Diffusion in almost-Riemannian geometry 2/3

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2D almost-Riemannian structure

Definition

A 2-ARS is the generalized Riemannian structure obtained locally by declaring that a pair of smooth vector fields X_1, X_2 which:

- can become collinear
- but satisfy the Hörmander (or Lie Bracket generating) condition

$$\forall\ q \quad \dim(\text{span}_q\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]] \ldots\} = 2$$

  is an orthonormal frame.

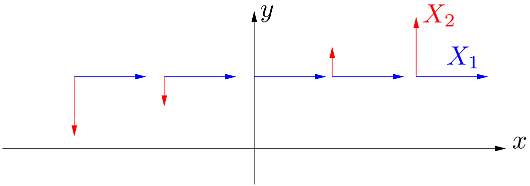
- where X_1 and X_2 are linearly independent, they define a Riemannian metric
- on the set $\mathcal{Z}$ where X_1 and X_2 are parallel we are not Riemannian (we will see that g, dA, K explodes on $\mathcal{Z}$)

When the structure is defined globally by a pairs of vector fields only then it is called **free**

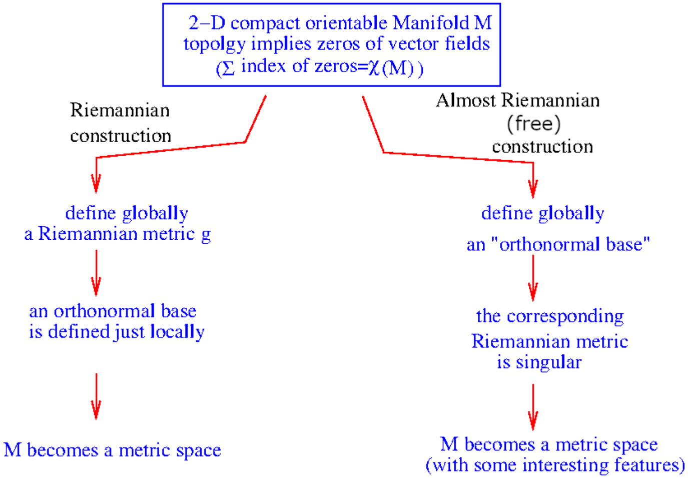
Example 1: the Grushin plane

The Grushin plane that is the generalized Riemannian structure on the plane for which an orthonormal frame is given by

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad (x, y) \in \mathbf{R}^2$$



Free structures are particularly important



(only) on the torus the two constructions coincide

Theorem

For every q_0 there exists a system of coordinates and a local orthonormal frame around q_0 such that

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix}$$

Moreover integral curves of X_1 are normal Pontryagin extremals

- $\mathcal{Z} = \{(x_1, x_2) \mid f(x_1, x_2) = 0\}$.
- on $M \setminus \mathcal{Z}$ one has

$$g_{(x_1,x_2)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{f(x_1,x_2)^2} \end{pmatrix},$$

$$dA_{(x_1,x_2)} = \frac{1}{|f(x_1, x_2)|} dx_1\, dx_2,$$

$$K(x_1, x_2) = \frac{f(x_1, x_2)\partial_{x_1}^2 f(x_1, x_2) - 2\left(\partial_{x_1} f(x_1, x_2)\right)^2}{f(x_1, x_2)^2}.$$

For Grushin $f = x_1$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x_1^2} \end{pmatrix}$, $dA = \frac{1}{|x_1|} dx_1\, dx_2, \quad K = -\frac{2}{x_1^2}$

Since

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f(x_1, x_2) \end{pmatrix}\right] = \begin{pmatrix} 0 \\ \partial_1 f(x_1, x_2) \end{pmatrix}$$

we have that in $(0,0)$ the structure is:

- step 1 if $f(0,0) \neq 0$
- step 2 if $f(0,0) = 0$ and $\partial_1 f(0,0) \neq 0$
- etc..

If the step is s at q then there exists $U(q)$ such that for every $\bar{q} \in U(q)$ we have that the step in $\bar{q}$ is $\leq$ than the step in q.

We are going to specify the normal forms at the different type of points

The different type of points

The different type of points 1/3

- if $\blacktriangle_1(q) = T_qM$ (i.e., if $q \notin \mathcal{Z}$)
  we say that q is a **Riemannian point**. In this case we have the normal form

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ e^{\phi(x_1,x_2)} \end{pmatrix}$$

- if $\blacktriangle_1(q) \neq T_qM$ and $\blacktriangle_2(q) = T_qM$ (i.e., if the step is 2 at q)
  we say that q is a **Grushin point**. In this case we have the normal form

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x_1 e^{\phi(x_1,x_2)} \end{pmatrix}$$

in both cases we could normalize $\phi(0, x_2) = 0$.

Lemma

If the step is at most 2 on M (i.e., if only Riemannian and Grushin points are present, i.e., if $\blacktriangle(q) + [\blacktriangle, \blacktriangle](q) = T_qM, \quad \forall q$) then $\mathcal{Z}$ is an embedded 1-D submanifold

→proof.

these are the nice 2-ARS.

The different type of points 2/3

- if $\blacktriangle_1(q) \neq T_qM$, $\blacktriangle_2(q) \neq T_qM$ and $\blacktriangle_3(q) = T_qM$ (i.e., if the step is 3 at q) we say that q is a **Tangency point**.

Lemma (application of Thom's transversality theorem)

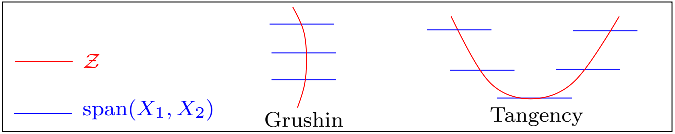
Generically (i.e. for an open and dense subset of all the 2-ARS on M w.r.t. the C^2-topology (standard if M is compact or Whitney if M is non compact) one has that

$$f(x_1, x_2) = 0 \text{ and } \partial_{x_1} f(x_1, x_2) = 0 \text{ occur only at isolated points.}$$

moreover at these points one has $\partial_{x_2} f \neq 0$ and $\partial^2_{x_1} f \neq 0$.

- generically $\mathcal{Z}$ is a 1-D embedded submanifold
- generically only Riemannian, Grushin and Tangency points are present
- generically at tangency points we have the normal form.

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ (x_2 - x_1^2\psi(x_1))e^{\phi(x_1,x_2)} \end{pmatrix}, \quad \psi(0) \neq 0.$$

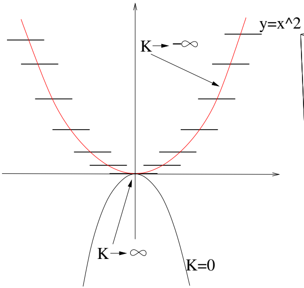


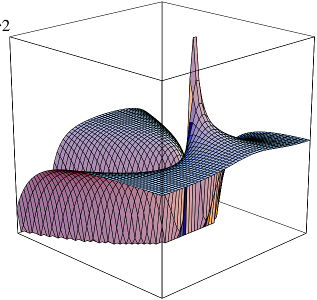
The different type of points 3/3: n example of tangency point

$$M = \mathbf{R}^2, \quad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ y - x^2 \end{pmatrix},$$

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(y-x^2)^2} \end{pmatrix}, \quad dA = \frac{1}{|y-x^2|} dx \wedge dy,$$

$$K(x,y) = \frac{-2\,\left(3\,x^2 + y\right)}{\left(x^2 - y\right)^2}.$$





The presence of Tangency points in generic structures renders their study quite a challenge.....

Properties of the singular set
(beyond the fact that it is a 1-D manifold in the generic case)

How big is the singular set ?

Theorem

Let μ be a smooth volume on M (not the Riemannian one!!!). Then $\mathcal{Z}$ has zero μ-volume

→Proof.

As a consequence 2-ARS are Riemannian in an open and dense subset of M.

Basic properties of the singular set 3

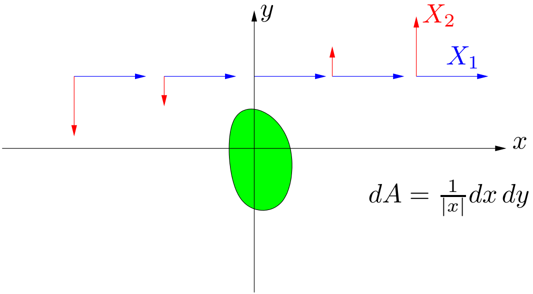
There are sets of finite diameter and infinite area (w.r.t. the intrinsic distance and area)

Theorem

Let Ω be a bounded open set such that $\Omega \cap \mathcal{Z} \neq \emptyset$. Then

$$\operatorname{diam}(\Omega) \leq \infty \text{ and } \int_{\Omega \setminus \mathcal{Z}} dA = \infty$$

Example on the Grushin Plane



Minimizers

Minimizers are computed with the Pontryagin Maximum Principle

Consider the problem: $\dot{q} = u_1(t)X_1(q) + u_2(t)X_2(q)$

$$\int_0^T \sqrt{u_1(t)^2 + u_2(t)^2}\, dt \to \min$$

$$q(0) \in \mathcal{S}, \quad q(T) \in \mathcal{T}$$

($\mathcal{S}$, and $\mathcal{T}$ zero or 1-dimensional manifolds)

If $(q(\cdot), u(\cdot))$ is a minimizer defined on $[0, T]$ and parameterized by constant velocity, then there exists a Lipschitz covector $p(\cdot)$ such that one or both the following conditions are satisfied:

(ABN) $\langle p(t), X_i(q(t)\rangle \equiv 0, \ \ i = 1, 2, \qquad p(0) \neq 0$

(NOR) $u_i(t) = \langle p(t), X_i(q(t)\rangle \ \ i = 1, 2,$ and $q(\cdot)$ and $p(\cdot)$ are solution to the Hamiltonian system corresponding to

$$H = \frac{1}{2}(\langle p(t), X_1(q(t)\rangle^2 + \langle p(t), X_2(q(t)\rangle^2)$$

Moreover $\langle p(0), T_{q(0)}\mathcal{S}\rangle = 0$ and $\langle p(T), T_{q(T)}\mathcal{T}\rangle = 0$

→$H = 1/2$ when trajectories are parameterized by arclength.

→normals extremals are smooth and small pieces are minimizers (**geodesics**)

→abnormal extremals could be nonsmooth (for a general SR problem) and could be nonminimizing....

Lemma

If γ *defined in* $[a, b]$ *is an abnormal extremal then it is trivial i.e.,* $\gamma(t) \equiv q_0 \in \mathcal{Z}$.

→proof

As a consequence for 2-ARs, geodesics are smooth.

Basic properties of Geodesics 1

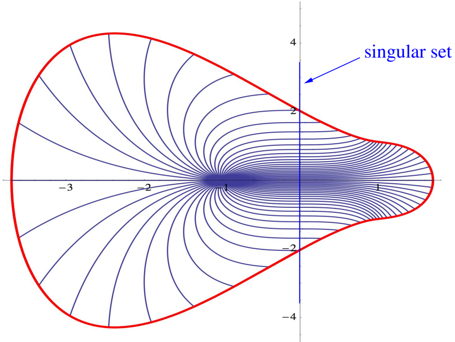
(archlength) geodesics are projections on the q space of Hamiltonian solutions of:

$$H(p,q) = \frac{1}{2}\left(\langle p, X_1(q)\rangle^2 + \langle p, X_2(q)\rangle^2\right)$$

corresponding to the level set $H = 1/2$.

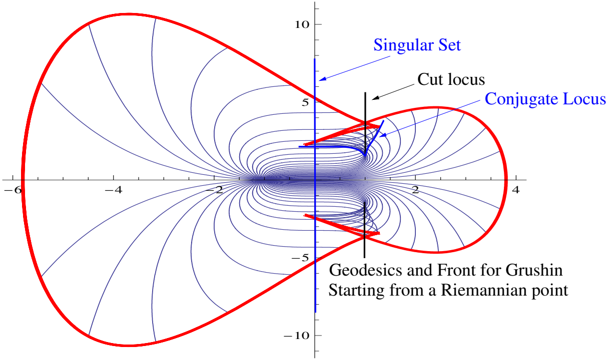
For 2-ARS all Riemannian quanities expodes, but geodesics are smooth and can cross the singular set with no singularities

Example (geodesics from $(-1,0)$ on the Grushin plane, $T = 2.7$)



Basic properties of Geodesics 2

the presence of a singular set permits the conjugate locus to be nonempty even if $k < 0, \forall \;\; q$
[Agrachev, Boscain, Sigalotti, DCDS, 2008]

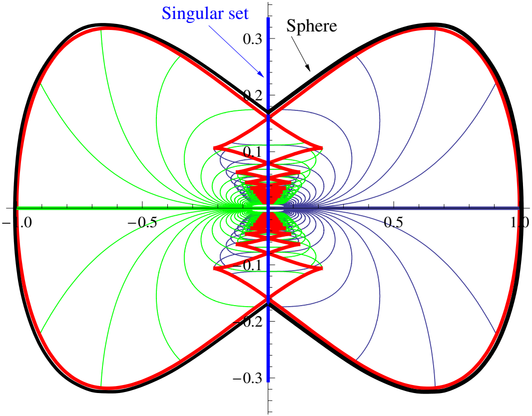


$k = -\frac{2}{x^2}$

$\rightarrow$The "length" of a sphere intersecting the singular set is ∞.
$\rightarrow$For people interested in singularity theory, this is the sole example of generic singularity in the analytic category that I am able to build with trigonometric functions..... it is almost-Riemannian

Basic properties of geodesics 3

Small spheres starting from the singular set are never smooth. (fixed a starting point on $\mathcal{Z}$, for every ε there exists a geodesics shorter than ε that already lost optimality)



Geodesics and front for the Grushin plane, starting from the singular set.

A Gauss Bonnet Theorem

Theorem (Gauss-Bonnet)

Let M be a 2D, compact, orientable Riemannian manifold. Then $\int_M K dA = 2\pi\ \chi(M) = 2\pi(2 - 2\mathbf{g})$

can we extend such a result in the AR context?

Definition

A 2-ARS is orientable if $\mathbf{U}$ is orientable as vector bundle or equivalent if we can take all matrices of changes of orthonormal frames in $SO(2)$.

This concept is unrelated from the orientability of M. There exists non orientable structures on orientable manifolds and viceversa.

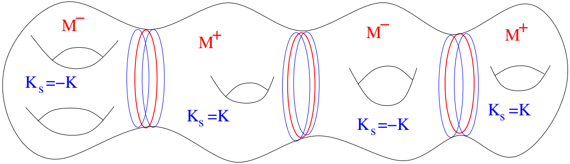
Definition

Let M be an orientable manifold and consider an **oriented, 2-step** 2-ARS on it. Let $\omega \in \Lambda^2(M)$ be a never vanishing two-form, defining an orientation on M. Define

$$M^{\pm} = \{p \in \Omega_i \setminus \mathcal{Z} | \ \ i \in I, \pm\omega(X_i, Y_i)(p) > 0\} \tag{1}$$

We call signed curvature the function (defined on $M \setminus \mathcal{Z}$)

$$K_s(p) = \begin{cases} K(p), & \text{if } p \in M^+, \\ -K(p), & \text{if } p \in M^-, \end{cases} \tag{2}$$



For every $\varepsilon > 0$, let $M_\varepsilon = \{p \in M | \ d(p, \mathcal{Z}) > \varepsilon\}$ We say that K_s is integrable on M if

$$\lim_{\varepsilon \to 0} \int_{M_\varepsilon} K_s \ dA \tag{3}$$

exists exists and is finite. In this case we denote such limit by $\int \mathcal{K}_s dA$.

Theorem

Let M be a compact oriented two-dimensional manifold and consider an oriented **2-step 2-ARS** *on it. Then*

$$\int \mathcal{K}_s dA = 2\pi(\chi(M^+) - \chi(M^-)), \tag{4}$$

where χ denotes the Euler characteristic.

Free structures

Lemma

If the 2-step 2-ARS is free then it is orientable and $(\chi(M^+) - \chi(M^-)) = 0$.

Corollary

Let M *be a compact oriented two-dimensional manifold and consider a free 2-step 2-ARS on it. Then*

$$\int \mathcal{K}_s dA = 0 \tag{5}$$

This is a deep fact: If the metric is defined globally by a couple of vector fields, then $\int \mathcal{K}_s dA = 0$. This is what happens in Riemannian geometry: on the torus!

$\rightarrow$we “force the manifold to be parallelizable by accepting singularities”

Manifold M (compact orientable)
topolgy implies zeros of vector fields
(Σ index of zeros$=\chi(M)$)

Riemannian construction

A Riemannian structure without singularities is defined locally on charts by vector fields

$$\int_M K \, dA = 2\pi\chi(M)$$

(topological information)

Almost Riemannian (trivialized) construction

A Riemannian structure defined globally by the vector fields

$$\int_M K_s \, dA = 0$$

has singularities
(topological information)

(only) on the torus the two constructions coincide

If the structure is not free then one should consider the Euclidean bundle $\mathbf{U}$ defining the ARS. Then if $\mathbf{U}$ is orientable we have

$$P\int_M K dA_s = e(\mathbf{U})$$

Extensions in presence of tangency points are possible, but less natural.