Diffusion in almost-Riemannian geometry

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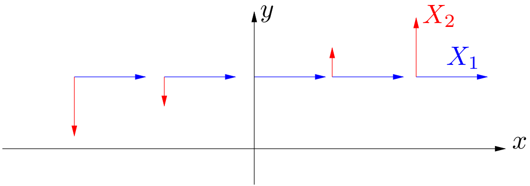
Almost Riemannian structures are generalized Riemannian structures that include the Grushin plane for which a (generalized) orthonormal frame is given by

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad (x, y) \in \mathbf{R}^2$$



On $\mathbf{R}^2 \setminus \{x = 0\}$ we have $g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}$, $dA = \frac{1}{|x|} dx\, dy, \quad K = -\frac{2}{x^2}$

origin of 2-ARS

- **context of Hypoelliptic operators:**
  - Baouendi, '67, Grushin '70, Franchi-Lanconelli '84 ($\tilde{\Delta} = \partial_x^2 + x^2 \partial_y^2$).
- **example of rank varying sub-Riemannian stuctures**
  - Gromov, Bellaiche, '96.
- **context of optimal control:**
  - control of three level quantum system (2006, Charlot, Chambrion, U.B.)
  - orbital transfer in space mechanics (2009, Bonnard, Caillau et al.)

plan of the 3 lectures

- definition of 2D-Almost Riemannian Manifold
- normal forms
- properties of the singular set
- geodesics
- a Gauss-Bonnet theorem
- heat and Schroedinger evolution for the Laplace-Beltrami operator

$$\Delta(\cdot) = \mathrm{div}(\mathrm{grad}(\cdot))$$

We restrict to dimension 2 (it is already rich enough). Some results can be extended to higher dimension

I will follow the Chapter 9 of

[1] A. Agrachev, D. Barilari, and U. Boscain. A Comprehensive Introduction to sub-Riemannian Geometry, volume 181 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2020.

Definition of a 2D Riemannian manifold

2D-Riemannian Manifold (M, g)

- M 2D-differentiable manifold
- g_q is a positive definite, symmetric, bilinear form on $T_q M$, $q \to g_q$ is smooth

→ $\|v\| := \sqrt{g_q(v, v)}, \qquad g_q(v, w) = \|v\| \|w\| \cos(\theta), \quad v, w \in T_q M$

→ Given $\gamma : [0, T] \to M$, A.C., one defines $\ell(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}, \dot{\gamma})} dt$

→ Distance between two points

$d(q_0, q_1) = \inf\{\ell(\gamma) | \ \gamma(0) = q_0, \gamma(T) = q_1\} \Rightarrow$ Metric Structure

(compatible with the topological structure of M)

In Riemannian geometry, the problem of finding the distance between two points can be locally written as an Optimal Control Problem:

$X_1(q), X_2(q)$ local orthonormal moving frame i.e.

$$g(X_1, X_1) = 1,\ g(X_1, X_2) = 0,\ \ g(X_2, X_2) = 1.$$

$$\begin{cases} \dot\gamma(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \gamma(0) = q_0, \quad \gamma(T) = q_1 \\ \min \int_0^T \|\dot\gamma(t)\| dt = \min \int_0^T \sqrt{g(\dot\gamma(t), \dot\gamma(t))} dt = \min \int_0^T \sqrt{u_1(t)^2 + u_2(t)^2} dt \end{cases}$$

observ. 1: X_1 and X_2 are always linearly independent
observ. 2: this construction is global only if M is parallelizable (the only compact orientable is the torus)

2D almost-Riemannian structure

Definition

A 2-ARS is the generalized Riemannian structure obtained locally by declaring that a pair of smooth vector fields X_1, X_2 which:

- can become collinear
- but satisfy the Hörmander (or Lie Bracket generating) condition

$$\forall\, q \quad \dim(\mathrm{span}_q\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]] \dots\} = 2$$

is an orthonormal frame.

- where X_1 and X_2 are linearly independent, they define a Riemannian metric
- on the set $\mathcal{Z}$ where X_1 and X_2 are parallel we are not Riemannian (we will see that g, dA, K explodes on $\mathcal{Z}$)

However

$$d(q_0, q_1) = \inf\{\int_0^T \sqrt{u_1(t)^2 + u_2(t)^2}dt \mid \dot\gamma = u_1(t)X_1 + u_2(t)X_2, \\ \gamma(0) = q_0,\ \gamma(T) = q_1\}$$

is well-defined and continuous and gives to M a structure of metric space compatible with its original topological structure.

This is essentially the Chow theorem (which states that Hormander condition implies finiteness and continuity of the distance)

$\rightarrow$notice that now $\dot\gamma = u_1(t)X_1 + u_2(t)X_2$, is no longer a definition of $u_1(t)$ and $u_2(t)$ but it is a constraint on the dynamics.

The globalization can be made in different equivalent ways:

- by writing compatibility conditions between charts

$$\begin{cases} Y_1(q) = \cos(\theta)X_1(q) + \sin(\theta)X_2(q), \\ Y_2(q) = -\sin(\theta)X_1(q) + \cos(\theta)X_2(q). \end{cases}$$

- defining a 2-ARS as an Euclidean bundle

Definition

Let M be a 2D connected smooth manifold. A 2D-*almost-Riemannian structure* on M is a pair $(\mathbf{U}, f)$ where

- $\mathbf{U}$ is an Euclidean bundle over M of rank 2. We denote each fiber by U_q, the scalar product on U_q by $(\cdot|\cdot)q$.
- $f : \mathbf{U} \to TM$ is a smooth map that is a morphism of vector bundles i.e. $f(U_q) \subseteq T_qM$ i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{f} & TM \\ & \searrow^{\pi_{\mathbf{U}}} & \downarrow \pi \\ & & M \end{array} \tag{1}$$

  and f is linear on fibers.
- $\blacktriangle = \{f(\sigma) \mid \sigma : M \to \mathbf{U} \text{ smooth section}\}$, is a bracket-generating family of vector fields.

Definition

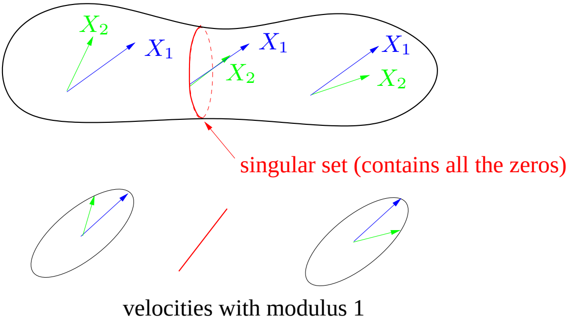
An *orthonormal frame* for the 2D almost-Riemannian structure on Ω is the pair of vector fields $\{F_1, F_2\} := \{f \circ \sigma_1, f \circ \sigma_2\}$ where $\{\sigma_1, \sigma_2\}$ is an orthonormal frame for $(\cdot \mid \cdot)_q$ on a local trivialization $\Omega \times \mathbf{R}^2$ of $\mathbf{U}$.

On a local trivialization $\Omega \times \mathbf{R}^2$, the map f can be written as $f(q, u) = u_1 F_1(q) + u_2 F_2(q)$.

Free structures

→If the structure is defined by a single pair of smooth vector fields we say that the structure is "free" (free structures are particularly interesting).

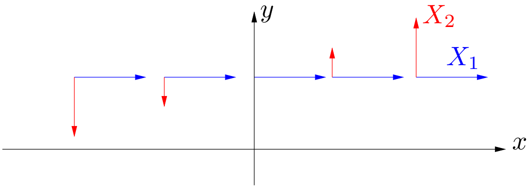
→Recall that zeros of vector fields are related to the topology of the manifold e.g. if M is compact orientable, the Riemannian metric defined by two globally defined vector fields is always singular (excepted for the torus)



Example 1: the Grushin plane

The Grushin plane that is the generalized Riemannian structure on the plane for which an orthonormal frame is given by

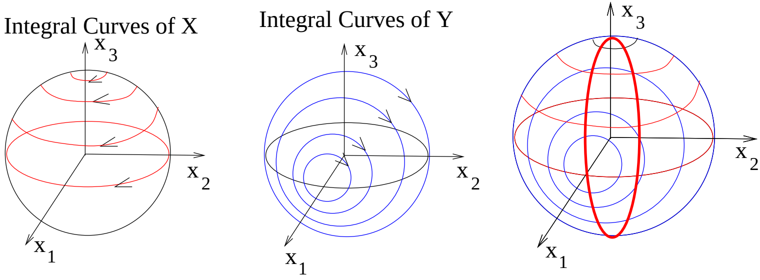
$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}, \qquad (x, y) \in \mathbf{R}^2$$



On $\mathbf{R}^2 \setminus \{x = 0\}$ we have $g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}$, $dA = \frac{1}{|x|} dx\, dy, \quad K = -\frac{2}{x^2}$

Example 2: The Quantum Sphere

$$M = S^2, \quad X = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ -x_3 \\ y \end{pmatrix}, \tag{2}$$



It is called the quantum sphere because this problem describes a controlled 3 level quantum problem (STIRAP process)

Let us recall some definitions and notations

- The *distribution* is defined as $\blacktriangle(q) = \{X(q) \mid X \in \blacktriangle\} = f(U_q) \subseteq T_qM$.
- The *step* of the structure at $q \in M$ is the minimal $s \in \mathbf{N}$, $s \geq 1$ such that $\blacktriangle_s(q) = T_qM$, where $\blacktriangle_1 := \blacktriangle$, $\blacktriangle_{i+1} := \blacktriangle_i + [\blacktriangle_1, \blacktriangle_i]$, for $i \geq 1$.
- The *(almost-Riemannian) norm* of a vector $v \in \blacktriangle_q$ is

$$\|v\| := \min\{|u|,\ u \in U_q \ \text{ s.t. } \ v = f(q,u)\}.$$

→example

- An *admissible curve* is a Lipschitz curve $\gamma : [0,T] \to M$ such that there exists a measurable and essentially bounded function $u : [0,T] \ni t \mapsto u(t) \in U_{\gamma(t)}$, called *control function*, such that $\dot{\gamma}(t) = f(\gamma(t), u(t))$, for a.e. $t \in [0,T]$.
  →example
- The *minimal control* of an admissible curve γ is

$$u^*(t) := \operatorname{argmin}\{|u|,\ u \in U_{\gamma(t)} \ \text{ s.t. } \ \dot{\gamma}(t) = f(\gamma(t), u)\}$$

  (for all t differentiability point of γ).

  →The minimal control is measurable.
- The *(almost-Riemannian) length* of an admissible curve $\gamma : [0,T] \to M$ is

$$\ell(\gamma) := \int_0^T \|\dot{\gamma}(t)\| dt = \int_0^T |u^*(t)| dt.$$

- The *(almost-Riemannian) distance* between two points $q_0, q_1 \in M$ is

$$d(q_0, q_1) = \inf\{\ell(\gamma) \mid \gamma : [0,T] \to M \text{ admissible},\ \gamma(0) = q_0,\ \gamma(T) = q_1\}. \quad (3)$$

is well-defined and continuous and gives to M a structure of metric space compatible with its original topological structure.

Some properties of length minimizers (local)

Existence. As a corollay of the Filippov theorem one has:

Theorem

Let $q_0 \in M$. There exists $\varepsilon > 0$ such that for every $q_1 \in B_{q_0}(\varepsilon)$ there exists a length minimizing curve joining q_0 to q_1.

$\rightarrow$this is as in Riemannian geometry. However even for ε small, the minimizer could be not unique.

Geodesics. Cannot be computed as in the Riemannian case with

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0$$

because this equation needs as initial condition $q^i(0)$ and $\dot{q}^i(0)$, but the $\dot{q}^i(0)$ are not all independent

Minimizers are computed with the Pontryagin Maximum Principle

Consider the problem

$$\dot{q} = u_1(t)X_1(q) + u_2(t)X_2(q)$$

$$\int_0^T \sqrt{u_1(t)^2 + u_2(t)^2}\, dt \to \min$$

$$q(0) \in \mathcal{S}, \quad q(T) \in \mathcal{T}$$

($\mathcal{S}$, and $\mathcal{T}$ zero or 1-dimensional manifolds)

If $(q(\cdot), u(\cdot))$ is a minimizer defined on $[0, T]$ and parameterized by constant velocity, then there exists a Lipschitz covector $p(\cdot)$ such that one or both the following conditions are satisfied:

ABN $\langle p(t), X_i(q(t)\rangle \equiv 0, \;\; i = 1, 2, \quad p(0) \neq 0$

NOR $u_i(t) = \langle p(t), X_i(q(t)\rangle \;\; i = 1, 2,$ and $q(\cdot)$ and $p(\cdot)$ are solution to the Hamiltonian system corresponding to

$$H = \frac{1}{2}(\langle p(t), X_1(q(t)\rangle^2 + \langle p(t), X_2(q(t)\rangle^2)$$

Moreover $\langle p(0), T_{q(0)}\mathcal{S}\rangle = 0$ and $\langle p(T), T_{q(T)}\mathcal{T}\rangle = 0$

→ $H = 1/2$ when trajectories are parameterized by arclength.

→small pieces of normals are minimizers

Theorem

For every q_0 there exists a system of coordinates and a local orthonormal frame around q_0 such that

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix}$$

Moreover integral curves of X_1 are normal Pontryagin extremals

$\rightarrow$Proof

Since

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \partial_1 f(x_1, x_2) \end{pmatrix}$$

we have thatin $(0, 0)$ the structure is:

- step 1 if $f(0, 0) \neq 0$
- step 2 if $f(0, 0) = 0$ and $\partial_1 f(0, 0) \neq 0$
- etc..

If the step is s at q then there exists $U(q)$ such that for every $\bar{q} \in U(q)$ we have that the step in $\bar{q}$ is $\leq$ than the step in q.

We are going to specify the normal forms at the different type of points

How big is the singular set ?

Theorem

Let μ be a smooth volume on M (not the Riemannian one!!!). Then $\mathcal{Z}$ has zero μ-volume

$\rightarrow$Proof.

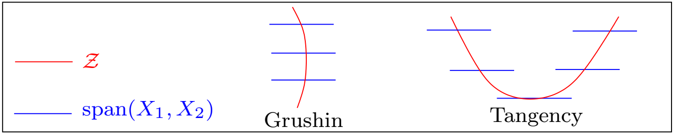
As a consequence 2-ARS are Riemannian in an open and dense subset of M.

Basic properties of the singular set 2

→**Under generic conditions** (when M is compact, for all systems in an open and dense subset of all 2D-ARS in the C^∞ topology)

- the singular set $\mathcal{Z}$ is a 1-dimensional embedded submanifold
- There are three type of points (here $\blacktriangle(q) = Span(X_1, X_2)$):
  - Riemannian points where X_1, X_2 are linearly independent,
  - Grushin points where $\blacktriangle(q)$ is 1-dimensional and $\dim(\blacktriangle(q) + [\blacktriangle, \blacktriangle](q)) = 2$

    (in this case $\blacktriangle(q)$ is transversal to $\mathcal{Z}$)
  - tangency points where $\dim(\blacktriangle(q) + [\blacktriangle, \blacktriangle](q)) = 1$ and the missing direction is obtained with one more bracket. ($\blacktriangle(q)$ is tangent to $\mathcal{Z}$ and these points are isolated)



- In a compact manifold $\mathcal{Z}$ is a set of non-intersecting circles

Basic properties of the singular set 3

There are sets of finite diameter and infinite area (w.r.t. the intrinsic distance and area)

Theorem

Let Ω be a bounded open set such that $\Omega \cap \mathcal{Z} \neq \emptyset$. Then

$$\mathrm{diam}(\Omega) \leq \infty \ \textit{and} \ \int_{\Omega \setminus \mathcal{Z}} dA = \infty$$

Example on the Grushin Plane