Colour-Kinematic Duality Redux

Perfect off-shell 'BRST-Lagrangian CK duality'

- YM BRST-action with manifest off-shell CK duality

$$S^{\text{YM}}_{\text{BRST CK-dual}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_{\mu}{}^{a}, b^{a}, \bar{c}^{a}, c^{a}, \underbrace{B_{\mu\nu\rho}{}^{a}, \bar{K}_{\mu}{}^{a}, \ldots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} gauge group Killing form and structure constants

- C_{ij}, F^{ijk} are differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^{i}_{k)l} = 0 \qquad F_{[ij|l} F^{l}_{|k]m} = 0$$

BRST Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories **Factorisation** **Daughter theories**

$c_{IJ}\phi^I \,\Box\, \phi^J + f_{IJK}\,\phi^I\,\phi^J\,\phi^K$

$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \,\Box\, \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\,\tilde{\phi}^{\tilde{I}}\,\tilde{\phi}^{\tilde{J}}\,\tilde{\phi}^{\tilde{K}}$

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Factorise (meiosis)

$$c_{ab} C_{ij} \Phi^{ai} \Box \Phi^{aj} + f_{abc} F_{ijk} \Phi^{ai} \Phi^{bj} \Phi^{ck}$$

$$c_{IJ} \phi^{I} \Box \phi^{J} + f_{IJK} \phi^{I} \phi^{J} \phi^{K}$$

$$\tilde{c}_{\tilde{I}\tilde{J}} \tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}} \tilde{\phi}^{\tilde{I}} \tilde{\phi}^{\tilde{J}} \tilde{\phi}^{\tilde{K}}$$

$$\tilde{c}_{\tilde{a}\tilde{b}} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} \Phi^{\tilde{a}\tilde{\imath}} \Box \tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\Phi}^{\tilde{a}\tilde{\imath}} \tilde{\Phi}^{\tilde{b}\tilde{\jmath}} \tilde{\Phi}^{\tilde{c}\tilde{k}}$$

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Syngamatic reproduction of factorable theories

Parent theories | **Factorisation** | **Daughter theories**

Double copy

Fusion (Syngamy)

$$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$$

Factorise (meiosis)

$$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$$

$$c_{IJ}\phi^{I}\,\Box\,\phi^{J} + f_{IJK}\phi^{I}\phi^{J}\phi^{K}$$

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$$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$$

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$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$

Factorise (meiosis)

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Zeroth copy

$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$

BRST Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$$c_{IJ}\phi^{I}\,\Box\,\phi^{J} + f_{IJK}\phi^{I}\phi^{J}\phi^{K}$$

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Factorise (meiosis)

Factorisation

$$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$$

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Daughter theories

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$$\tilde{c}_{\tilde{a}\tilde{b}}C_{ij}\Phi^{\tilde{a}i}\Box\Phi^{\tilde{a}j} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}F_{ijk}\Phi^{\tilde{a}i}\Phi^{\tilde{b}j}\Phi^{\tilde{c}k}$$

$$c_{ab}\tilde{C}_{ij}\Phi^{a\tilde{\imath}}\Box\Phi^{a\tilde{\jmath}} + f_{abc}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{a\tilde{\imath}}\Phi^{b\tilde{\jmath}}\Phi^{c\tilde{k}}$$

$$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$$

Key examples

Yang–Mills squared

- Pair kinematics: $S^{\text{YM}}_{\text{BRST-CK}} \otimes \tilde{S}^{\text{YM}}_{\text{BRST-CK}} \to S^{\mathcal{N}=0}_{\text{DC}}$ ($\mathcal{N}=0$ supergravity)

$$A^{ia} = (A_\mu{}^a, \text{ghosts}, \text{auxiliaries}) \qquad S^{\text{YM}}_{\text{CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \text{ghosts}, \text{auxiliaries}) \qquad S^{\mathcal{N}=0}_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

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$G \times \tilde{G}$ bi-adjoint scalar theory

- Pair colours: $S^{\text{YM}}_{\text{BRST-CK}} \otimes \tilde{S}^{\text{YM}}_{\text{BRST-CK}} \to S^{\text{bi-adj}}_{\text{DC}}$

$$S^{\text{bi-adj}}_{\text{DC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

BRST double copy fields and action

- Factor BRST complex (field space) into colour and kinematics:

$$\mathfrak{g} \otimes \mathfrak{V} \otimes \mathfrak{S} \longrightarrow \mathfrak{V} \otimes \mathfrak{V} \otimes \mathfrak{S}$$

fields				antifields		
factorisation	role	L_∞ deg	dim	factorisation	L_∞ deg	dim
$\lambda = [\mathrm{g},\mathrm{g}]\mathrm{s}_x \frac{1}{2}\lambda(x)$	ghost for ghost	-1	$\frac{d}{2}-3$	$\lambda^+ = [\mathrm{a},\mathrm{a}]\mathrm{s}^+_x \frac{1}{2}\lambda^+(x)$	4	$\frac{d}{2}+3$
$\Lambda = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}_x\Lambda_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$\Lambda^+ = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}^+_x\Lambda^+_\mu$	3	$\frac{d}{2}+2$
$\gamma = [\mathrm{g},\mathrm{n}]\mathrm{s}_x\gamma(x)$	NL field of Λ_μ	0	$\frac{d}{2}-2$	$\gamma^+ = [\mathrm{a},\mathrm{n}]\mathrm{s}^+_x\gamma^+(x)$	3	$\frac{d}{2}+2$
$B = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}_x\frac{1}{2}B_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$B^+ = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathrm{s}^+_x\frac{1}{2}B^+_{\mu\nu}(x)$	2	$\frac{d}{2}+1$
$\alpha = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}_x\alpha_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\alpha^+ = [\mathrm{n},\mathrm{v}^\mu]\mathrm{s}^+_x\alpha^+_\mu(x)$	2	$\frac{d}{2}+1$
$\varepsilon = [\mathrm{g},\mathrm{a}]\mathrm{s}_x\varepsilon(x)$	anti-ghost of Λ_μ	1	$\frac{d}{2}-1$	$\varepsilon^+ = [\mathrm{g},\mathrm{a}]\mathrm{s}^+_x\varepsilon^+(x)$	2	$\frac{d}{2}+1$
$\bar\Lambda = [\mathrm{a},\mathrm{v}^\mu]\mathrm{s}_x\bar\Lambda_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar\Lambda^+ = [\mathrm{g},\mathrm{v}^\mu]\mathrm{s}^+_x\bar\Lambda^+_\mu(x)$	1	$\frac{d}{2}$
$\bar\gamma = [\mathrm{a},\mathrm{n}]\mathrm{s}_x\bar\gamma(x)$	NL field of $\bar\Lambda_\mu$	2	$\frac{d}{2}$	$\bar\gamma^+ = [\mathrm{g},\mathrm{n}]\mathrm{s}^+_x\bar\gamma^+(x)$	1	$\frac{d}{2}$
$\bar\lambda = [\mathrm{a},\mathrm{a}]\mathrm{s}_x\frac{1}{2}\bar\lambda(x)$	anti-ghost of $\bar\Lambda_\mu$	3	$\frac{d}{2}+1$	$\bar\lambda^+ = [\mathrm{g},\mathrm{g}]\mathrm{s}^+_x\frac{1}{2}\bar\lambda^+(x)$	0	$\frac{d}{2}-1$
$X = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}_xX_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$X^+ = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}^+_xX^+_\mu(x)$	3	$\frac{d}{2}+2$
$\beta = (\mathrm{g},\mathrm{n})\mathrm{s}_x\beta(x)$	NL field of X_μ	0	$\frac{d}{2}-2$	$\beta^+ = (\mathrm{a},\mathrm{n})\mathrm{s}^+_x\beta^+(x)$	3	$\frac{d}{2}+2$
$h = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}_x\frac{1}{2}h_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$h^+ = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathrm{s}^+_x\frac{1}{2}h^+_{\mu\nu}(x)$	2	$\frac{d}{2}+1$
$\varpi = (\mathrm{n},\mathrm{v}^\mu)\mathrm{s}_x\varpi_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\varpi^+ = (\mathrm{n},\mathrm{v}^\mu)\mathrm{s}^+_x\varpi^+_\mu(x)$	2	$\frac{d}{2}+1$
$\pi = (\mathrm{n},\mathrm{n})\mathrm{s}_x\frac{1}{2}\pi(x)$	NL field of ϖ_μ	1	$\frac{d}{2}-1$	$\pi^+ = (\mathrm{n},\mathrm{n})\mathrm{s}^+_x\frac{1}{2}\pi^+(x)$	2	$\frac{d}{2}+1$
$\delta = (\mathrm{g},\mathrm{a})\mathrm{s}_x\delta(x)$	anti-ghost of X_μ	1	$\frac{d}{2}-1$	$\delta^+ = (\mathrm{g},\mathrm{a})\mathrm{s}^+_x\delta^+(x)$	2	$\frac{d}{2}+1$
$\bar X = (\mathrm{a},\mathrm{v}^\mu)\mathrm{s}_x\bar X_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar X^+ = (\mathrm{g},\mathrm{v}^\mu)\mathrm{s}^+_x\bar X_\mu(x)$	1	$\frac{d}{2}$
$\bar\beta = (\mathrm{a},\mathrm{n})\mathrm{s}_x\bar\beta(x)$	NL field of $\bar X_\mu$	2	$\frac{d}{2}$	$\bar\beta^+ = (\mathrm{g},\mathrm{n})\mathrm{s}^+_x\bar\beta^+(x)$	1	$\frac{d}{2}$

BRST Double-Copy: Fields and Action

- Double copy BRST-action uniquely determined:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{DC}} =& \tfrac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \tfrac{1}{2} \varpi_\mu \Box \varpi^\mu + \xi^2 (\partial^\mu \varpi_\mu)^2 + \tfrac{1}{2} \pi \Box \pi \\
& - 2\xi \varpi^\nu \Box^{\frac{1}{2}} \partial^\mu h_{\mu\nu} - 2\xi \pi \Box^{\frac{1}{2}} \partial_\mu \varpi^\mu + 2\xi^2 \pi \partial_\mu \partial_\nu h^{\mu\nu} \\
& - 2\bar{X}_\mu \Box X^\mu - \delta \Box \delta - 2\bar{\beta} \Box \beta \\
& + \tfrac{1}{2} B_{\mu\nu} \Box B^{\mu\nu} - 2\bar{\Lambda}_\mu \Box \Lambda^\mu + \alpha_\mu \Box \alpha^\mu + \xi^2 (\partial^\mu \alpha_\mu)^2 + \varepsilon \Box \varepsilon - \bar{\lambda} \Box \lambda - 2\bar{\gamma} \Box \gamma \\
& - 2\xi \alpha^\nu \Box^{\frac{1}{2}} \partial^\mu B_{\mu\nu} - 2\xi \gamma \Box^{\frac{1}{2}} \partial_\mu \bar{\Lambda}^\mu + 2\xi \bar{\gamma} \Box^{\frac{1}{2}} \partial_\mu \Lambda^\mu \\
& - 2\xi \beta \Box^{\frac{1}{2}} \partial_\mu \bar{X}^\mu + 2\xi \bar{\beta} \Box^{\frac{1}{2}} \partial_\mu X^\mu + \cdots
\end{aligned}
$$

Graviton-dilaton

KR 2-form

- Canonical field redefinition to Fierz-Pauli + Kalb-Ramond + dilaton action

$$QB = d\Lambda, \qquad Q\Lambda = d\lambda, \qquad Q\lambda = 0$$

BRST Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to true $S_{\text{BRST}}^{\mathcal{N}=0}$?

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- Semi-classical equivalence (requires off-shell BRST CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \rightarrow \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

$$\sum \frac{nc}{d} \quad \rightarrow \quad \sum \frac{n\tilde{n}}{d}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points

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- Quantum consistency: is there some Q such that

$$QS_{\text{DC}}^{\mathcal{N}=0} = 0, \qquad Q^2 = 0$$

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Answer: double-copy operator Q_{DC} (requires off-shell BRST CK duality)

BRST Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BRST-action implies double copy BRST operator Q_{DC}

$$QA^{ia} = Q^{i}{}_{j}A^{ja} + Q^{i}{}_{jk}f^{a}{}_{bc}A^{jb}A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}i} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}}\tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}}\tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}\tilde{A}^{\tilde{b}\tilde{\jmath}}\tilde{A}^{\tilde{c}\tilde{k}}$$

$$Q_{\text{DC}} = \underbrace{Q^{i}{}_{j}A^{j\tilde{\imath}} + Q^{i}{}_{jk}F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}}A^{i\tilde{\jmath}} + F^{i}{}_{jk}Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_R}$$

- Yang-Mills gauge $\Rightarrow$ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

$$Q_{\text{2-form}}B = \Lambda, \quad Q_{\text{2-form}}\Lambda = \lambda \quad Q_{\text{2-form}}\lambda = 0$$

Cf. [Anastasiou-LB-Duff-Hughes-Nagy '14]

BRST Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc} and $QS^{\text{YM}}_{\text{BRST}} = 0$, $Q^2 = 0$ can only rely on generic properties of f^{abc}:

$$Q_{\text{DC}} S_{\text{DC}} = 0, \qquad Q^2_{\text{DC}} = 0$$

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- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \rightarrow \quad \text{local susy}$$

- Straightforward supersymmetric completion

Generalisations

The double copy to all orders

- Given CK duality of the tree-level physical S-matrix we can run our argument:
  - Non-linear sigma model [Chen-Du '13] $\rightarrow$ special Galileon
  - Orbifolding, fundamental couplings. . . [Johansson-Ochirov '14] $\rightarrow$ plethora of supergravity theories
  - Bagger–Lambert–Gustavsson [Bargheer-He-McLoughlin '12; Huang-Johansson '12]$\rightarrow$ $D = 3$ maximal supergravity

Super Yang–Mills and Supergravity

BRST-Lagrangian CK duality for super Yang–Mills

- Irreducible super Yang–Mills multiplets are CK duality respecting
  Cf. [Bjerrum-Bohr–Damgaard–Vanhove '09]

- Susy Ward identities: CK gluons $+$ susy $\Rightarrow$ CK gluini
  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

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- CK dual BRST-Lagrangian then follows with (essentially) no new ideas

$$A^{Ia} = (A^{ia}, \Psi^{xa}) = (A_\mu{}^a, \psi_\alpha{}^a, \text{ghosts}, \text{aux})$$

Gluino

- Off-shell CK duality implies supersymmetry directly (exercise!)

$$S^{\mathrm{SYM}}_{\mathrm{BRST}} = \int C_{IJ} c_{ab} A^{Ia} \Box A^{Ja} + F_{IJK} f_{abc} A^{Ia} A^{Jb} A^{Kc}$$

$$\delta_\epsilon A^{ia} = F^i{}_{xy} \Psi^{xa} \epsilon^y, \quad \delta_\epsilon \Psi^{xa} = F^x{}_{jy} A^{ja} \epsilon^y$$

Super Yang–Mills and Supergravity

BRST-Lagrangian double copy

- (Type I super Yang–Mills)^2 = Type IIA/B supergravity

$$A^{Ia} = ({A_\mu}^a, {\psi_\alpha}^a, \text{ghosts}, \text{aux})$$

R-R field strengths

$$A^{J\tilde{J}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \Psi_{\alpha\nu}, \Psi_{\mu\beta}, F_{\alpha\beta}, \text{ghosts}, \text{aux})$$

gravitini

- Local NS-R sector susy follows from super Yang–Mills factors

$$\mathcal{Q}_\alpha {A_\mu}^a = {\delta^a}_b {\gamma_{\mu\alpha}}^\beta {\psi_\beta}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = {\gamma_{(\mu\alpha}}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen–Kallosh χ ghosts

$$\bar{c} \otimes \psi \sim \bar{\eta}\,, \quad c \otimes \psi \sim \eta\,, \quad b \otimes \psi \sim \chi$$

- Similar for R–NS sector

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory

$$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$

  - The BRST transformation of the gluino has no linear contribution, $Q_{\mathrm{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory

$$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$

  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths
- Type IIA/B action can be written in terms of field strengths, e.g.

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \tfrac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Super Yang–Mills and Supergravity

Sen's mechanism from double copy Ramond–Ramond sector

- Double copy R–R field strengths are *elementary* fields that correctly reproduce scattering amplitudes through their Feynman diagrams

$$\mathcal{L}^{\text{DC}}_{\text{R–R}} = \overline{F}^{\alpha\beta}\,\Box^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots$$

← direct from double copy

$F_{\alpha\beta} \sim \sum_{p=0}^{d} \frac{1}{p!}(\gamma^{\mu_1\cdots\mu_p}c)F_{\mu_1\cdots\mu_p}$ }

$$\to \quad -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots$$

Aux. (D-p-1)-form B }

$$\to \quad -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots$$

Undo Feynman gauge }

$$\to \quad -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots$$

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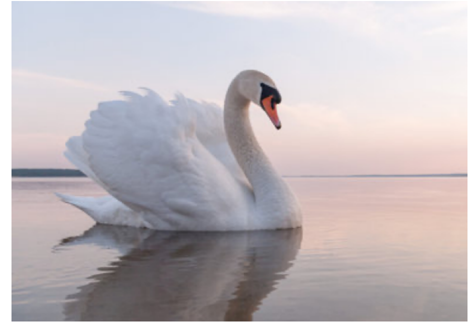
$$\begin{aligned}\mathcal{L}^{\mathsf{DC}}_{\mathsf{R\text{–}R}} &= \overline{F}^{\alpha\beta}\,\Box^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots \\ &\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots \\ &\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots \\ &\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots\end{aligned}$$

- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]

- Sen's mechanism was motivated by IIB string field theory, where the R–R sector is naturally given in terms of bispinors - natural double copy shadow

Lecture 3: Homotopy CK Duality and Double Copy

The Homotopy Algebra of Colour-Kinematics Duality



- CK duality: kinematic algebra
  Hands on quantum field theory

- A: $BV_\infty^\square$ homotopy algebra
  Abstract mathematics [BJKMSW '22 (to appear)]

Gluons, quarks. . .
Scattering amplitudes
Symmetries of Nature

$BV_\infty^\square$-algebras — Homotopy transfer — Exact CK duality

$BV_\infty^\square$-operad — Koszul resolution — $BV^\square$-operad

Symmetric monoidal category → Cat(Hodge)

Twisted tensor product of (co)algebras

$H^\square$ — $H^\square$-modules — Hodge Complexes

Hopf algebras with central element $\square$

Natural generalisation of familiar Hodge–de Rham complex

Homotopy algebras
Koszul duality
Operads
Symmetric monoidal categories

- CK duality: a symmetry of Nature as a mug is a donut!

Batalin–Vilkovisky formalism

- In the case of open symmetries, BRST complex only up to e.o.m.

$$Q^2_{\mathrm{BRST}}\Phi|_{\Phi_0} = 0$$

- $\Rightarrow$ extend the putative BRST complex

$$\mathfrak{F}_{\mathsf{BV}} = T^*[1]\mathfrak{F}_{\mathsf{BRST}}$$

- Fields Φ^A local coordinates on $\mathfrak{F}_{\mathsf{BRST}}$, antifields Φ^+_A are fibre coordinates

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- Fields Φ^A local coordinates on $\mathfrak{F}_{\mathrm{BRST}}$, antifields Φ^+_A are fibre coordinates
- Using canonical symplectic structure on $\mathfrak{F}_{\mathrm{BV}}$ we extend Q_{BRST} and S_{BRST} to Q_{BV} and S_{BV} ($\{-,-\}$ is degree -1 $\to$ Gerstenhaber algebra)

$$Q_{\mathrm{BV}} \;=\; \{S_{\mathrm{BV}}, -\}, \quad S_{\mathrm{BV}}|_{\mathfrak{F}_{\mathrm{BRST}}} \;=\; S_{\mathrm{BRST}} \quad \pi_* Q_{\mathrm{BV}} \;=\; Q_{\mathrm{BRST}}$$

$$Q_{\mathrm{BV}}S_{\mathrm{BV}} \;=\; \{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} \;=\; 0$$

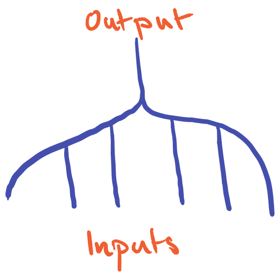
$$S_{\mathrm{BV}}[\phi, \phi^+] \;=S_{\mathrm{classical}}[\phi] + \phi^+_A(Q_{\mathrm{BV}}\phi)^A$$

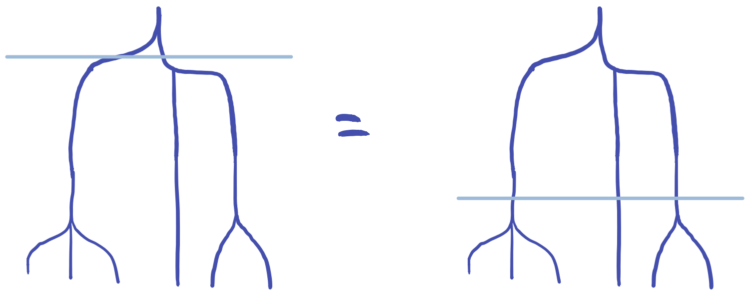
- Gauge-fixing fermion Ψ gives symplectomorphism

$$\Phi \mapsto \Phi + \{\Psi, \Phi\}, \quad \Phi^+_A \mapsto \Phi^+_A \;+\; \frac{\delta}{\delta\Phi^A}\Psi$$

Operads and homotopy algebras

- Operads: encode types of algebras (symmetric operad: monoid in the monoidal category of $\mathbb{S}$-modules)





- Powerful abstract reasoning for deducing concrete statements (e.g. Koszul duality)
- Given a chain complex with algebraic structure, can this structure be transferred to homotopy equivalent chain complexes?
- Yes, if we allow for a richer algebraic structure of higher operations: homotopy algebras

Operads and homotopy algebras

- I will only leave the door to ∞-algebras ajar: you should push it open!
- Hossenfelder warned physicists to not get "Lost in Math", but getting lost can be fun! (And is the only way to discover something truly unexpected)

Operads and homotopy algebras

- I will only leave the door to ∞-algebras ajar: you should push it open!
- Hossenfelder warned physicists to not get "Lost in Math", but getting lost can be fun! (And is the only way to discover something truly unexpected)
- You won't be lost in the dark, there are excellent books:

  'Operads in algebra, topology and physics', Martin Markl, Steve Shnider and Jim Stasheff

  'Algebraic Operads', Jean-Louis Loday and Bruno Vallette

  'Algebraic Structure of String Field Theory', Martin Doubek, Branislav Jurčo, Martin Markl and Ivo Sachs

Homotopy algebras

- Informally: generalise familiar (associate, commutative, Lie. . .) algebras to include higher products satisfying higher relations up to homotopies

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- Lie algebras → L_∞-algebras [Zwiebach '93; Hinich–Schechtman '93]:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \dots$
Relations *Antisymmetry + Jacobi*	Relations *Antisymmetry + homotopyJacobi*

$$[[x,y],z] + (-1)^{x(y+z)}[[y,z],x] + (-1)^{y(x+z)}[[x,z],y] = -[[x,y,z]]$$

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- Associative algebras $\to$ A_∞-algebras [Stasheff '63]
- Commutative algebras $\to$ C_∞-algebras [Kadeishvili '88]
- BV algebras $\to$ BV_∞-algebras [Galvez-Carrillo–Tonks–Vallette '09]
- Lie algebras $\to$ EL_∞-algebras [LB-Kim-Saemann '21]

Homotopy Maurer-Cartan theory

- Inner product $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on dgLa $(\mathfrak{g}, d)$:

$$\langle x, dy \rangle = (-)^{1+x+y+xy} \langle y, dx \rangle, \qquad \langle x, [y,z] \rangle = (-)^{z(x+y)} \langle z, [x,y] \rangle$$

- Cyclic structure $\langle -,- \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$ on L_∞-algebra $(\mathfrak{L}, \mu_i)$:

$$\langle x_1, \mu_i(x_2, \ldots x_{i+1}) \rangle = (-)^{i+i(x_1+x_{i+1})+x_{i+1}\sum_{j=1}^{i} x_j} \langle x_{i+1}, \mu_i(x_1, \ldots x_i) \rangle$$

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- (Homotopy) Maurer-Cartan element $a \in \mathfrak{g}$ ($a \in \mathfrak{L}$) from (h) MC-action:

$$f_a = da + \tfrac{1}{2}[a,a] = 0, \qquad S_{\mathrm{MC}} = \tfrac{1}{2}\langle a, da \rangle + \tfrac{1}{3!}\langle a, [a,a] \rangle$$

$$F_a = \sum_k \tfrac{1}{k!}\mu_k(a, a, \ldots, a) = 0, \qquad S_{\mathrm{hMC}} = \sum_k \tfrac{1}{(k+1)!}\langle a, \mu_k(a, a, \ldots, a) \rangle$$

- Covariant derivative, Bianchi identity and gauge transformations:

$$D_a x = \sum_k \frac{(-1)^k}{k!}\mu_{k+1}(x, a, \ldots, a), \quad D_a F_a = 0, \quad \delta_c a = D_a c$$

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathrm{Sym}^p(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

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- Chevalley–Eilenberg formulation of L_∞-algebra $\mathfrak{L}$ with basis t_a:

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$$Qt^a = -\sum_n \tfrac{1}{n!} \mu_n{}^a{}_{a_1 \cdots a_n} t^{a_1} \cdots t^{a_n}, \qquad Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi}$$

- Any BV field theory with Q_{BV} corresponds to a cyclic L_∞-algebra in the CE picture, see e.g. [Jurčo-Raspollini-Saemann-Wolf '18]

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\mathsf{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}_0^{\mathsf{YM}} & \oplus & \mathfrak{L}_1^{\mathsf{YM}} & \oplus & \mathfrak{L}_2^{\mathsf{YM}} & \oplus & \mathfrak{L}_3^{\mathsf{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathsf{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathsf{Id}} & b^+ & &
\end{array}
$$

- Define 'superfield' $\mathcal{A}$ as degree 1 element in $(\mathfrak{L}^{\mathsf{YM}}[1])^* \otimes \mathfrak{L}^{\mathsf{YM}}$

$$S_{\mathrm{BV}}^{\mathrm{YM}}[\phi, \phi^+] = S_{\mathrm{hMC}}[\mathcal{A}]$$

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- L_∞ quasi-isomorphisms $\longrightarrow$ physical equivalence (field redefinitions etc)
  - Strictification: $\mu_i = 0, i > 2 \rightarrow$ cubic theory
  - Minimal model: $\mu_1 = 0 \rightarrow$ tree scattering amplitudes

See [Jurčo-Raspollini-Saemann-Wolf '18; Jurčo-Macrelli-Saemann-Wolf '19]

Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathrm{YM}}$ factorises [BJKMSW '21]

$$\mathfrak{L}^{\mathrm{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

- $\mathfrak{colour}$: gauge group Lie algebra
- $\mathfrak{kinematics}$: graded vector space of Poincaré representations of fields

$$\underset{c}{\mathbb{R}[-1]} \quad \oplus \quad \underset{(A_\mu,\, b)}{(\mathbb{R}^d \oplus \mathbb{R})} \quad \oplus \quad \underset{\bar{c}}{\mathbb{R}[1]} \quad \oplus \quad \underset{B_{\mu\nu\rho}\cdots}{\text{Auxiliaries}}$$

- $\mathfrak{scalar}$: A_∞-algebra of a scalar field theory

$$\langle -, - \rangle_{\mathrm{YM}} = \langle -, - \rangle_{\mathfrak{colour}} \langle -, - \rangle_{\mathfrak{kinematics}} \langle -, - \rangle_{\mathfrak{scalar}}$$

- Double copy:

$$\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar} \longrightarrow \mathfrak{kinematics} \otimes_\tau \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$$

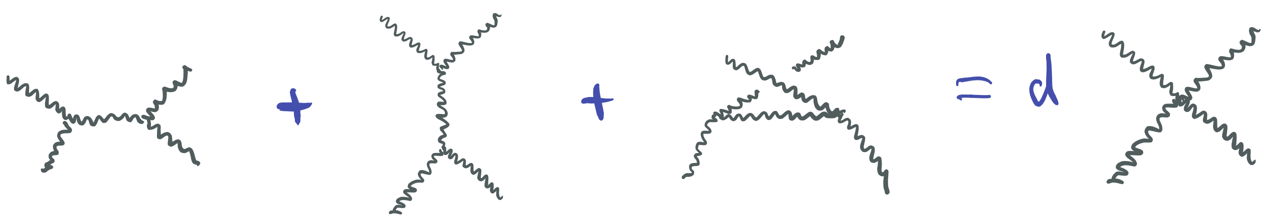
Homotopy algebra of CK duality

Michel Reiterer [1912.03110]

- Remarkable proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra:
  1. There is a degree -1 unary map h on Zeitlin-Costello complex $\mathfrak{ZC}$

     $$h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

  2. There is a $BV_\infty^\square$-algebra on $\mathfrak{ZC}$ that deforms the BV_∞-algebras
  3. Every such h is second order (in the graded sense) up to homotopy
  4. Minimal model and strictification then implies CK duality of physics tree-level S-matrix



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- Very special: only $D = 4$ and strictly tree-level

- A little mysterious, e.g. requires a seven term identity

  $$\Delta^2\square = (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id}\otimes\Delta\square) - (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id}\otimes\mathrm{id}\otimes\square)$$

- Reiterer is right, but let's make him righter!

Homotopy algebra of CK duality

- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra (cf. [Getzler '93])

$$\mathfrak{L}_{\rm CK}^{\sf YM} = \mathfrak{g} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar} \equiv \mathfrak{g} \otimes \mathfrak{Kin}$$

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- A $BV^{\Box}$-algebra $(V, d, - \cdot -, [-,-], h)$ is a graded $\mathfrak{H}$-module V equipped with two $\mathfrak{H}$-linear unary operators $d, h : V \to V$ of degrees 1 and -1, respectively, and two $\mathfrak{H}$-linear binary operators $- \cdot -, [-,-] : V \otimes_K V \to V$ (where $V \otimes_K V$ has been given the canonical $\mathfrak{H}$-module structure using the Hopf coproduct) of degrees 0 and -1, respectively, such that

$$d^2 = 0, \qquad h^2 = 0, \qquad dh + hd = \Box$$

$$\begin{aligned} d(x \cdot y) - dx \cdot y - (-1)^{x} x \cdot dy &= 0 \\ h(x \cdot y) - hx \cdot y - (-1)^{x} x \cdot hy &= [x, y] \\ d[x, y] - [dx, y] - (-1)^{x}[x, dy] &= \Box(x \cdot y) - \Box x \cdot y - x \cdot \Box y \end{aligned}$$

and h is second order, $\mathfrak{H}$ is a cocommutative Hopf algebra with central element $\Box$ compatible with a Hodge-twisted Hopf algebra [BJKMSW (to appear)]

Homotopy algebra of CK duality

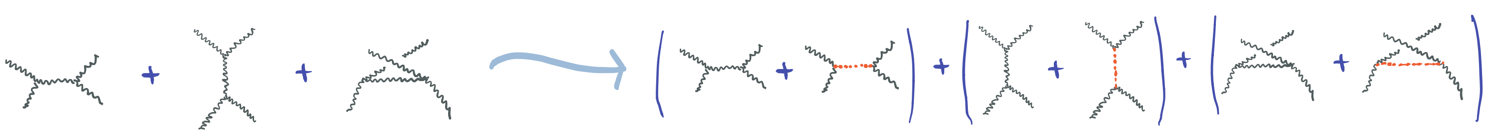
- The homotopy $BV^{\square}$-algebra depends on the ambient category
- In the usual category of chain complexes d is privileged: $BV^{\square}_{\infty}$

Homotopy algebra of CK duality

- The homotopy $BV^{\square}$-algebra depends on the ambient category
- In the usual category of chain complexes d is privileged: $BV^{\square}_{\infty}$
- Cat(Hodge): symmetric monoidal category of Hodge complexes (modules over Hodge-twisted Hopf algebras with central element $\square$)

$$\Delta^2\square = (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \Delta\square) - (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \mathrm{id} \otimes \square)$$

- Then d and h are a part of the ambient structure
- Koszul res. in Cat(Hodge) is a $BV^{\square}_{H\infty}$-algebra: the algebra of CK duality
- $BV^{\square} \to BV^{\square}_{H\infty}$-algebra: integrating out auxiliary fields



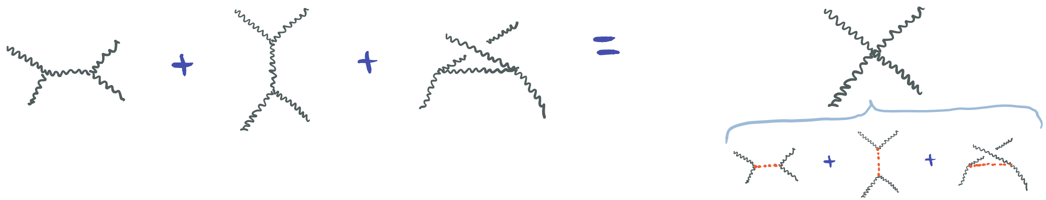
Homotopy algebra of CK duality

- Higher products:

$m^0_n(x_1, \ldots x_n)$	Colour-stripped vertices of gauge-fixed action
$m^0_{p,q}(x_1, \ldots x_n)$	Tolotti-Weinzerl corrections for tree on-shell CK duality
$m^0_{p,q,r}(x_1, \ldots x_n)$	Field red. vertices correcting for off-shell CK duality

- Axioms (homotopy 'Jacobi' relations) $\Rightarrow$ off-shell CK duality

$$[[x,y],z] \pm [[y,z],x] \pm [[z,x],y] = (\Box_{xy} + \Box_{yz} + \Box_{xz}) m^0_{1,2}(x,y,z)$$



- Purely tree-level calculations and one identity at any order:

$$\sum_{p+q=n+2} n\text{-point tree with two internal } (p\text{-ary and } q\text{-ary}) \text{ vertices}$$

$$= n\text{-point tree with one internal } (n\text{-ary}) \text{ vertex}$$

Where next?

- AdS background [Zhou '21; Diwakar-Herderschee-Roiban-Teng '21 ...] $\to$ Hopf algebra of universal enveloping algebra of AdS isometries
- Bagger-Lambert-Gustavsson CK duality [Bargheer-He-McLoughlin '12; Huang-Johansson '12] $\to$ m-ary $BV^{\square}$ operads
- Matter coupling [Johansson-Ochirov '14] $\to$ many-sorted $BV^{\square}$ operads
- String theory (modular envelope over) $BV_{\infty}^{L_0}$

$$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

Cf. BV_∞ structure on TVOA [Galvez-Carrillo–Tonks–Vallette '09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman '93]

Thanks for listening