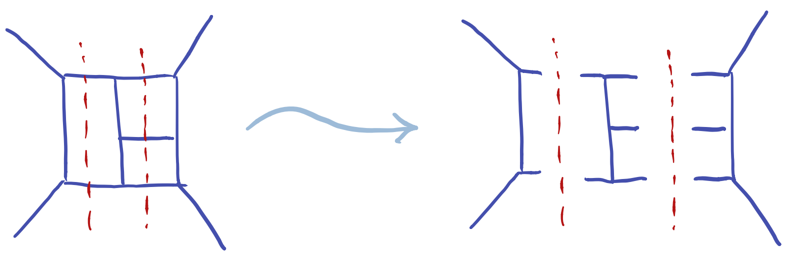
Double copy intuition

- Proof: [Bern-Dennen-Huang-Kiermaier '10]
- Inductive: $A^{3,0}_{\mathcal{N}=0} = 2i(A^{3,0}_{\text{gluon}})^2$
- Recursion via complex momentum shifts $p \mapsto p + zq$:

$$A^{n,0}_{\mathcal{N}=0} \mapsto A^{n,0}_{\mathcal{N}=0}(z) = \sum_{\alpha} \frac{1}{s_\alpha} A^{\alpha}_{\mathcal{N}=0}(z_\alpha) \tilde{A}^{\alpha}_{\mathcal{N}=0}(z_\alpha)$$



- Loop generalisation: unitary cuts



- Cut-constructible part amplitude correct: higher dimensions to get the rest
- Unitarity of Yang–Mills integrand essential for this argument (point 7)

Generalisations

Distinct theoris (CK dual)

$$\sum_{i \in \text{cubic diag}} \frac{n_i n_i}{d_i} \longrightarrow \sum_{i \in \text{cubic diag}} \frac{n_i \tilde{n}_i}{d_i}$$

Inputs: Matter-coupled (super) Yang-Mills, $D = 3$ Chern-Simons-Matter, QCD, Higgsed theories, Z-theory, $(DF)^2$ theories ...

Outputs: ϕ^3 theory, Maxwell/scalar/Yang-Mills supergravity, gauged supergravity (Minkowski vacua), non-linear sigma model, pure gravity, Born-Infeld, conformal gravity, strings ...
[Hodges '11; Cachazo, He, Yuan '13 '14, Dolan, Goddard '13; Naculich '14 '15, Cachazo, He, Yuan '13 '14; Chiodaroli et al '14 '15; Johansson, Ochirov '15 '16; Chiodaroli, Günaydin, Johansson, Roiban '17; Carrasco, Mafra, Schlotterer '16; Johansson, Nohle '17; Azevedo, Chiodaroli, Johansson, Schlotterer '18...]

Examples: Magic square of $D = 3$ supergravities

$$\mathcal{N}_L + \mathcal{N}_R \;\; \text{sugra (+ matter)} = (\mathcal{N}_L \;\; \text{super YM}) \times (\mathcal{N}_R \;\; \text{super YM})$$

- Find $D = 3$ supergravities with global symmetries given by Freudenthal-Rosenfeld-Tits magic square:

$\mathbb{A}_{L/R}(\mathcal{N}_{L/R})$	$\mathbb{R}(1)$	$\mathbb{C}(2)$	$\mathbb{H}(4)$	$\mathbb{O}(8)$
$\mathbb{R}(1)$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{su}(2,1)$	$\mathfrak{sp}(4,2)$	$\mathfrak{f}_{4(-20)}$
$\mathbb{C}(2)$	$\mathfrak{su}(2,1)$	$\mathfrak{su}(2,1)\times\mathfrak{su}(2,1)$	$\mathfrak{su}(4,2)$	$\mathfrak{e}_{6(-14)}$
$\mathbb{H}(3)$	$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
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[LB, Duff, Hughes, Nagy '13]

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[LB, Duff, Hughes, Nagy '13]

- $D = 3$ $\mathcal{N}$-extended super YM over the division algebras $\mathcal{N} = \dim \mathbb{A}$

$$\mathfrak{sugra}(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3\mathbb{A}_L \otimes \mathbb{A}_R$$

- Generalises to all $3 \leq D \leq 10$: square $(\mathbb{A}_L, \mathbb{A}_R) \to$ pyramid $(\mathbb{A}_D, \mathbb{A}_L, \mathbb{A}_R)$ [Anastasiou-LB-Hughes-Nagy '15]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

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- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]

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- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]
- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

  "Enhanced" cancellations

  [Bern-Davies-Dennen '14]
- Such cancellations not seen for $\mathcal{N} = 8$ at 5 loops: implications unclear

Implications and applications

- Classical (non)perturbative solutions and gravity wave astronomy [Monteiro–O'Connell–White '14; Cardoso–Nagy–Nampuri '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16; Berman–Chacón–Luna–White '18; Kosower–Maybee–O'Connell '18; Bern–Cheung–Roiban–Shen–Solon–Zeng '19; Bern–Luna–Roiban–Shen–Zeng '20; Chacón-Nagy-White '21. . .]

- Geometric/world-sheet picture: ambitwistor string theories theories and scattering equations, e.g. non-trivial gluon and spacetime backgrounds [Cachazo–He–Yuan '13 '14; Mason–Skinner '13; Adamo–Casali–Skinner '13; Adamo–Casali–Mason–Nekovar '17 '18; Geyer–Monteiro '18; Geyer–Mason '19; Geyer-Monteiro-Stark–Muchão '21. . .]

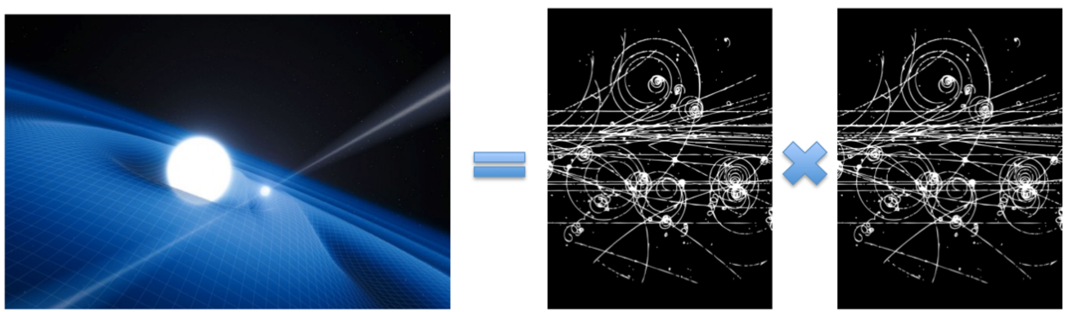
- Surprising applications: gauge structure of the conjectured $(4, 0)$ phase of M-theory and twin non-Lagrangian S-folds theories [LB '18; LB-Duff-Marrani '19]

Lecture 2: Off-shell field theory colour–kinematics and double copy

Gravity = Gauge × Gauge

Longstanding open questions

- Does CK duality (in some appropriate sense) hold to all orders?
- Does the double copy hold: is Einstein really the square of Yang–Mills?
- Is this restricted to the S-matrix or more general?



Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to *Jacobian counter terms* [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

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  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms for unitarity
  - Generalised unitarity proof of double copy doesn't straightforwardly apply

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- Double copy of BV action is manifestly valid $\rightarrow$ double copy to all loops
- Perturbative quantum Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form and dilaton *is* the square Yang–Mills theory [BJKMSW '20, '21]

Colour–kinematics duality and double copy: recap

Two key ideas:

- Realising CK duality and the double copy at the level of field theory:

Colour–kinematics duality and double copy: recap

Two key ideas:

- Realising CK duality and the double copy at the level of field theory:

  1. CK duality manifesting actions and kinematic algebras
     [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16] [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21; Cheung-Mangan '21; Ben-Shahar-Johansson '21; Brandhuber-Chen-Johansson-Travaglini-Wen '21. . .]

  2. Field theory product of BRST gauge theories and Lagrangian double-copy
     [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20; BJKMSW '20, '21]

- Today: the YM BV action admits a natural form of anomalous CK duality that immediately implies the double copy to all orders

Lighting overview

Step 1. Cubic tree-level off-shell CK duality manifesting Yang-Mills BRST-action:

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

Step 2. BRST-action double-copy:

$$S_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Step 3. Double-copy BRST operator:

$$(Q_{\text{YM}}, \tilde{Q}_{\text{YM}}) \longrightarrow Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

Step 4. Assuming tree-level physical CK duality, perturbative quantum equivalence:

$$Q_{\text{DC}}{}^2 = Q_{\text{DC}} S_{\text{DC}} = 0 \quad \Rightarrow \quad S_{\text{DC}} \cong S^{\mathcal{N}=0}_{\text{BRST}}$$

Corollary: Loop amplitude (integrands) computed from Feynman diagrams of $S^{\text{YM}}_{\text{BRST-CK}}$ manifest CK duality, *up to counterterms needed for unitarity*, and double-copy correctly to give amplitudes of $\mathcal{N} = 0$ supegravity

Step 1: Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$S^{\text{YM}}_{\text{on-shell CK}} = \sum_{n=2}^{\infty} \int \mathcal{L}^{(n)}_{\text{YM}} \sim A\Box A + \partial AAA + \frac{\Box}{\Box} AAAA + \frac{\partial^3}{\Box^2} AAAAA + \cdots$$

[Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975]

Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [BJKMSW '21]

$$
\begin{aligned}
S^{\text{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int d^D x\, & \tfrac{1}{2} A_\mu \,\square\, A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& \tfrac{1}{2} B^{\mu\nu\kappa} \,\square\, B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + \tfrac{1}{2} B^{\mu\nu\kappa} \,\square\, B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu} \,\square\, \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \,\square\, \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \,\square\, \bar{C}_{\mu\nu\kappa\lambda} + \\
& + gC^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2} \partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g\bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right) + \cdots
\end{aligned}
$$

[Bern–Dennen–Huang–Kiermaier '10]

- Purely cubic Feynman diagrams $\longrightarrow$

$$
A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

Colour-Kinematic Duality Redux

Generalise to off-shell BRST CK duality

- Does not imply loop-level CK duality, e.g. unphysical off-shell modes propagate in the loops

- To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops:

Colour-Kinematic Duality Redux

Generalise to off-shell BRST CK duality

- Does not imply loop-level CK duality, e.g. unphysical off-shell modes propagate in the loops
- To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops:

  1. Longitudinal gluons - gauge choice
  2. Ghosts - BRST Ward identities
  3. Off-shell - nonlocal field redefinitions (invisible on-shell)

- 3. $\Rightarrow$ induces Jacobian counterterms that cancel spurious modes

[BJKMSW '21]

Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Relax transversality $p_n \cdot \varepsilon_n \neq 0 \Rightarrow$ tree CK duality fails

- By analogy can compensate with new *non-zero* vertices [BJKMSW '20]:

- Add them to the action without changing the physics [BJKMSW '20]

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing functional

$$\begin{array}{lcccccc} \text{Gauge-fixing func. } G[A]: & \partial \cdot A & \mapsto & G'[A] & = & \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: & b & \mapsto & b' & = & b + Y \end{array}$$

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A)Y[A]$$

- Can add through the gauge-fixing functional

$$\begin{array}{rcccl} \text{Gauge-fixing func. } G[A]: & \partial \cdot A & \mapsto & G'[A] & = \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: & b & \mapsto & b' & = b + Y \end{array}$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

Colour-Kinematic Duality Redux

Tree-level CK duality for ghosts

- Use on-mass-shell BRST Ward identities

$$Q_{\mathrm{YM}}^{\mathrm{lin}} A_{\mathsf{phys}} = 0, \quad Q_{\mathrm{YM}}^{\mathrm{lin}} A_{\mathsf{f}} = c, \quad Q_{\mathrm{YM}}^{\mathrm{lin}} b = \bar{c}$$

- Analogous to global SUSY Ward identities

$$0 \; = \; \langle 0 | [Q_{\mathrm{YM}}^{\mathrm{lin}}, O_1 \cdots O_n] | 0 \rangle$$

- Transfers CK duality onto ghosts through

$$\mathcal{L}_{\mathsf{ghost}}^{\mathsf{YM}} = \bar{c} Q_{\mathrm{YM}} (\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce new auxiliary gluons and ghosts [BJKMSW '20, '21]:

$$
\begin{aligned}
\mathcal{L}^{\text{YM}}_{\text{BRST CK-dual}} = \; & \tfrac{1}{2} A_{a\mu} \Box A^{\mu a} - \bar{c}_a \Box c^a + \tfrac{1}{2} b_a \Box b^a + \xi \, b_a \sqrt{\Box}\, \partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
& - \tfrac{1}{2} B^{\mu\nu\kappa}_a \Box B^a_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
& - g f_{abc} \Big\{ K^{a\mu}_1 (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \Big\} \\
& + g f_{abc} \Big\{ K^{a\mu}_2 \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \Big\} + \cdots
\end{aligned}
$$

- Cubic Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes and ghosts (on-shell)

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- Relaxing on-shell to off-shell momenta CK duality violated by terms

$$p_i^2 F_i$$

for each external momentum p_i (unphysical gluons and ghosts)

- Can compensate with terms $\propto F\Box\Phi$ with non-local field redefinition

$$\Phi \mapsto \Phi + F, \qquad \Phi\Box\Phi \mapsto \Phi\Box\Phi + F\Box\Phi + \cdots$$

so that off-shell tree-level BRST CK duality is manifest $\rightarrow$ loop CK duality [BJKMSW '21]

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  so that off-shell tree-level BRST CK duality is manifest $\rightarrow$ loop CK duality [BJKMSW '21]

- Price to pay: Jacobian determinants $\rightarrow$ counterterms ensuring unitarity

- In this sense, this manifest loop CK duality is anomalous on the physical Hilbert space (but is exact on the complete pre-Hilbert space)

Colour-Kinematic Duality Redux

Perfect off-shell 'BRST-Lagrangian CK duality'

- BV YM action with manifest *off-shell* CK duality

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} + A^{+}_{ia}\left(Q^{i}{}_{j} A^{ja} + Q^{i}_{jk} f^{a}_{bc} A^{jb} A^{kc}\right)$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_{\mu}{}^{a}, b^{a}, \bar{c}^{a}, c^{a}, \underbrace{G_{\mu\nu\rho}{}^{a}, \bar{K}_{\mu}{}^{a}, \ldots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} gauge group Killing form and structure constants

- C_{ij}, F^{ijk} are differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^{i}_{k)l} = 0 \qquad F_{[ij|l} F^{l}_{|k]m} = 0$$

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
- The F_{ijk} are the structure constants of a *kinematic Lie algebra* mirroring the usual colour structure constants f_{abc}. Cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]
- Corollary: loop amplitude integrands are CK dual automatically
- Anomalous, in a controlled manner, due to Jacobian counterterms that ensure (generalised) unitarity

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
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- Corollary: loop amplitude integrands are CK dual automatically
- Anomalous, in a controlled manner, due to Jacobian counterterms that ensure (generalised) unitarity
- Shift in point of view:
  - A consistent field theory formulation of CK duality
  - Anomaly: generalised unitarity proof of loop double copy doesn't go through, at least not straightforwardly
  - Departure from standard articulation of loop integrand CK duality: all desiderata *except* generalised unitarity
  - Latter replaced with off-shell CK duality of BV action (without Jacobian counterterms): alternative proof of double copy

BV Lagrangian Syngamy

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$c_{IJ}\phi^I \Box \phi^J + f_{IJK}\phi^I\phi^J\phi^K$

$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$

Factorisation

$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$

$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$

Daughter theories

$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$

$\tilde{c}_{\tilde{a}\tilde{b}}C_{ij}\Phi^{\tilde{a}i}\Box\Phi^{\tilde{a}j} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}F_{ijk}\Phi^{\tilde{a}i}\Phi^{\tilde{b}j}\Phi^{\tilde{c}k}$

$c_{ab}\tilde{C}_{ij}\Phi^{a\tilde{\imath}}\Box\Phi^{a\tilde{\jmath}} + f_{abc}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{a\tilde{\imath}}\Phi^{b\tilde{\jmath}}\Phi^{c\tilde{k}}$

$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$

BV Lagrangian Syngamy

Yang–Mills squared

- $S^{\text{YM}}_{\text{BRST-CK}} \otimes \tilde{S}^{\text{YM}}_{\text{BRST-CK}} \to \mathcal{N} = 0$ supergravity

$$A^{ia} = (A_\mu{}^a, \text{ghosts}, \text{auxiliaries}) \qquad S^{\text{YM}}_{\text{CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \text{ghosts}, \text{auxiliaries}) \qquad S^{\mathcal{N}=0}_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

- $G \times \tilde{G}$ bi-adjoint scalar theory,

$$S^{\text{bi-adj}}_{\text{DC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

BV Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence of $S_{\text{DC}}^{\mathcal{N}=0}$ (requires on-shell tree-level CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \to \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

$$\sum \frac{nc}{d} \quad \to \quad \sum \frac{n\tilde{n}}{d}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points

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$$QS_{\text{DC}} = 0, \qquad Q^2 = 0$$

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Answer: double-copy operator Q_{DC} (requires off-shell BRST CK duality)

BV Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BV action implies double copy BRST operator Q_{DC}

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} + A^{+}_{ia} \left(Q^{i}{}_{j} A^{ja} + Q^{i}_{jk} f^{a}_{bc} A^{jb} A^{kc} \right)$$

$$QA^{ia} = Q^{i}{}_{j} A^{ja} + Q^{i}{}_{jk} f^{a}{}_{bc} A^{jb} A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}i} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \tilde{A}^{\tilde{b}\tilde{\jmath}} \tilde{A}^{\tilde{c}\tilde{k}}$$

$$\underbrace{\underbrace{Q^{i}{}_{j} A^{j\tilde{\imath}} + Q^{i}{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{i\tilde{\jmath}} + F^{i}{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_R}}_{Q_{\text{DC}}}$$

- Yang-Mills gauge $\Rightarrow$ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

Cf. [Anastasiou-LB-Duff-Hughes-Nagy '14]

BV Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc}

$$Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0, \qquad Q_{\mathrm{DC}}^2 = 0$$

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- Einstein *is* the square of Yang–Mills (at least perturbatively)

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- Einstein *is* the square of Yang–Mills (at least perturbatively)

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \rightarrow \quad \text{local susy}$$

- Straightforward supersymmetric completion

§4.

Generalisations

Generalisations

The double copy to all orders

- Given CK duality of the tree-level physical S-matrix we can run our argument:
  - Non-linear sigma model [Chen-Du ‘13] $\to$ special Galileon
  - Fundamental couplings [Johansson-Ochirov ‘14] $\to$ plethora of supergravity theories
  - Bagger–Lambert–Gustavsson [Bargheer-He-McLoughlin ’12; Huang-Johansson ’12]$\to$ $D = 3$ maximal supergravity

Super Yang–Mills and Supergravity

BRST-Lagrangian CK duality for super Yang–Mills

- Irreducible super Yang–Mills multiplets are CK duality respecting
  Cf. [Bjerrum-Bohr–Damgaard–Vanhove '09]

- Susy Ward identities: CK gluons $+$ susy $\Rightarrow$ CK gluini
  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

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- CK dual BRST-Lagrangian then follows with (essentially) no new ideas

Super Yang–Mills and Supergravity

BRST-Lagrangian double copy

- (Type I super Yang–Mills)^2 = Type IIA/B supergravity

$$A^{ia} = ({A_\mu}^a, {\psi_\alpha}^a, \text{ghosts}, \text{aux})$$

$$A^{i\tilde{\jmath}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \Psi_{\alpha\nu}, \Psi_{\mu\beta}, F_{\alpha\beta}, \text{ghosts}, \text{aux})$$

- Local NS-R sector susy follows from super Yang–Mills factors

$$\mathcal{Q}_\alpha {A_\mu}^a = {\delta^a}_b {\gamma_{\mu\alpha}}^\beta {\psi_\beta}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = {\gamma_{(\mu\alpha}}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen–Kallosh χ ghosts

$$\bar{c} \otimes \psi \sim \bar{\eta}, \quad c \otimes \psi \sim \eta, \quad b \otimes \psi \sim \chi$$

- Similar for R–NS sector

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory
    $$\begin{aligned} \text{IIA:} \quad & \overline{16} \otimes 16 &=& \ 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad & 16 \otimes 16 &=& \ 10 \oplus 120 \oplus 126 \end{aligned}$$
  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

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- Type IIA/B action can be written in terms of field strengths, e.g.

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \tfrac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Sen's mechanism from double copy Ramond–Ramond sector

- Double copy R–R field strengths are *elementary* fields that correctly reproduce scattering amplitudes through their Feynman diagrams

$$
\begin{aligned}
\mathcal{L}^{\mathsf{DC}}_{\mathsf{R\text{–}R}} &= \overline{F}^{\alpha\beta}\,\Box^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots \\
&\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots
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&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\square B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots
\end{aligned}$$

- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]

- Sen's mechanism was motivated by IIB string field theory, where the R–R sector is naturally given in terms of bispinors - natural double copy shadow

Homotopy CK Duality and Double Copy

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

- Lie algebras $\to$ L_∞-algebras, first arose in string field theory:

Vector space	Graded vector space
$\mathfrak{g} = V_0$	$\mathfrak{L} = \bigoplus_n V_n$
Bracket	Higher brackets
$\mu_2 = [-,-]$	$\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \dots$
Relations	Relations
$Antisymmetry + Jacobi$	$Antisymmetry + homotopy Jacobi$

[Zwiebach '93; Hinich–Schechtman '93]

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Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \ldots$
Relations *Antisymmetry + Jacobi*	Relations *Antisymmetry + homotopyJacobi*

[Zwiebach '93; Hinich–Schechtman '93]

- Associative algebras $\rightarrow$ A_∞-algebras [Stasheff '63]

- Commutative algebras $\rightarrow$ C_∞-algebras [Kadeishvili '88]

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathrm{Sym}^p(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

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- Chevalley–Eilenberg formulation of L_∞-algebra $\mathfrak{L}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{L}) = \bar{T}(\mathfrak{L}[1]^*)$$

$$Qt^a = -\sum_n \tfrac{1}{n!} \mu_n{}^a{}_{a_1 \cdots a_n} t^{a_1} \cdots t^{a_n} , \qquad Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi}$$

- Any BV field theory with operator Q_{BV} corresponds to an L_∞-algebra in the CE picture, see e.g. [Jurco-Raspollini-Saemann-Wolf '18]

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\mathsf{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}_0^{\mathsf{YM}} & \oplus & \mathfrak{L}_1^{\mathsf{YM}} & \oplus & \mathfrak{L}_2^{\mathsf{YM}} & \oplus & \mathfrak{L}_3^{\mathsf{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathsf{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathsf{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths + gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \to$ cyclic structure $\langle -,-\rangle_{\mathsf{YM}}$ on $\mathfrak{L}^{\mathsf{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle$

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c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathsf{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathsf{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths + gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \rightarrow$ cyclic structure $\langle -,-\rangle_{\mathsf{YM}}$ on $\mathfrak{L}^{\mathsf{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\dots,a)\rangle$
- L_∞ quasi-isomorphisms $\longrightarrow$ physical equivalence (field redefinitions etc)
  - Strictification: $\mu_i = 0, i > 2 \rightarrow$ cubic theory
  - Minimal model: $\mu_1 = 0 \rightarrow$ tree scattering amplitudes

Cf. [Jurčo-Raspollini-Saemann-Wolf '18; Jurčo-Macrelli-Saemann-Wolf '19]

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

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[BLKMSW '21]

- $\mathfrak{colour}$: gauge group Lie algebra

- $\mathfrak{kinematics}$: graded vector space of Poincaré representations of fields

$$\begin{array}{ccccccc} \mathbb{R}[-1] & \oplus & (\mathbb{R}^d \oplus \mathbb{R}) & \oplus & \mathbb{R}[1] & \oplus & \text{Auxiliaries} \\ c & & (A_\mu, b) & & \bar{c} & & B_{\mu\nu\rho}\cdots \end{array}$$

- $\mathfrak{scalar}$: A_∞-algebra of a scalar field theory

$$\langle -, - \rangle_{\mathsf{YM}} = \langle -, - \rangle_{\mathfrak{colour}} \langle -, - \rangle_{\mathfrak{kinematics}} \langle -, - \rangle_{\mathfrak{scalar}}$$

Homotopy algebra of CK duality

Michel Reiterer [1912.03110]

- Proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra!
- Relies on the existence of a degree -1 unary map h on Zeitlin-Costello BV complex for Yang–Mills (think order formulation with A, F^+) satisfying

$$h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

- h exists and is a second-order derivation up to homotopy $\Rightarrow$
  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
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  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
  - On-shell tree-level CK duality for physical gluons
- Very special: only $D = 4$, no loop desiderata (ghosts, gauge-fixing)
- A little mysterious: $BV_\infty^\square$-algebra generalise famous BV_∞-algebras (homotopy BV-algebras [Galvez-Carrillo–Tonks–Vallette '09]), where e.g.

$$\Delta^2\square = (\mathrm{id} + \sigma_{(123)} + \sigma_{(123)}^2)(\mathrm{id} \otimes \Delta\square) - (\mathrm{id} + \sigma_{(123)} + \sigma_{(123)}^2)(\mathrm{id} \otimes \mathrm{id} \otimes \square)$$

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The homotopy algebra of CK duality [BJKMSW 'to appear 21]

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$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \underbrace{\mathfrak{kinematics} \otimes_{\tau} \mathfrak{scalar}}_{\mathfrak{Kin}\ \equiv\ BV^{\square}\text{-algebra}}$$

- $BV^{\square}$-algebra comes with two products $- \cdot -$ and $[-,-]$ and three unary operators

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  - The homotopy $BV^{\square}$-algebra depends on the ambient category
  - In the usual category of chain complexes d is privileged
  - Introduce symmetric monoidal category of Hodge complexes (modules over twisted Hopf algebras with central element $\square$)

$$d^2 = h^2 = 0, \quad dh + hd = \square$$

    Coassociativity $\Rightarrow$ the seven-term identity
  - In this category, both d and h are a part of the ambient structure

Homotopy algebra of CK duality

The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations

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The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations
- Computational efficiency:
  - Purely tree-level calculations
  - One identity at any order (the rest follow axiomatically)

$$\sum_{p+q=n+2} n\text{-point tree with two internal } (p\text{-ary and } q\text{-ary}) \text{ vertices}$$
$$= n\text{-point tree with one internal } (n\text{-ary}) \text{ vertex}$$

  - But, work with Feynman diagrams - marry with on-shell methods?

Future work

- AdS background [Zhou ‘21; Diwakar-Herderschee-Roiban-Teng ‘21 ...] $\rightarrow$ Hopf algebra of universal enveloping algebra of AdS isometries
- Bagger-Lambert-Gustavsson CK duality [Bargheer-He-McLoughlin ’12; Huang-Johansson ’12] $\rightarrow$ m-ary $BV^{\square}$ operads
- Matter coupling [Johansson-Ochirov ’14] $\rightarrow$ many-sorted $BV^{\square}$ operads
- String theory (modular envelope over) $BV_{\infty}^{L_0}$

  $$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

  Cf. BV_{∞} structure on TVOA [Galvez-Carrillo–Tonks–Vallette ‘09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman ‘93]
- Counterterms?

Thanks for listening