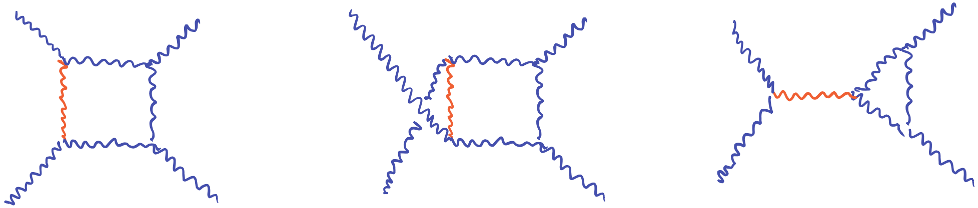
Going to loops

- Obvious generalisation



- Quickly becomes very difficult since CK duality relations become functional due the graph automorphisms $\Rightarrow$ numerator ansatz:
  1. The numerators are polynomials in momenta and polarization vectors
  2. Power-counting matches those of Feynman-gauge Feynman rules
  3. Diagrams with only trivalent vertices
  4. Relabeling maps numerators to numerators
  5. Diagram symmetries respected
  6. The cuts of ansatz match a spanning set of unitarity cuts for Yang–Mills
  7. CK duality manifest in integrand

- 1-6 are are manifested by Feynman diagrams (3 requires aux fields)

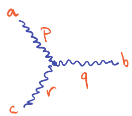
- 6 $\Leftrightarrow$ unitary theory + ansatz verification

- 1-7 cannot be satisfied at two loops: something has to give [Bern-Davies-Nohle '15]

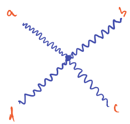
Recap: Scattering amplitudes

- Vary BRST-action $\int \mathrm{d}^d x \left\{ -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \frac{1}{2}(\partial^\mu A^a_\mu)^2 - \bar{c}_a \partial^\mu (\nabla_\mu c)^a \right\}$

$$= -\frac{i\eta_{\mu\nu}\delta^{ab}}{p^2}$$

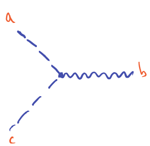


$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$



$$= -ig^2 \left[f^{abx} f_x{}^{cd} \eta^{\mu\nu\rho\sigma} + f^{adx} f_x{}^{bc} \eta^{\mu\sigma\nu\rho} + f^{acx} f_x{}^{db} \eta^{\mu\rho\nu\sigma} \right]$$

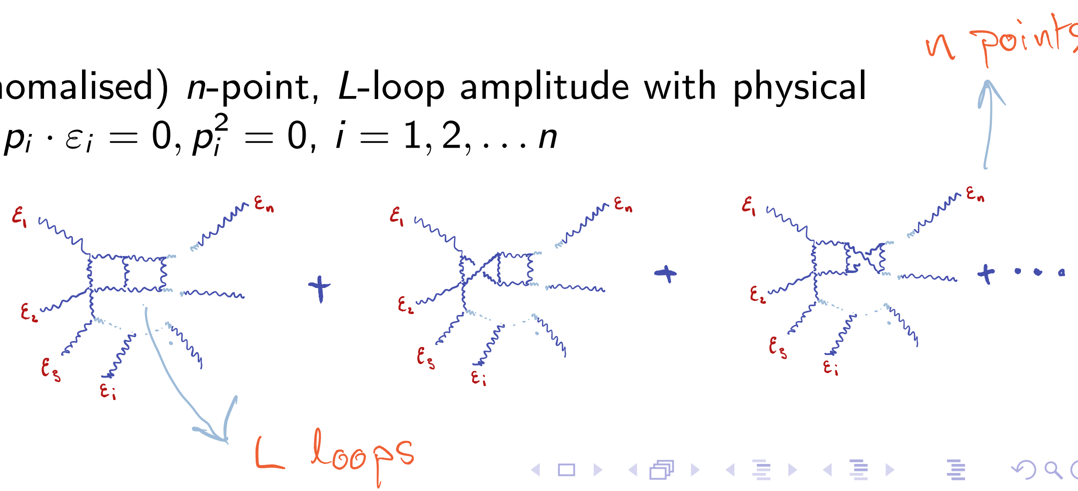
$$= \frac{i\delta^{ab}}{p^2}$$



$$= -gf^{abc} p^\mu$$

- Construct (unnrenomalised) n-point, L-loop amplitude with physical external states $\varepsilon_i, p_i \cdot \varepsilon_i = 0, p_i^2 = 0, \ i = 1, 2, \dots n$

$$A^{n,L}_{\mathrm{YM}} =$$



Recap: Bern-Carrasco-Johansson CK duality

- There is an organisation of the n-point L-loop gluon amplitude:

$$A^{n,L}_{\text{YM}} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}$$

[Bern-Carrasco-Johansson '08]

- Fails at two loops if one insists that n_i have all the properties expected from the Feynman diagram calculation

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[Bern-Carrasco-Johansson '08]

- Fails at two loops if one insists that n_i have all the properties expected from the Feynman diagram calculation

- What does this have to do with gravity?

Gravity

- Einstein–Hilbert action: perturbatively expanded $g = \eta + \kappa h$

$$S^{\mathrm{EH}}_{\mathrm{classical}} = \frac{1}{2\kappa^2} \int \star R$$
$$\sim \int d^D x \left\{ \partial\partial hh + \kappa h \partial\partial hh + \kappa^2 hh\partial\partial hh + \kappa^3 hhh\partial\partial hh \cdots \right\}$$

- Invariant under gauge transformations (expanding $\delta_\theta g = \mathcal{L}_\theta(g)$)

$$\delta_\theta h = \frac{1}{\kappa}\mathcal{L}_\theta(g)|_{\kappa=0}, \qquad \delta_\theta h_{\mu\nu} = 2\partial_{(\mu}\theta_{\mu)} = \partial_\mu\theta_\nu + \partial_\nu\theta_\nu$$

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All order interactions

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- Gauge fix using BRST formalism $\theta \to X$, $Q^2 = 0$:

$$Qg = \mathcal{L}_X(g), \quad QX = \mathcal{L}_X X, \quad Q\bar{X} = \pi, \quad Q\pi = 0$$

with gauge-fixing fermion $\Psi = -\bar{X}\cdot(\frac{\xi}{2}\pi - G[h])$:

$$S^{\mathrm{EH}}_{\mathrm{BRST}} = S^{\mathrm{EH}}_{\mathrm{classical}} + \int \star Q\Psi$$

Gravity Feynman diagrams

- Recall, gluon three point vertex:

$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$

Gravity Feynman diagrams

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- Compare graviton three point [De Wit '69; Carrasco '15 (TASI lectures)]:

$$\begin{aligned}
&\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1{}^\lambda k_1{}^\rho + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1{}^\lambda k_1{}^\rho - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1{}^\lambda k_1{}^\rho + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}k_1{}^\sigma k_1{}^\rho + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1{}^\tau k_1{}^\rho + \eta^{\mu\tau}\eta^{\nu\sigma}k_2{}^\lambda k_1{}^\rho + \eta^{\mu\sigma}\eta^{\nu\tau}k_2{}^\lambda k_1{}^\rho + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2{}^\mu k_1{}^\rho + \\
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\end{aligned}$$

Gravity Feynman diagrams

- Oh-shell three-point amplitude $(- - +)$:

$$A^{3,0}_{\text{graviton}} = 4[(\varepsilon_1 \cdot \varepsilon_3)(p \cdot \varepsilon_2) - (\varepsilon_2 \cdot \varepsilon_3)(q \cdot \varepsilon_1)]^2 = i2[A^{3,0}_{\text{gluon}}]^2$$

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- Vast simplification $\longrightarrow$ hidden structure

Gravity $=$ Gauge $\times$ Gauge

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

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- Gives an amplitude of $\mathcal{N} = 0$ supergravity

$$A^{n,L}_{\mathcal{N}=0} = \sum_{i \in \text{cubic diag}} \int_L \frac{n_i n_i}{S_i d_i}$$

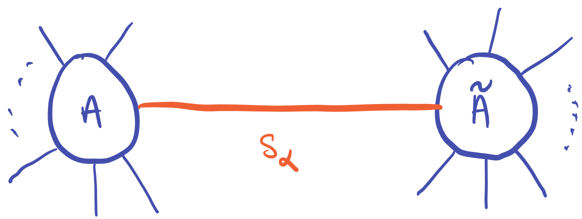
$$S_{\mathcal{N}=0} = \frac{1}{2\kappa^2} \int \star R - \frac{1}{d-2} d\varphi \wedge \star d\varphi - \frac{1}{2} \mathrm{e}^{-\frac{4}{d-2}\varphi} dB \wedge \star dB$$

where B is the Kalb-Ramond 2-form [See K. Waldorf lectures], φ is the dilaton [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

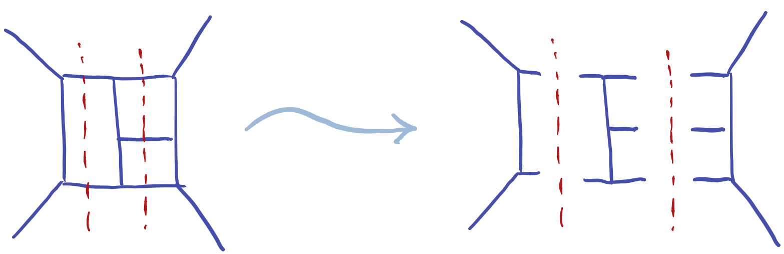
Double copy intuition

- Proof: [Bern-Dennen-Huang-Kiermaier '10]
- Inductive: $A^{3,0}_{\mathcal{N}=0} = 2i(A^{3,0}_{\text{gluon}})^2$
- Recursion via complex momentum shifts $p \mapsto p + zq$:

$$A^{n,0}_{\mathcal{N}=0} \mapsto A^{n,0}_{\mathcal{N}=0}(z) = \sum_{\alpha} \frac{1}{s_\alpha} A^{\alpha}_{\mathcal{N}=0}(z_\alpha) A^{\alpha}_{\mathcal{N}=0}(z_\alpha)$$



- Loop generalisation: unitary cuts



- Cut-constructible part amplitude correct: higher dimensions to get the rest
- Unitarity of Yang–Mills integrand essential for this argument (point 7)

Generalisations

- First expand the domain of CK duality, e.g. minimally coupled fermions

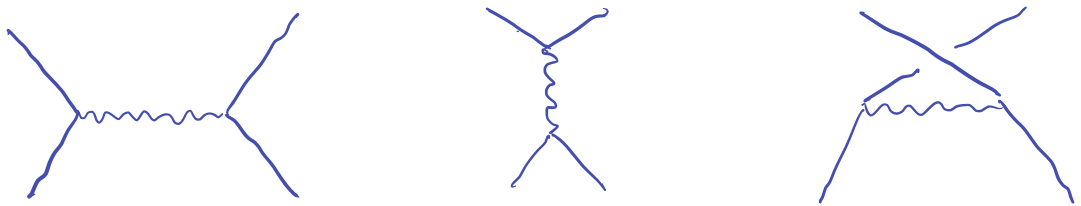
$$S^{Dirac} = \int \star \mathrm{tr} \bar{\psi} \not{D} \psi$$

Generalisations

- First expand the domain of CK duality, e.g. minimally coupled fermions

$$S^{Dirac} = \int \star \mathrm{tr} \bar{\psi} \not{D} \psi$$

- CK duality ⇒ supersymmetry [Chiodaroli-Jin-Roiban '13]



$$\frac{i}{2}\left(\frac{(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)c_s}{s} + \frac{(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)c_s}{t} + \frac{(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)c_s}{u}\right)$$

- CK duality requires:

$$(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4) + (\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4) + (\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4) = 0$$

- Fierz identity: $\mathcal{N} = 1$ super Yang–Mills in $D = 3, 4, 6, 10$
- Compactify, Higgs, orbifold, and quarks...

Generalisations

Distinct theories
(CK dual)

$$\sum_{i \in \text{cubic diag}} \frac{n_i n_i}{d_i} \longrightarrow \sum_{i \in \text{cubic diag}} \frac{n_i \tilde{n}_i}{d_i}$$

Inputs: Matter-coupled (super) Yang-Mills, $D = 3$ Chern-Simons-Matter, QCD, Higgsed theories, Z-theory, $(DF)^2$ theories . . .

Outputs: ϕ^3 theory, Maxwell/scalar/Yang-Mills supergravity, gauged supergravity (Minkowski vacua), non-linear sigma model, pure gravity, Born-Infeld, conformal gravity, strings . . .
[Hodges '11; Cachazo, He, Yuan '13 '14, Dolan, Goddard '13; Naculich '14 '15, Cachazo, He, Yuan '13 '14; Chiodaroli et al '14 '15; Johansson, Ochirov '15 '16; Chiodaroli, Günaydin, Johansson, Roiban '17; Carrasco, Mafra, Schlotterer '16; Johansson, Nohle '17; Azevedo, Chiodaroli, Johansson, Schlotterer '18. . .]

Examples: Magic square of $D = 3$ supergravities

$$\mathcal{N}_L + \mathcal{N}_R \ \text{ sugra (+ matter)} = (\mathcal{N}_L \ \text{ super YM}) \times (\mathcal{N}_R \ \text{ super YM})$$

- Find $D = 3$ supergravities with global symmetries given by Freudenthal-Rosenfeld-Tits magic square:

$\mathbb{A}_{L/R}(\mathcal{N}_{L/R})$	$\mathbb{R}(1)$	$\mathbb{C}(2)$	$\mathbb{H}(4)$	$\mathbb{O}(8)$
$\mathbb{R}(1)$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{su}(2,1)$	$\mathfrak{sp}(4,2)$	$\mathfrak{f}_{4(-20)}$
$\mathbb{C}(2)$	$\mathfrak{su}(2,1)$	$\mathfrak{su}(2,1)\times\mathfrak{su}(2,1)$	$\mathfrak{su}(4,2)$	$\mathfrak{e}_{6(-14)}$
$\mathbb{H}(3)$	$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
$\mathbb{O}(8)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(8)}$

[LB-Duff-Hughes-Nagy '13]

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[LB-Duff-Hughes-Nagy '13]

- $D = 3$ $\mathcal{N}$-extended super YM over the division algebras $\mathcal{N} = \dim \mathbb{A}$

$$\mathfrak{sugra}(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3\mathbb{A}_L \otimes \mathbb{A}_R$$

- Generalises to all $3 \leq D \leq 10$: square $(\mathbb{A}_L, \mathbb{A}_R) \rightarrow$ pyramid $(\mathbb{A}_D, \mathbb{A}_L, \mathbb{A}_R)$ [Anastasiou-LB-Hughes-Nagy '15]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

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- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]

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- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]

- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

  "Enhanced" cancellations

  [Bern-Davies-Dennen '14]

- Such cancellations not seen for $\mathcal{N} = 8$ at 5 loops: implications unclear

Implications and applications

- Classical (non)perturbative solutions and gravity wave astronomy [Monteiro-O'Connell-White '14; Cardoso-Nagy-Nampuri '16; Luna-Monteiro-Nicholson-Ochirov-O'Connell-Westerberg-White '16; Berman-Chacón-Luna-White '18; Kosower-Maybee-O'Connell '18; Bern-Cheung-Roiban-Shen-Solon-Zeng '19; Bern-Luna-Roiban-Shen-Zeng '20; Chacón-Nagy-White '21. . .]

- Geometric/world-sheet picture: ambitwistor string theories theories and scattering equations, e.g. non-trivial gluon and spacetime backgrounds [Cachazo-He-Yuan '13 '14; Mason-Skinner '13; Adamo-Casali-Skinner '13; Adamo-Casali-Mason-Nekovar '17 '18; Geyer-Monteiro '18; Geyer-Mason '19; Geyer-Monteiro-Stark-Muchão '21. . .]

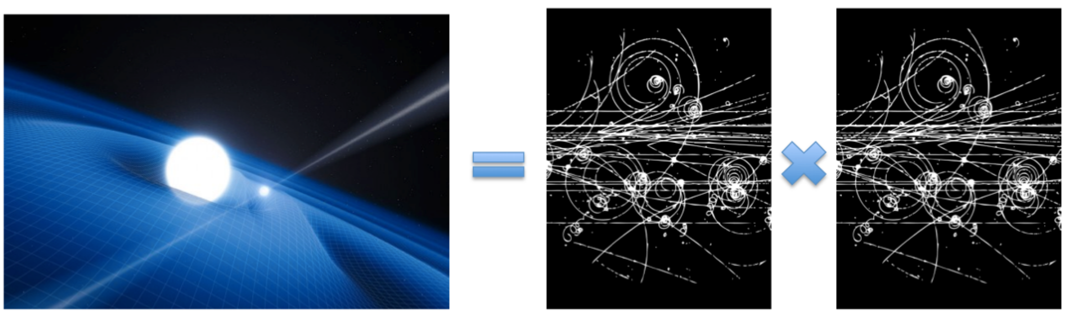
- Surprising applications: gauge structure of the conjectured $(4, 0)$ phase of M-theory and twin non-Lagrangian S-folds theories [LB '18; LB-Duff-Marrani '19]

Lecture 2: Off-shell field theory colour–kinematics and double copy

Gravity = Gauge × Gauge

Longstanding open questions

- Does CK duality (in some appropriate sense) hold to all orders?
- Does the double copy hold: is Einstein really the square of Yang–Mills?
- Is this restricted to the S-matrix or more general?



Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to Jacobian counter terms [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

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- Natural, but non-standard, notion of CK duality:
  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms for unitarity
  - Generalised unitarity proof of double copy doesn't straightforwardly apply

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  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms for unitarity
  - Generalised unitarity proof of double copy doesn't straightforwardly apply
- Double copy of BV action is manifestly valid → double copy to all loops
- Perturbative quantum Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form and dilaton *is* the square Yang–Mills theory [BJKMSW '20, '21]

Colour–kinematics duality and double copy: recap

Two key ideas:

- Realising CK duality and the double copy at the level of field theory:

Colour–kinematics duality and double copy: recap

Two key ideas:

- Realising CK duality and the double copy at the level of field theory:

  1. CK duality manifesting actions and kinematic algebras
     [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16]
     [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21; Cheung-Mangan '21; Ben-Shahar-Johansson '21; Brandhuber-Chen-Johansson-Travaglini-Wen '21. . .]

  2. Field theory product of gauge theories and Lagrangian double-copy
     [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20; BJKMSW '20, '21]

- The Chern-Simons BRST-action is automatically off-shell CK dual
  [Johansson, Ben-Shahar '21]

Lighting overview

Step 1. Cubic tree-level off-shell CK duality manifesting Yang-Mills BRST-action:

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

Step 2. BRST-action double-copy:

$$S_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Step 3. Double-copy BRST operator:

$$(Q_{\text{YM}}, \tilde{Q}_{\text{YM}}) \longrightarrow Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

Step 4. Assuming tree-level physical CK duality, perturbative quantum equivalence:

$$Q_{\text{DC}}{}^2 = Q_{\text{DC}} S_{\text{DC}} = 0 \quad \Rightarrow \quad S_{\text{DC}} \cong S^{\mathcal{N}=0}_{\text{BRST}}$$

Corollary: Loop amplitude (integrands) computed from Feynman diagrams manifest CK duality, up to counterterms needed for unitarity, and double-copy correctly to give amplitudes of $\mathcal{N} = 0$ supegravity

Step 1: Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- At tree-level CK duality

$$A^{n,0}_{\mathrm{YM}} = \sum_{i\in \text{cubic diag}} \frac{c_i n_i}{d_i} \quad \longrightarrow \quad A^{n,0}_{\mathrm{YM}} = \sum_{i\in \text{cubic diag}} \frac{c_i n_i^{\mathrm{BCJ}}}{d_i}$$

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- For $\delta n_i := n_i - n_i^{\mathrm{BCJ}}$ we have a vanishing 'amplitude'

$$\delta A^{n,0} = \sum_{i \in \text{cubic diag}} \frac{c_i \delta n_i}{d_i} = 0$$

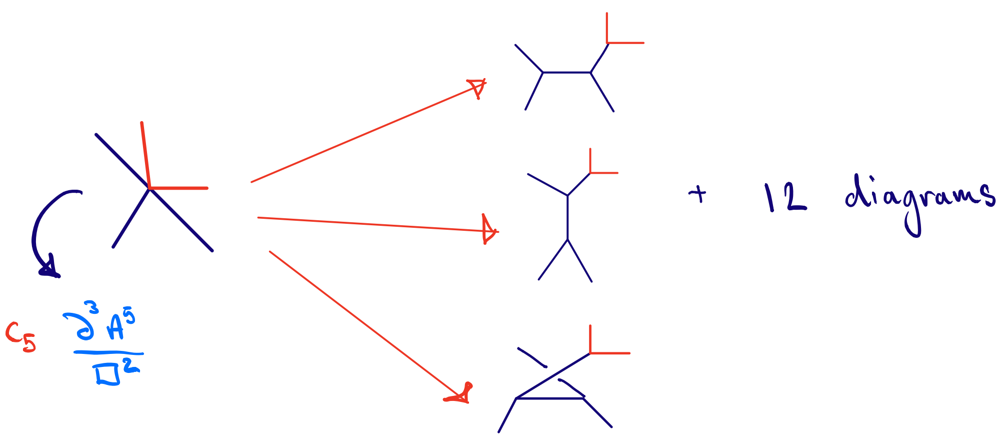
- Can be generated by adding zero vertex to action

Step 1: Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$S^{\text{YM}}_{\text{on-shell CK}} = \sum_{n=2}^{\infty} \int \mathcal{L}^{(n)}_{\text{YM}} \sim A\Box A + \partial AAA + \frac{\Box}{\Box} AAAA + \underbrace{\frac{\partial^3}{\Box^2} AAAAA}_{=0 \text{ by Jacobi}} + \cdots$$



[Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975]

Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [BJKMSW '21]

$$
\begin{aligned}
S^{\text{YM}}_{\text{on-shell CK}} = \operatorname{tr} \int d^D x\, & \tfrac{1}{2} A_\mu \Box A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g\left(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu}\right)[A^\mu, A^\nu] \\
& + C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + g C^{\mu\nu}[A_\mu, A_\nu] + g \partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2} \partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g \bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right) + \cdots
\end{aligned}
$$

4-point aux. field

5-point aux. fields

[Bern–Dennen–Huang–Kiermaier '10]

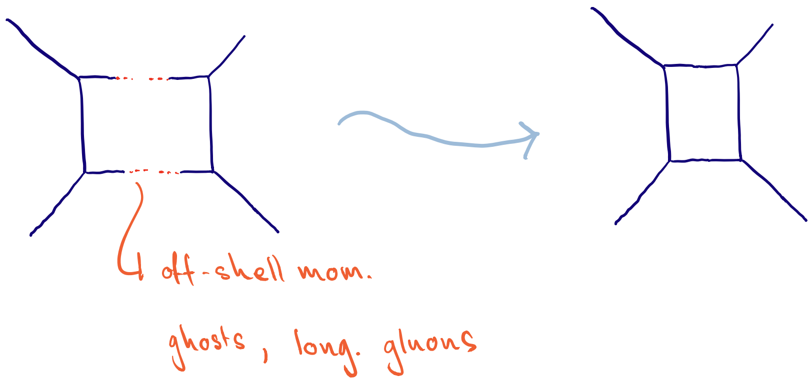
- Purely cubic Feynman diagrams $\longrightarrow$

$$
A^{n,0}_{\text{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

Colour-Kinematic Duality Redux

Generalise to off-shell BRST CK duality

- Does not imply off-shell or loop-level CK duality, e.g. unphysical off-shell modes propagate in the loops



- To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops [BJKMSW '20]

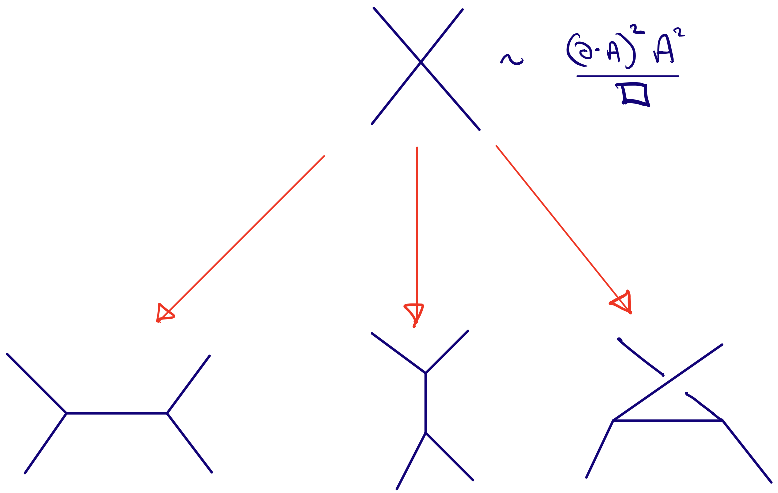
Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Relax transversality $p_i \cdot \varepsilon_i \neq 0$ for external states $\Rightarrow$ CK duality fails

$$A^{n,0}_{\mathrm{YM}} \quad \to \quad \hat{A}^{n,0}_{\mathrm{YM}}$$

- By analogy can compensate with new vertices [BJKMSW '20]:



- Non-zero non-zero: they $\hat{A}^{n,0}_{\mathrm{YM}} \to \hat{A}'^{n,0}_{\mathrm{YM}}$ ($A^{n,0}_{\mathrm{YM}}$ invariant)

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- New vertices are necessarily of the form

$$(\partial \cdot A) Y[A]$$

Colour-Kinematic Duality Redux

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$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing fermion $\Psi' = \Psi - 2\xi \bar{c} Y$

$$\begin{aligned}
\text{Gauge-fixing } G[A]: \quad & \partial \cdot A & \mapsto \quad & G'[A] & = \quad & \partial \cdot A - 2\xi Y \\
\text{Nakanishi-Lautrup } b: \quad & b & \mapsto \quad & b' & = \quad & b + Y \\
\text{BRST action } S^{\text{YM}}_{\text{BRST}} \quad & S^{\text{YM}}_{\text{BRST}} & \mapsto \quad & S'^{\text{YM}}_{\text{BRST}} & = \quad & S^{\text{YM}}_{\text{BRST}} + \int (\partial \cdot A) Y + \cdots
\end{aligned}$$

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\end{array}$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

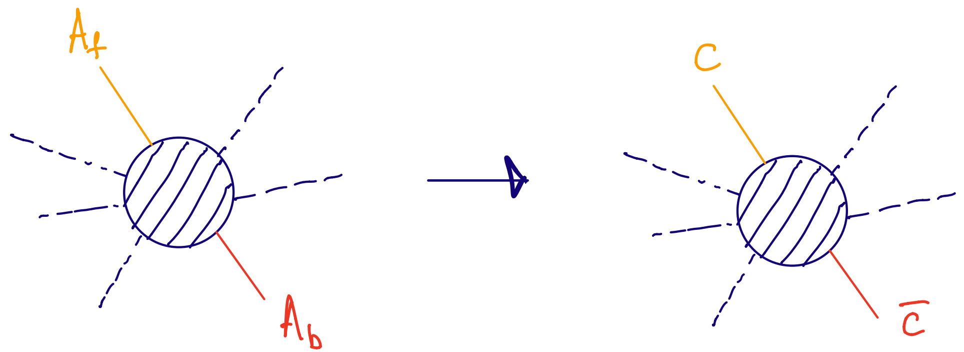
Colour-Kinematic Duality Redux

Tree-level CK duality for ghosts

- Use on-mass-shell BRST Ward identities

$$Q^{\rm lin} A_{\sf phys} = 0, \quad Q^{\rm lin} A_{\sf f} = c, \quad Q^{\rm lin} b = \bar{c}$$

$$0 \;=\; \langle 0|[Q^{\rm lin}, O_1 \cdots O_n]|0\rangle$$



- Transfers CK duality onto ghosts through

$$S^{\sf YM}_{\sf ghost} = \int \star {\rm tr}\, \bar{c} Q(\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce new auxiliary gluons and ghosts [BJKMSW '20, '21]:

$$
\begin{aligned}
\mathcal{L}^{\text{YM}}_{\text{BRST CK-dual}} = \; & \tfrac{1}{2} A_{a\mu} \Box A^{\mu a} - \bar{c}_a \Box c^a + \tfrac{1}{2} b_a \Box b^a + \xi\, b_a \sqrt{\Box}\, \partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
& - \tfrac{1}{2} B^{\mu\nu\kappa}_a \Box B^a_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
& - g f_{abc} \Big\{ K^{a\mu}_1 (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \Big\} \\
& + g f_{abc} \Big\{ K^{a\mu}_2 \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \Big\} + \cdots
\end{aligned}
$$

long. aux. fields

ghost aux. fields

- Cubic Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes and ghosts (on-shell)

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- Relaxing on-shell to off-shell momenta $p_i^2 \neq 0$ CK duality violated by terms

$$p_i^2 F_i$$

for each external momentum p_i (including unphysical gluons and ghosts)

- Compensate with terms $F\Box\Phi$ introduced via non-local field redefinition:

$$\Phi \mapsto \Phi + F, \qquad \Phi\Box\Phi \mapsto \Phi\Box\Phi + F\Box\Phi + \cdots$$

- Off-shell tree-level BRST CK duality is manifest $\rightarrow$ loop CK duality [BJKMSW '21]

Colour-Kinematic Duality Redux

Price to pay

- Jacobian determinants $\rightarrow$ counterterms ensuring unitarity

$$\det\left(\mathbb{1} + g\,\frac{\delta f(\phi)}{\delta\phi}\right) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \hbar\left(\bar{c}_I c^I + g\bar{c}_I \frac{\delta G^I}{\delta\phi^J} c^J\right)}$$

where $\phi \mapsto \phi + gf(\phi)$

- No reason to think such terms will preserve CK duality!

Colour-Kinematic Duality Redux

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  where $\phi \mapsto \phi + gf(\phi)$

- No reason to think such terms will preserve CK duality!

- In this sense, this manifest loop CK duality is anomalous on the physical Hilbert space (but is exact on the complete pre-Hilbert space)

- Recall, two two-loop CK duality (w/ conditions 1-7) is impossible

- We understand this impossibility as an anomaly of the CK duality of the BRST-action

Colour-Kinematic Duality Redux

Perfect off-shell 'BRST-Lagrangian CK duality'

- YM BRST-action with manifest off-shell CK duality

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_\mu{}^a, b^a, \bar{c}^a, c^a, \underbrace{B_{\mu\nu\rho}{}^a, \bar{K}_\mu{}^a, \ldots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} gauge group Killing form and structure constants

- C_{ij}, F^{ijk} are differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^a_{c)d} = 0 \qquad f_{[ab|d} f^d_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^i_{k)l} = 0 \qquad F_{[ij|l} F^l_{|k]m} = 0$$

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
- The F_{ijk} are the structure constants of a *kinematic Lie algebra* mirroring the usual colour structure constants f_{abc}. Cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]
- Corollary: loop amplitude integrands are CK dual automatically
- Anomalous, in a controlled manner, due to Jacobian counterterms that ensure (generalised) unitarity

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
- The F_{ijk} are the structure constants of a *kinematic Lie algebra* mirroring the usual colour structure constants f_{abc}. Cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]
- Corollary: loop amplitude integrands are CK dual automatically
- Anomalous, in a controlled manner, due to Jacobian counterterms that ensure (generalised) unitarity
- Shift in point of view:
  - A consistent field theory formulation of CK duality
  - Anomaly: generalised unitarity proof of loop double copy doesn't go through, at least not straightforwardly
  - Departure from standard articulation of loop integrand CK duality: all desiderata *except* generalised unitarity
  - Latter replaced with off-shell CK duality of BV action (without Jacobian counterterms): alternative proof of double copy

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$$c_{IJ}\phi^I \,\Box\, \phi^J + f_{IJK}\,\phi^I\,\phi^J\,\phi^K$$

$$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \,\Box\, \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\,\tilde{\phi}^{\tilde{I}}\,\tilde{\phi}^{\tilde{J}}\,\tilde{\phi}^{\tilde{K}}$$

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories | **Factorisation**

Factorise (meiosis)

$$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$$

$$c_{IJ}\phi^{I}\,\Box\,\phi^{J} + f_{IJK}\phi^{I}\phi^{J}\phi^{K}$$

$$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}}\,\Box\,\tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$$

$$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$$

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories	**Factorisation**	**Daughter theories**
		$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$
	$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$	
$c_{IJ}\phi^{I}\Box\phi^{J} + f_{IJK}\phi^{I}\phi^{J}\phi^{K}$		
$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}}\Box\tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$		
	$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$	

Factorise (meiosis)

Fusion (syngamy)

Double copy

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$c_{IJ}\phi^I \Box \phi^J + f_{IJK}\phi^I\phi^J\phi^K$

$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$

Factorise (meiosis)

Factorisation

$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$

$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$

Fusion (syngamy)

Daughter theories

Double copy

$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$

$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$

Zeroth copy

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$c_{IJ}\phi^I \Box \phi^J + f_{IJK}\phi^I\phi^J\phi^K$

$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$

Factorise (meiosis)

Factorisation

$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$

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$c_{ab}\tilde{C}_{ij}\Phi^{a\tilde{\imath}}\Box\Phi^{a\tilde{\jmath}} + f_{abc}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{a\tilde{\imath}}\Phi^{b\tilde{\jmath}}\Phi^{c\tilde{k}}$

$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$

Zeroth copy

Key examples

Yang–Mills squared

- Pair kinematics: $S^{\text{YM}}_{\text{BRST-CK}} \otimes \tilde{S}^{\text{YM}}_{\text{BRST-CK}} \to S^{\mathcal{N}=0}_{\text{DC}}$ ($\mathcal{N}=0$ supergravity)

$$A^{ia} = (A_\mu{}^a, \text{ghosts}, \text{auxiliaries}) \qquad S^{\text{YM}}_{\text{CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \text{ghosts}, \text{auxiliaries}) \qquad S^{\mathcal{N}=0}_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

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$G \times \tilde{G}$ bi-adjoint scalar theory

- Pair colours: $S^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} \otimes \tilde{S}^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} \to S^{\text{bi-adj}}_{\mathrm{DC}}$

$$S^{\text{bi-adj}}_{\mathrm{DC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

BRST double copy fields and action

- Factor BRST complex (field space) into colour and kinematics:

$$\mathfrak{g} \otimes \mathfrak{V} \otimes \mathfrak{S} \longrightarrow \mathfrak{V} \otimes \mathfrak{V} \otimes \mathfrak{S}$$

fields				antifields		
factorisation	role	L_∞ deg	dim	factorisation	L_∞ deg	dim
$\lambda = [\mathrm{g},\mathrm{g}]\mathrm{s}_x \frac{1}{2}\lambda(x)$	ghost for ghost	-1	$\frac{d}{2}-3$	$\lambda^+ = [\mathrm{a},\mathrm{a}]\mathrm{s}_x^+ \frac{1}{2}\lambda^+(x)$	4	$\frac{d}{2}+3$
$\Lambda = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}_x\Lambda_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$\Lambda^+ = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}_x^+\Lambda_\mu^+$	3	$\frac{d}{2}+2$
$\gamma = [\mathrm{g},\mathrm{n}]\mathrm{s}_x\gamma(x)$	NL field of Λ_μ	0	$\frac{d}{2}-2$	$\gamma^+ = [\mathrm{a},\mathrm{n}]\mathrm{s}_x^+\gamma^+(x)$	3	$\frac{d}{2}+2$
$B = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}_x\frac{1}{2}B_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$B^+ = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}_x^+\frac{1}{2}B_{\mu\nu}^+(x)$	2	$\frac{d}{2}+1$
$\alpha = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}_x\alpha_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\alpha^+ = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}_x^+\alpha_\mu^+(x)$	2	$\frac{d}{2}+1$
$\varepsilon = [\mathrm{g},\mathrm{a}]\mathrm{s}_x\varepsilon(x)$	anti-ghost of Λ_μ	1	$\frac{d}{2}-1$	$\varepsilon^+ = [\mathrm{g},\mathrm{a}]\mathrm{s}_x^+\varepsilon^+(x)$	2	$\frac{d}{2}+1$
$\bar\Lambda = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}_x\bar\Lambda_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar\Lambda^+ = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}_x^+\bar\Lambda_\mu^+(x)$	1	$\frac{d}{2}$
$\bar\gamma = [\mathrm{a},\mathrm{n}]\mathrm{s}_x\bar\gamma(x)$	NL field of $\bar\Lambda_\mu$	2	$\frac{d}{2}$	$\bar\gamma^+ = [\mathrm{g},\mathrm{n}]\mathrm{s}_x^+\bar\gamma^+(x)$	1	$\frac{d}{2}$
$\bar\lambda = [\mathrm{a},\mathrm{a}]\mathrm{s}_x\frac{1}{2}\bar\lambda(x)$	anti-ghost of $\bar\Lambda_\mu$	3	$\frac{d}{2}+1$	$\bar\lambda^+ = [\mathrm{g},\mathrm{g}]\mathrm{s}_x^+\frac{1}{2}\bar\lambda^+(x)$	0	$\frac{d}{2}-1$
$X = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}_x X_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$X^+ = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}_x^+X_\mu^+(x)$	3	$\frac{d}{2}+2$
$\beta = (\mathrm{g},\mathrm{n})\mathrm{s}_x\beta(x)$	NL field of X_μ	0	$\frac{d}{2}-2$	$\beta^+ = (\mathrm{a},\mathrm{n})\mathrm{s}_x^+\beta^+(x)$	3	$\frac{d}{2}+2$
$h = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}_x\frac{1}{2}h_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$h^+ = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}_x^+\frac{1}{2}h_{\mu\nu}^+(x)$	2	$\frac{d}{2}+1$
$\varpi = (\mathrm{n},\mathrm{v}^\mu)\mathrm{s}_x\varpi_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\varpi^+ = (\mathrm{n},\mathrm{v}^\mu)\mathrm{s}_x^+\varpi_\mu^+(x)$	2	$\frac{d}{2}+1$
$\pi = (\mathrm{n},\mathrm{n})\mathrm{s}_x\frac{1}{2}\pi(x)$	NL field of ϖ_μ	1	$\frac{d}{2}-1$	$\pi^+ = (\mathrm{n},\mathrm{n})\mathrm{s}_x^+\frac{1}{2}\pi^+(x)$	2	$\frac{d}{2}+1$
$\delta = (\mathrm{g},\mathrm{a})\mathrm{s}_x\delta(x)$	anti-ghost of X_μ	1	$\frac{d}{2}-1$	$\delta^+ = (\mathrm{g},\mathrm{a})\mathrm{s}_x^+\delta^+(x)$	2	$\frac{d}{2}+1$
$\bar X = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}_x\bar X_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar X^+ = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}_x^+\bar X_\mu(x)$	1	$\frac{d}{2}$
$\bar\beta = (\mathrm{a},\mathrm{n})\mathrm{s}_x\bar\beta(x)$	NL field of $\bar X_\mu$	2	$\frac{d}{2}$	$\bar\beta^+ = (\mathrm{g},\mathrm{n})\mathrm{s}_x^+\bar\beta^+(x)$	1	$\frac{d}{2}$

BRST Double-Copy: Fields and Action

- Double copy BRST-action uniquely determined:

$$
\begin{aligned}
\mathcal{L}_{\rm DC} =& \tfrac{1}{2} h_{\mu\nu} \,\square\, h^{\mu\nu} + \tfrac{1}{2} \varpi_\mu \,\square\, \varpi^\mu + \xi^2 (\partial^\mu \varpi_\mu)^2 + \tfrac{1}{2} \pi \,\square\, \pi \\
& - 2\xi \varpi^\nu \,\square^{\frac{1}{2}} \partial^\mu h_{\mu\nu} - 2\xi \pi \,\square^{\frac{1}{2}} \partial_\mu \varpi^\mu + 2\xi^2 \pi \partial_\mu \partial_\nu h^{\mu\nu} \\
& - 2\bar{X}_\mu \,\square\, X^\mu - \delta \,\square\, \delta - 2\bar{\beta} \,\square\, \beta \\
& + \tfrac{1}{2} B_{\mu\nu} \,\square\, B^{\mu\nu} - 2\bar{\Lambda}_\mu \,\square\, \Lambda^\mu + \alpha_\mu \,\square\, \alpha^\mu + \xi^2 (\partial^\mu \alpha_\mu)^2 + \varepsilon \,\square\, \varepsilon - \bar{\lambda} \,\square\, \lambda - 2\bar{\gamma} \,\square\, \gamma \\
& - 2\xi \alpha^\nu \,\square^{\frac{1}{2}} \partial^\mu B_{\mu\nu} - 2\xi \gamma \,\square^{\frac{1}{2}} \partial_\mu \bar{\Lambda}^\mu + 2\xi \bar{\gamma} \,\square^{\frac{1}{2}} \partial_\mu \Lambda^\mu \\
& - 2\xi \beta \,\square^{\frac{1}{2}} \partial_\mu \bar{X}^\mu + 2\xi \bar{\beta} \,\square^{\frac{1}{2}} \partial_\mu X^\mu \;+ \cdots
\end{aligned}
$$

Graviton-dilaton

KR 2-form

- Canonical field redefinition to Fierz-Pauli + Kalb-Ramond + dilaton action

$$
QB = d\Lambda, \qquad Q\Lambda = d\lambda, \qquad Q\lambda = 0
$$

BRST Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to true $S_{\text{BRST}}^{\mathcal{N}=0}$?

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- Semi-classical equivalence (requires off-shell BRST CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \rightarrow \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

BCJ numerators

$$\sum \frac{nc}{d} \quad \rightarrow \quad \sum \frac{n\tilde{n}}{d}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points

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$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \rightarrow \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

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- Quantum consistency: is there some Q such that

$$QS^{\mathcal{N}=0}_{\text{DC}} = 0, \qquad Q^2 = 0$$

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$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \to \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

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- Quantum consistency: is there some Q such that

$$QS_{\rm DC}^{\mathcal{N}=0} = 0, \qquad Q^2 = 0$$

Answer: double-copy operator $Q_{\rm DC}$ (requires off-shell BRST CK duality)

BRST Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BRST-action implies double copy BRST operator Q_{DC}

$$QA^{ia} = Q^{i}{}_{j}A^{ja} + Q^{i}{}_{jk}f^{a}{}_{bc}A^{jb}A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}i} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}}\tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}}\tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}\tilde{A}^{\tilde{b}\tilde{\jmath}}\tilde{A}^{\tilde{c}\tilde{k}}$$

$$Q_{\text{DC}} = \underbrace{Q^{i}{}_{j}A^{j\tilde{\imath}} + Q^{i}{}_{jk}F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}}A^{i\tilde{\jmath}} + F^{i}{}_{jk}Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_R}$$

- Yang-Mills gauge $\Rightarrow$ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

$$Q_{\text{2-form}}B = \Lambda, \quad Q_{\text{2-form}}\Lambda = \lambda \quad Q_{\text{2-form}}\lambda = 0$$

Cf. [Anastasiou-LB-Duff-Hughes-Nagy '14]

BV Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc} and $QS^{\mathrm{YM}}_{\mathrm{BRST}} = 0$, $Q^2 = 0$ can only rely on generic properties of f^{abc}:

$$Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0, \qquad Q^2_{\mathrm{DC}} = 0$$

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- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

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$$Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0, \qquad Q^2_{\mathrm{DC}} = 0$$

- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \to \quad \text{local susy}$$

- Straightforward supersymmetric completion

Generalisations

The double copy to all orders

- Given CK duality of the tree-level physical S-matrix we can run our argument:
  - Non-linear sigma model [Chen-Du '13] $\rightarrow$ special Galileon
  - Orbifolding, fundamental couplings... [Johansson-Ochirov '14] $\rightarrow$ plethora of supergravity theories
  - Bagger–Lambert–Gustavsson [Bargheer-He-McLoughlin '12; Huang-Johansson '12] $\rightarrow$ $D = 3$ maximal supergravity

Super Yang–Mills and Supergravity

BRST-Lagrangian CK duality for super Yang–Mills

- Irreducible super Yang–Mills multiplets are CK duality respecting
  Cf. [Bjerrum-Bohr–Damgaard–Vanhove '09]

- Susy Ward identities: CK gluons $+$ susy $\Rightarrow$ CK gluini
  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

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  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

- CK dual BRST-Lagrangian then follows with (essentially) no new ideas

$$A^{Ia} = (A^{ia}, \Psi^{xa}) = (A_\mu{}^a, \psi_\alpha{}^a, \text{ghosts}, \text{aux})$$

Gluino

- Off-shell CK duality implies supersymmetry directly (exercise!)

$$S^{\mathrm{SYM}}_{\mathrm{BRST}} = \int C_{IJ} c_{ab} A^{Ia} \square A^{Ja} + F_{IJK} f_{abc} A^{Ia} A^{Jb} A^{Kc}$$

$$\delta_\epsilon A^{ia} = F^i{}_{xy} \Psi^{xa} \epsilon^y, \quad \delta_\epsilon \Psi^{xa} = F^x{}_{jy} A^{ja} \epsilon^y$$

Super Yang–Mills and Supergravity

BRST-Lagrangian double copy

- (Type I super Yang–Mills)^2 = Type IIA/B supergravity

$$A^{Ia} = ({A_\mu}^a, {\psi_\alpha}^a, \text{ghosts}, \text{aux})$$

$$A^{J\tilde{J}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \Psi_{\alpha\nu}, \Psi_{\mu\beta}, F_{\alpha\beta}, \text{ghosts}, \text{aux})$$

R-R field strengths

gravitini

- Local NS-R sector susy follows from super Yang–Mills factors

$$\mathcal{Q}_\alpha {A_\mu}^a = {\delta^a}_b {\gamma_{\mu\alpha}}^\beta {\psi_\beta}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = {\gamma_{(\mu\alpha}}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen–Kallosh χ ghosts

$$\bar{c} \otimes \psi \sim \bar{\eta}\,, \quad c \otimes \psi \sim \eta\,, \quad b \otimes \psi \sim \chi$$

- Similar for R–NS sector

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory
  $$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$
  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory
  $$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$
  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths
- Type IIA/B action can be written in terms of field strengths, e.g.

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \tfrac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Super Yang–Mills and Supergravity

Sen's mechanism from double copy Ramond–Ramond sector

- Double copy R–R field strengths are *elementary* fields that correctly reproduce scattering amplitudes through their Feynman diagrams

$$\mathcal{L}^{\mathrm{DC}}_{\text{R–R}} = \overline{F}^{\alpha\beta}\,\Box^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots$$

← direct from double copy

$F_{\alpha\beta} \sim \sum_{p=0}^{d} \frac{1}{p!}(\gamma^{\mu_1\cdots\mu_p}c)F_{\mu_1\cdots\mu_p}$ $\Big\} \to \quad -\frac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots$

Aux. (D-p-1)-form B $\Big\} \to \quad -\frac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \frac{1}{2}B\wedge\star\Box B + \cdots$

Undo Feynman gauge $\Big\} \to \quad -\frac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \frac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots$

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\mathcal{L}^{\mathsf{DC}}_{\mathsf{R\text{–}R}} &= \overline{F}^{\alpha\beta}\,\square^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots \\
&\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\square^{-1}\mathrm{d}F\right) + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\square B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots
\end{aligned}
$$

- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]

- Sen's mechanism was motivated by IIB string field theory, where the R–R sector is naturally given in terms of bispinors - natural double copy shadow

Lecture 3: Homotopy CK Duality and Double Copy

BV formalism

- In the case of open symmetries, BRST complex only up to e.o.m.

$$Q^2_{\mathrm{BRST}}\Phi|_{\Phi_0} = 0$$

- $\Rightarrow$ extend the putative BRST complex

$$\mathfrak{F}_{\mathrm{BV}} = T^*[1]\mathfrak{F}_{\mathrm{BRST}}$$

- Fields Φ^A local coordinates on $\mathfrak{F}_{\mathrm{BRST}}$, antifields Φ^+_A are fibre coordinates
- Using canonical symplectic structure on $\mathfrak{F}_{\mathrm{BV}}$ we extend Q_{BRST} and S_{BRST} to Q_{BV} and S_{BV}, requiring

$$Q_{\mathrm{BV}}|_{\mathfrak{F}_{\mathrm{BRST}}} = Q_{\mathrm{BRST}}, \quad Q_{\mathrm{BV}} = \{S_{\mathrm{BV}}, -\}$$

$$Q_{\mathrm{BV}}S_{\mathrm{BV}} = \{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$$

BV formalism: gauge-fixing and quantization

- Before quantization: imposing gauge-fixing in the BV formalism
- Gauge-fixing S_{BV} means evaluating it on an appropriate Lagrangian submanifold of $\mathfrak{F}_{\mathrm{BV}}$
- We eliminate the antifields by introducing a gauge-fixing fermion Ψ:

$$\Phi^{+}_{A} \;=\; \frac{\delta}{\delta \Phi^{A}} \Psi$$

- Gauge-independence of the expectation values for observables: BV quantum master equation

$$\{S^{\hbar}_{\mathrm{BV}}, S^{\hbar}_{\mathrm{BV}}\} - 2i\hbar \Delta_{\mathrm{BV}} S^{\hbar}_{\mathrm{BV}} \;=\; 0$$

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

Homotopy Algebras and BV Lagrangian Field Theories

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- Lie algebras $\to$ L_∞-algebras, first arose in string field theory:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \dots$
Relations $Antisymmetry + Jacobi$	Relations $Antisymmetry + homotopy Jacobi$

[Zwiebach '93; Hinich–Schechtman '93]

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[Zwiebach '93; Hinich–Schechtman '93]

- Associative algebras $\to$ A_∞-algebras [Stasheff '63]

- Commutative algebras $\to$ C_∞-algebras [Kadeishvili '88]

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathrm{Sym}^p(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

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- Chevalley–Eilenberg formulation of L_∞-algebra $\mathfrak{L}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{L}) = \bar{T}(\mathfrak{L}[1]^*)$$

$$Qt^a = -\sum_n \tfrac{1}{n!} \mu_n{}^a{}_{a_1 \cdots a_n} t^{a_1} \cdots t^{a_n} , \qquad Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi}$$

- Any BV field theory with operator Q_{BV} corresponds to an L_∞-algebra in the CE picture, see e.g. [Jurco-Raspollini-Saemann-Wolf '18]

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\mathsf{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}_0^{\mathsf{YM}} & \oplus & \mathfrak{L}_1^{\mathsf{YM}} & \oplus & \mathfrak{L}_2^{\mathsf{YM}} & \oplus & \mathfrak{L}_3^{\mathsf{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathsf{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathsf{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths + gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \to$ cyclic structure $\langle -,-\rangle_{\mathsf{YM}}$ on $\mathfrak{L}^{\mathsf{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\dots,a)\rangle$

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- BV action $\sim \sum \frac{1}{(i+1)!} \langle a, \mu_i(a, \ldots, a) \rangle$
- L_∞ quasi-isomorphisms $\longrightarrow$ physical equivalence (field redefinitions etc)
  - Strictification: $\mu_i = 0, i > 2 \rightarrow$ cubic theory
  - Minimal model: $\mu_1 = 0 \rightarrow$ tree scattering amplitudes

Cf. [Jurčo-Raspollini-Saemann-Wolf '18; Jurčo-Macrelli-Saemann-Wolf '19]

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

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- $\mathfrak{colour}$: gauge group Lie algebra

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[BLKMSW '21]

- $\mathfrak{colour}$: gauge group Lie algebra

- $\mathfrak{kinematics}$: graded vector space of Poincaré representations of fields

$$\underset{c}{\mathbb{R}[-1]} \quad \oplus \quad \underset{(A_\mu, b)}{(\mathbb{R}^d \oplus \mathbb{R})} \quad \oplus \quad \underset{\bar{c}}{\mathbb{R}[1]} \quad \oplus \quad \underset{B_{\mu\nu\rho}\cdots}{\text{Auxiliaries}}$$

- $\mathfrak{scalar}$: A_∞-algebra of a scalar field theory

$$\langle -, - \rangle_{\mathsf{YM}} = \langle -, - \rangle_{\mathfrak{colour}} \langle -, - \rangle_{\mathfrak{kinematics}} \langle -, - \rangle_{\mathfrak{scalar}}$$

Homotopy algebra of CK duality

Michel Reiterer [1912.03110]

- Proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra!
- Relies on the existence of a degree -1 unary map h on Zeitlin-Costello BV complex for Yang–Mills (think order formulation with A, F^+) satisfying

$$h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

- h exists and is a second-order derivation up to homotopy $\Rightarrow$
  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
  - On-shell tree-level CK duality for physical gluons

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- h exists and is a second-order derivation up to homotopy $\Rightarrow$
  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
  - On-shell tree-level CK duality for physical gluons
- Very special: only $D = 4$, no loop desiderata (ghosts, gauge-fixing)
- A little mysterious: $BV_\infty^\square$-algebra generalise famous BV_∞-algebras (homotopy BV-algebras [Galvez-Carrillo–Tonks–Vallette '09]), where e.g.

$$\Delta^2\square = (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \Delta\square) - (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \mathrm{id} \otimes \square)$$

Homotopy algebra of CK duality

The homotopy algebra of CK duality [BJKMSW 'to appear 21]

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- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra, cf. [Getzler '93]

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \underbrace{\mathfrak{kinematics} \otimes_{\tau} \mathfrak{scalar}}_{\mathfrak{Kin}\ \equiv\ BV^{\square}\text{-algebra}}$$

- $BV^{\square}$-algebra comes with two products $- \cdot -$ and $[-,-]$ and three unary operators

$$d^2 = h^2 = 0, \quad dh + hd = \square$$

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The homotopy algebra of CK duality [BJKMSW 'to appear 21]

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  - The homotopy $BV^{\square}$-algebra depends on the ambient category
  - In the usual category of chain complexes d is privileged

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  $$d^2 = h^2 = 0, \quad dh + hd = \square$$
  - The homotopy $BV^{\square}$-algebra depends on the ambient category
  - In the usual category of chain complexes d is privileged
  - Introduce symmetric monoidal category of Hodge complexes (modules over twisted Hopf algebras with central element $\square$)
  $$d^2 = h^2 = 0, \quad dh + hd = \square$$
  Coassociativity $\Rightarrow$ the seven-term identity
  - In this category, both d and h are a part of the ambient structure

Homotopy algebra of CK duality

The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations

Homotopy algebra of CK duality

The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations
- Computational efficiency:
  - Purely tree-level calculations
  - One identity at any order (the rest follow axiomatically)

$$\sum_{p+q=n+2} n\text{-point tree with two internal } (p\text{-ary and } q\text{-ary}) \text{ vertices}$$

$$= n\text{-point tree with one internal } (n\text{-ary}) \text{ vertex}$$

  - But, work with Feynman diagrams - marry with on-shell methods?

Future work

- AdS background [Zhou '21; Diwakar-Herderschee-Roiban-Teng '21 ...] $\to$ Hopf algebra of universal enveloping algebra of AdS isometries
- Bagger-Lambert-Gustavsson CK duality [Bargheer-He-McLoughlin '12; Huang-Johansson '12] $\to$ m-ary $BV^{\square}$ operads
- Matter coupling [Johansson-Ochirov '14] $\to$ many-sorted $BV^{\square}$ operads
- String theory (modular envelope over) $BV_{\infty}^{L_0}$

  $$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

  Cf. BV_{∞} structure on TVOA [Galvez-Carrillo–Tonks–Vallette '09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman '93]
- Counterterms?

Thanks for listening